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STRESSES IN FRAMED STRUCTURES

Compiled by a Staff of Specialists

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SECOND EDITION

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1942

PREFACE TO THE SECOND EDITION

The Second Edition of this volume—one of the six in the Structural Engineers' Handbook Library—has been prepared by the undersigned for the use of the practicing engineer and also for the student who wishes to consult a reference work which covers thoroughly modern civil engineering structures.

Whenever necessary, revised material has been introduced which embodies the latest design or construction procedure. All errors of which there is any record have been corrected. The Editor-in-Chief of the present revision will be grateful to all those who, finding other errors, will bring them to his attention.

Major changes from the First Edition include those in Sec. 1, where the recommendations of the American Society of Civil Engineers on design for wind forces have been incorporated and the most recent American Railway Engineering Association specifications for steel railway bridges have been added, also the typical train loadings recently developed by Dr. D. B. Steinman. In Sec. 2 a new illustrative problem for the design of a roof truss, which embodies the American Society of Civil Engineering method of design for wind forces, has been included, and a mill bent has been analyzed in accordance with the same recommendations. In Sec. 3, centrifugal and longitudinal forces acting on trains in motion are in accordance with the latest A.R.E.A. recommendations. In Sec. 4, wind forces acting on loaded and unloaded bridges are also in accordance with latest A.R.E.A. recommendations.

Section 6 on "Statically Indeterminate Frames" has been modernized and considerably amplified by inclusion of the moment distribution method of analysis.

A new Appendix C has been added which presents charts for the design of concrete beams and columns with variable moment of inertia.

Credit has been given in the text of this volume for data, details, or photographs used for purposes of supplementing the technical matter Mention is made here of the participation of the following Associate Editors in the preparation of the first edition: Charles A. Ellis, Robins Fleming, S. G. Roebald, H. S. Rogers, C. A. Willson, and Wilbur M. Wilson.

The thanks of the undersigned are due all who have so willingly cooperated in the preparation of this revision.

R. R. Zipprodt.

Bethesda, Md.,
October, 1942.

PREFACE TO THE FIRST EDITION

This volume is one of a series designed to provide the engineer and the student with a reference work covering thoroughly the design and construction of the principal kinds and types of modern civil engineering structures. An effort has been made to give such a complete treatment of the elementary theory that the books may also be used for home study.

The titles of the six volumes comprising this series are as follows:

Foundations, Abutments and Footings
Structural Members and Connections
Stresses in Framed Structures
Steel and Timber Structures
Reinforced Concrete and Masonry Structures
Movable and Long-span Steel Bridges

Each volume is a unit in itself, as references are not made from one volume to another by section and article numbers. This arrangement allows the use of any one of the volumes as a text in schools and colleges without the use of any of the other volumes.

Data and details have been collected from many sources and credit is given in the body of the books for all material so obtained. A few chapters, however, throughout the six volumes have been taken without special mention, and with but few changes, from Hool and Johnson's Handbook of Building Construction.

The Editors-in-Chief wish to express their appreciation of the spirit of cooperation shown by the Associate Editors and the Publishers. This spirit of cooperation has made the task of the Editors-in-Chief one of pleasure and satisfaction.

G. A. H.
W. S. K.

MADISON, WIS.,
September, 1923.

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SECTION 1

GENERAL THEORY

TRUSSES IN GENERAL

1. Definition of a Truss.—A truss is a structure composed of three or more members so designed and connected that the structure as a whole acts as a beam and the individual members are subjected primarily to longitudinal stress.

For truss members to be considered properly as subjected to longitudinal stress only, they must at least be free to turn at the joints—that is, they must be pin-connected.

If the pin-connected frame of Fig. 1*a* should be loaded as shown, it would deform in the direction indicated by the dotted lines and a com-

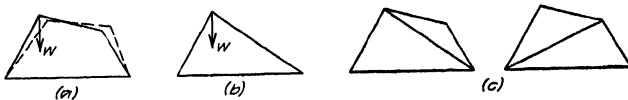



Fig. 1.

plete collapse of the structure would result. For this frame to be rigid when loaded, its joints must be capable of resisting moment, and hence the members themselves must carry bending moment in addition to longitudinal stress. Such is not the case, however, for the pin-connected frame shown in Fig. 1*b*. When this structure is loaded as shown, there will be no deformation in shape except that due to the very small changes in lengths of the members caused by longitudinal stress. Therefore the frame of Fig. 1*a* may be converted into a rigid structure by inserting a fifth member, as shown in Fig. 1*c*. From this explanation it should be clear that a truss, generally speaking, should be composed of a series of triangles.

2. Principal Elements of a Truss.—A common form of bridge truss is shown in Fig. 2. In this truss l is the *span* and h is the ***height*** or ***depth***

of the truss. The points *a*, *b*, *c*, *C*, *D*, etc. are called *joints*. The joints at points *B* and *F* are called *hip joints*.

The members along the lines *BF* and *ag* are the *chord members; BF* is the *upper chord* or *top chord* and *ag* is the *lower chord* or *bottom chord*.

The system of members connecting the chord members is called the *web system* and the individual members are called *web members*. Vertical web members are known as *verticals* and inclined web members are known as *diagonals*. The spaces between the verticals are called *panels* and the distances *ab*, *bc*, *BC*, etc., are called *panel lengths* and are generally equal. The diagonals *aB* and *Fg* at the ends of the truss are known as *end posts* and the verticals *Bb* and *Ff* are known as *hip verticals*. Tension members in the web system are sometimes called *ties*, while compression members in the web system are often called *posts* or *columns*.

If the dotted members *cD* and *De* were omitted, the diagonals *Cd* and *dE* would be in tension for certain loading conditions and in compression for other loadings—in other words, there would be a reversal of stress in these members. In order to avoid the necessity of designing a mem-

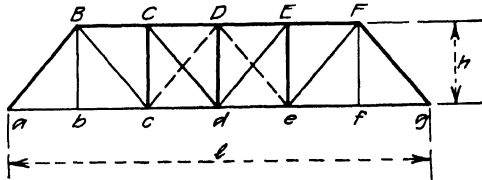



Fig. 2.

ber and its connections to resist both tension and compression, it is generally found desirable to place auxiliary diagonals, called *counters*, in all panels in which reversals of stress may occur. Then, under any given condition of loading, part of these diagonals are idle while the others are stressed, but in no case is the stress in any member changed back and forth from tension to compression under different conditions of loading.

Other things being equal, it is more economical to use a steel member for a given tensile stress than it is for an equal compressive stress. This is because of the buckling tendency in a compression member which is not present in a tension member. Due to this fact counters are generally made tension members.

3. Action of a Truss.—If a load is placed at joint *c* of the truss of Fig. 2, a part of this load is carried to the left reaction at *a* along members *cB* and *Ba* while the remainder of the load is carried to the right reaction at *g* along members *cD*, *Dd*, *dE*, *Ee*, *eF* and *Fg*. The top chord is in compression and the bottom chord is in tension. Under this particular condition of loading, members *Bb*, *Cc*, *Cd*, *De* and *Ff* are idle. If the load were fixed at joint *c*, all of these members might be removed and the truss would stand. However, if the load were moved to some other loca-

tion a part of these members would come into action and some of the other members would be unstressed. The hip verticals *Bb* and *Ff* are stressed only when there are loads at joints *b* and *f*.

4. General Classes of Trusses.—When the nature of the supporting forces is such that the reactions are vertical under vertical loading, and when the reactions can be determined by the methods of statics under any condition of loading, the framework is said to be a *simple truss*. If a structure is such that its reactions are inclined under vertical loading, it is known as an *arch*. When it is impossible to find the reactions and the stresses in the members by the methods of statics, the framework is called a *statically indeterminate structure*. In the following articles the discussion will be devoted primarily to simple trusses.

5. Types of Roof Trusses.—Some of the common forms of simple roof trusses are shown in Fig. 3. In the Howe or Triangular truss the top

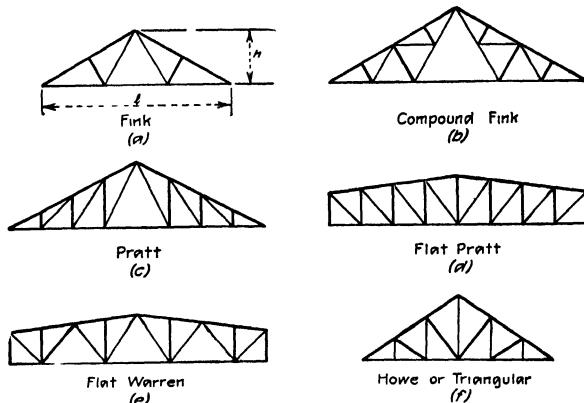



Fig. 3.

chord members and the diagonals are generally made of wood while the bottom chord members and the verticals are made of steel. All of the other trusses are generally made of steel throughout.

6. Pitch of Roof Truss.—The *pitch* or *rise* of a roof truss is a fraction representing the ratio of the center height or rise of the truss to its span length. In Fig. 3*a*, for example, the pitch is represented by the fraction $\frac{h}{l}$.

7. Classes of Bridges.—Bridges are classified with respect to purpose as *highway bridges* and *railroad bridges*. They are classified with respect to the location of the floor system as *through bridges*, *deck bridges* and *pony truss bridges*.

A through bridge is one in which the floor system is located at or near the *lower* chord. In a deck bridge, on the other hand, the floor system is located at or near the *upper* chord. In through and deck bridges

lateral bracing is provided at both the upper and lower chords. A pony truss bridge is a modified form of through bridge in which the trusses are so shallow that it is impossible to provide the upper lateral bracing.

The beam, girder, or truss structure including the floor system and lateral bracing comprise the *superstructure* of a bridge. The system of supporting elements, consisting of the abutments and piers, comprise the *substructure.*

8. Types of Bridge Trusses.—Some of the common forms of bridge trusses are shown in Fig. 4. For spans of moderate length either the Pratt, Warren or Howe truss is used. Of these three the Pratt truss is the most popular. The Warren truss is used extensively for pony

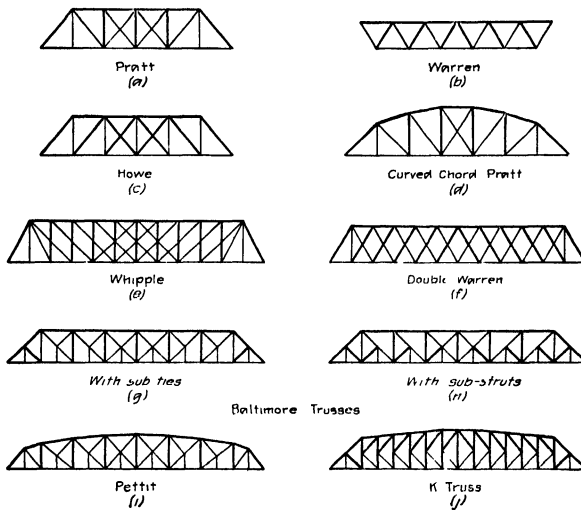



Fig. 4.

truss bridges while the Howe truss is used when it is advisable to make the compression members of wood. For longer spans it becomes economical to use the curved-chord Pratt truss shown in Fig. 4*d*.

Regardless of span length it has been found desirable to keep about the same inclination of web members as shown in these first four types. If, however, these trusses are used for very long spans and if the web members are maintained at about the same inclination, the cost of the floor system is very materially increased due to the increase in panel length. Therefore, it has been found necessary to develop other types of trusses which would have an economical inclination of web members and a relatively short panel length. The Whipple, Double Warren, Baltimore, Pettit and K trusses meet these requirements.

Both the Whipple and the Double Warren trusses are known as multiple intersection trusses. They are statically indeterminate and while

they were formerly used quite extensively they are now being replaced by the Baltimore, Pettit and K types (see Fig. 4).

The Baltimore and Pettit trusses are perhaps the most popular types for long span bridges at the present time. These two trusses are essentially of the same type since the Pettit truss is in reality just a curved-chord Baltimore truss. The chief objection to this type lies in the fact that the secondary stresses (see Sec. 5) are relatively high. For this reason the type of truss known as the K truss, shown in Fig. 4*j*, is being considered with increasing favor for long span bridges. In this type of bridge the secondary stresses are very small. It was largely due to this fact that the Quebec bridge was made of this type.

LOADS ON STRUCTURES

9. General Classes of Loads.—There are two general classes of loads for which a structure must be designed, namely, dead load and live load. The dead load includes the weight of the structure itself and any additional stationary loads which it must carry. The live load includes all loads which are movable or variable in amount.

Wind and snow are the live loads which act upon the outside of a roof. Ceilings and special loads such as traveling cranes are often fastened to the lower chords of roof trusses.

In the case of highway bridges, motor trucks, road rollers, and occasionally street cars constitute the primary live loads. Obviously, railroad trains are of major importance in the design of railroad bridges.

10. Dead Load.—The weight of a structure, which generally constitutes the total dead load or is a large part of it, is not definitely known until after the design is made. Hence it is evident that in designing, the weight must be assumed at first and the structure designed accordingly. If the resulting weight does not agree very closely with the assumed weight, the structure must be redesigned until the discrepancy between the assumed and the final weight is comparatively small.

To go through this process in the design of each new structure would require an immense amount of labor. In order to eliminate much if not all of this work various formulas have been developed for the purpose of approximating the weight of a given type of structure.

For wooden roof trusses the following formula recommended by Professor N. Clifford Ricker is quite commonly used:

$$w = 0.04L + 0.000167L^2$$

For steel roof trusses the formula which occurs in Ketchum's Structural Engineers Handbook is representative of good practice. This formula is as follows:

$$w = \frac{P}{45}\left(1 + \frac{L}{5\sqrt{A}}\right)$$

In each of the above formulas, w = weight of truss in pounds per square foot of horizontal covered area, P = capacity of truss in pounds per square foot of horizontal projection, A = distance between trusses in feet, and L = length of span in feet.

Formulas for the weights of simple span bridges under 300 ft. in length,[1] designed for Cooper's E-50 loading and in accordance with the American Railway Engineering Association specifications are given in "Modern Framed Structures," Part III. These formulas are as follows:

Deck plate girders

$$w = 12.5L + 100$$

Through plate girders with beams and stringers

$$w = 14L + 450$$

Through pin-connected trusses

$$w = 8L + 700$$

where L = length center to center of bearings. In the last formula 5,000 lb. should be added to the total weight if end floor beams are used. The plate girder formulas should be multiplied by 0.9 for Cooper's E-40 loading and by 1.1 for Cooper's E-60 loading. The pin-connected truss formula should be multiplied by $\frac{7}{8}$ for Cooper's E-40 loading and by $\frac{9}{8}$ for Cooper's E-60 loading.

A great many other formulas for the weights of structures might be given but those stated above are typical and are sufficient to illustrate the method of approximating the dead load.

11. Live Loads.—The live load which a structure must carry in any given case either is known or may be approximated quite readily after a study of the local conditions. Snow and wind loads usually govern the design of roofs, while railway and highway loadings govern the design of bridges.

11*a*. Snow Load.—In the design of bridges the effect of snow is so small in comparison with the other loads which the bridge must carry that it is generally neglected entirely. However, in the design of roofs, the weight of snow may become the major portion of the live load and therefore it must be considered.

The snow load for which a given roof must be designed is a variable quantity depending upon the climatic conditions and the pitch of the roof. It is usually assumed to vary from a maximum of 40 or 45 lb. per sq. ft. of roof surface in the northern parts of the United States to a value of zero for the southern states.

Ordinarily when snow load is considered in bridge design a value is assumed which is smaller than would be used for a roof situated in the

[1] Open floor, single track.

same locality. This is for the reason that the maximum highway or railway loading for the bridge and the maximum snow load are not apt to occur simultaneously.

11b. Wind Load.—Various formulas and rules have been presented in the past for the purpose of determining wind pressure upon a plane surface normal to the direction of the wind.

In the past, reliable results were believed to be given by the Duchemin formula, derived by a French army officer, Col. Duchemin, from experiments made in 1829. This formula is

$$P_n = P \frac{2 \sin \alpha}{1 + \sin^2 \alpha}$$

in which P = unit pressure in pounds per square foot on a surface perpendicular to the direction of the wind, P_n = component of pressure normal to the roof, and α = angle which the inclined surface makes with the direction of the wind. Since this formula has been widely used in the past, illustrative problems are provided in which it is applied.

Probably the most authoritative source of information at the present time is the *Final Report* of Sub-Committee No. 31, Committee on Steel, Structural Division of the American Society of Civil Engineers. This report, published in the *Transactions* of the Society, Vol. 105, p. 1713, 1940, recommends as follows:

(1) As a standard wind load for the United States and Canada the Sub-Committee recommends a uniformly distributed force of 20 lb per sq. ft. for the first 300 ft. above ground level, increased above this level by 2.5 lb. per sq ft for each additional 100 ft. of height, no omission of wind force being permitted for the lower parts of the building by reason of alleged shelter. Special wind-force specifications should be formulated locally for areas that are definitely known to be subject to hurricanes or tornadoes.

The foregoing recommendation applies only to those surfaces which are perpendicular to the direction of the wind, the wind being assumed to act horizontally. It has been recognized for some time that suction may exist for certain inclinations of a plane surface on the windward side and for all inclinations on the leeward side as well as for surfaces parallel to the wind. With respect to this effect, the report states:

(2) For proportioning the wind bracing of tall buildings it is not necessary to divide the wind force into pressure and suction effects, although this should generally be done for structures with rounded roofs, for mill or other buildings with large open interiors, and for walls in which large openings may occur. The effects of possible high local suction should be investigated in relation to secondary members and the attachment of roofing or siding.

(3) For plane surfaces inclined to the wind and not more than 300 ft. above the ground, the external wind force may be pressure or suction, depending on the exposure and the slope. For a windward slope inclined at not more than 20 deg. to the horizontal, a suction of 12 lb. per sq. ft. is recommended; for slopes between 20 and 30 deg. a suction uniformly diminishing from 12 lb. per sq. ft. to zero; and for slopes between 30 and 60 deg. a pressure increasing uniformly from zero to 9 lb. per sq. ft. For the leeward slope, for all inclinations in excess of zero, a suction of 9 lb. per sq. ft. is recommended.

(4) It is recommended that for a flat roof a normal external suction of not less than 12 lb. per sq. ft. should be considered as applied to the entire roof surface.

(5) On walls parallel to the wind it is recommended that an external suction of 9 lb. per sq. ft. should be considered.

Special recommendations for rounded roofs are:

(6) For roofs that are rounded or may be represented roughly by a circular arc passing through the two springings and the eaves, the wind force will depend not only upon the exposure and the ratio of rise to span of the equivalent circular arc but also upon whether the springings are elevated above the ground or are on the ground. Where the surfaces considered are not more than 300 ft. above ground level the recommended external wind force is as follows:

(*a*) On windward quarter of the roof arc, when the roof rests on elevated vertical supports and where the rise ratio is less than 0.20, a suction of 12 lb. per sq. ft. is recommended; and for a rise ratio varying from 0.20 to 0.60, a pressure increasing uniformly from zero to 12 lb. per sq. ft., or alternatively, for rise ratios between 0.20 and 0.35, a suction varying uniformly between these limits from 12 lb. per sq. ft. to zero is recommended. For roofs springing from the ground level a pressure, for rise ratios varying from zero to 0.60, uniformly increasing from zero to 11.4 lb. per sq. ft., is recommended.

(*b*) For the central half of the roof arc, where the roof rests on elevated vertical supports, with rise ratios varying from zero to 0.60, a suction uniformly varying from 11 lb. per sq. ft. to 20 lb. per sq ft is recommended; for roofs starting from ground level, a suction of 11 lb. per sq. ft , regardless of the rise ratio, is recommended.

(*c*) For the leeward quarter of the roof arc, for all values of the rise ratio greater than zero, a suction of 9 lb. per sq. ft. is recommended.

In addition to wind forces acting on the external surfaces, the report further recognizes the existence of internal wind forces of pressure or suction as stated below:

(7) Even for buildings that are nominally airtight, internal wind forces of either pressure or suction may exist, varying from 3 to 6 lb. per sq. ft. and depending on whether the openings are generally in the windward or in the leeward surfaces. Large internal pressure may arise due to the breaking of windows in the windward side of buildings by reason of flying gravel from the roof or other objects carried by the wind. Still larger internal forces of pressure or suction may arise when the windward or leeward side of a building is completely open. The Sub-Committee recommends that for buildings that are nominally airtight an internal pressure or suction of 4.5 lb. per sq. ft. should be considered as acting normal to the walls and the roof. For buildings

with 30 per cent or more of the wall surfaces open or subject to being open, an internal pressure of 12 lb. per sq. ft. or an internal suction of 9 lb. per sq. ft. is recommended; for buildings that have percentages of wall openings varying from zero to 30 per cent of the wall space, the recommendation is an internal pressure varying uniformly from 4.5 to 12 lb. per sq. ft. or an internal suction varying uniformly from 4.5 to 9 lb. per sq. ft.

With regard to the determination of the final design wind force in any specific instance, the following is quoted from the report:

(8) The Sub-Committee recommends that the design wind force applied to any surface of a building be a combination of (*a*) the aforementioned appropriate external wind force and (*b*) the appropriate indicated internal wind force.

(9) Where a series of roofs exists in one building, one roof being nominally masked by another, the structure as a whole should be designed for the full wind load on the first roof and for 80 per cent of the wind load on the other roofs. Any one roof should be designed for the full wind load.

(10) When wind surfaces are more than 300 ft. above the ground, the external and internal wind forces should be scaled up in the proportion that the prescribed wind force on plane surfaces normal to the wind fixed by recommendation (1) at the level under consideration bears to 20 lb per sq ft.

11c. Railway Loadings.– Railway bridges are generally designed for a load system consisting of two consolidation locomotives coupled together and followed by a uniform train load. In the early days of bridge engineering in this country it was customary for each railroad bridge engineer to select a loading which represented the heaviest train which would operate on his road and then to require that all bridges be designed for this loading. As might be expected there was a large variety of loadings which necessitated an unusual amount of tedious computation on the part of bridge designers.

In 1894 Theodore Cooper proposed a set of typical engine loadings in which the wheel spacings remained constant for all weights of trains while the wheel loads for light and medium weight trains were proportional to the wheel loads for heavy trains. Cooper's load systems were received with a great deal of favor and they are used extensively in the design of railway bridges. In 1923 Dr. D. B. Steinman[1] presented a different set of engine loadings known as the *M* loading, spaced and weighted more nearly to represent the arrangement of driving wheels used at that time.

Some bridge engineers have attempted to simplify the work of design still further by devising equivalent uniform load systems. An equiva-

[1] "Locomotive Loadings for Railroad Bridges," *Trans. Am. Soc. Civil Eng.*, Vol. LXXXVI, p. 606, 1923.

lent uniform load for calculating the stress in a given truss is one so chosen as to cause practically the same stresses as those due to the specified axle loads. While this method has certain advantages over the method in which concentrated loads are used, it has not been used so extensively.

If a railroad train were stopped on a bridge, the stresses in the members would be less than in the case where the train was moving rapidly over the structure. This may be caused by unbalanced locomotive drivers, rough and uneven track, flat or irregular wheels, or other minor influences. The stress produced in a member due to one or more of these causes is generally called *impact stress.*

While it is impossible to determine the exact amount of the stress due to impact, yet it is essential that this dynamic effect be considered in some way. Most bridge engineers agree that the best way of allowing for impact is to arbitrarily increase the live load stress according to some rule or formula supposed to represent the effect of impact. The 1940 A.R.E.A. Specifications for Steel Railway Bridges provides the following:

Impact

206. To the maximum computed static live load stresses, there shall be added the impact, consisting of

a. The rolling effect: The rolling effect is due to the rolling of the live load from side to side. It shall be taken as increasing the static live load on one rail by 20 per cent, with an equal decrease on the other rail. These loads shall be distributed to the supporting members.

b. The direct vertical effect:

With steam locomotives (hammer blow, track irregularities, and car impact), a percentage of the static live load stress equal to:

For L less than 100 ft . . . $100 - 0.60L$

For L 100 ft or more . $\frac{1800}{L - 40} + 10$

With electric locomotives (track irregularities and car impact), a percentage of the static live load stress equal to . . . $\frac{360}{L} + 12.5$

where L = length, in feet, center to center of supports for stringers, longitudinal girders, and trusses (chords and main members) or L = length of floor beams or transverse girders, in feet, for floor beams, floor beam hangers, subdiagonals of trusses, transverse girders, and supports for transverse girders.

207. For members receiving load from more than one track, the impact percentage shall be applied to the static live load on the number of tracks shown below:

Load received from:

Two tracks:

For L less than 175 ft.	Full impact on two tracks.
For L from 175 to 225 ft	Full impact on one track and a percentage of full impact on the other as given by the formula, $450 - 2L$.

For L greater than 225 feet.......... Full impact on one track and none on the other.

More than two tracks:

For all values of L.. .. . . Full impact on any two tracks.

11*d*. Highway Loadings.—The most modern practice in highway bridge design is represented by the "Standard Specifications for Highway Bridges," published by the American Association of State Highway Officials, Washington, D.C. The 1941 edition offers two optional standard trucks (H or H-S) and their corresponding lane loads which are equivalent to truck trains. The specifications also provide electric-railway and freight-car loadings for those highway bridges which carry electric-railway traffic.

12. Conversion of Surface Loads to Joint Loads.

12*a*. Roofs.—Since the snow and wind loads on a roof are applied to the whole roof surface and not to the joints of the roof trusses,

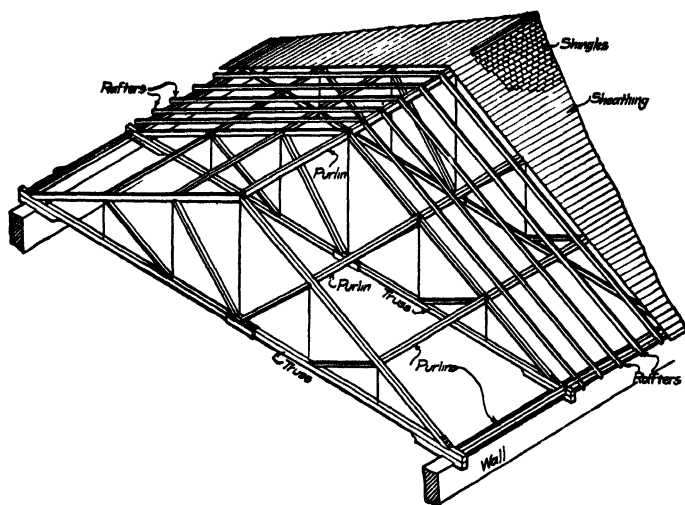



Fig. 5.

some means must be provided whereby these loads can be transferred from the roof surface to the truss joints.

A typical wooden roof framing system is shown in Fig. 5. The roof covering, shingles in this case, is laid upon sheathing which in turn is supported upon the rafters. These rafters are parallel to the planes of the trusses and are supported upon purlins. The purlins span the distance between trusses and transmit their loads to the trusses.

In this particular case the purlins occur only at truss joints. When the top chord panel lengths exceed a certain amount, it becomes econom-

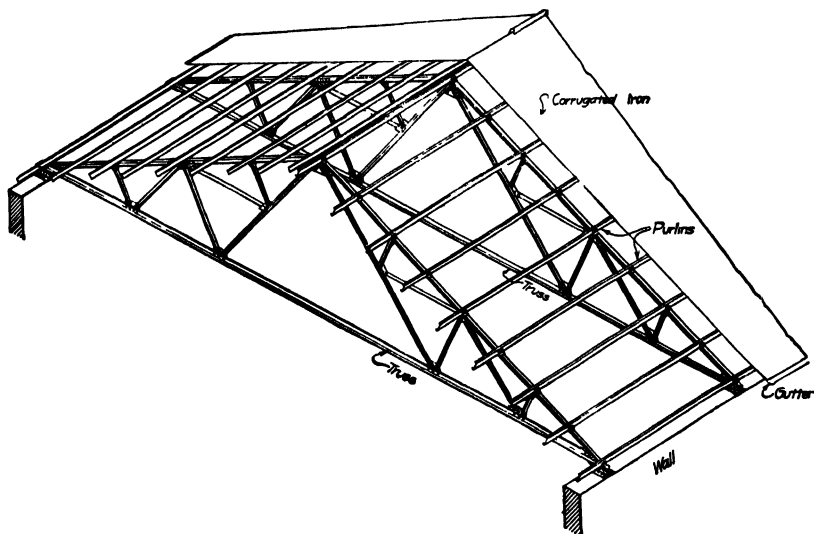



Fig. 6.

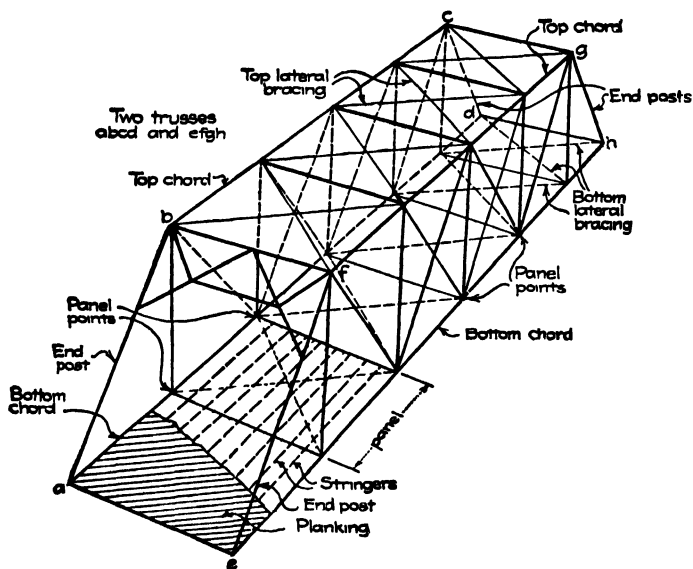



Fig. 7.

ical to reduce the span of the rafters by placing purlins at the centers of the top chord panels, even though this produces bending stresses in the top chord members of the trusses.

It may be seen from Fig. 5 that each intermediate roof truss carries the load acting upon a roof area whose horizontal projection has a width equal to the distance center to center of trusses and a length equal to the truss span length. Obviously, each of the end trusses carries the load on just half this roof area.

In the steel roof framing system shown in Fig. 6 corrugated iron is used for the roof covering. Sheathing and rafters are omitted and the corrugated iron is laid directly on the purlins. However, in this case, purlins are placed half-way between top chord joints as well as at the joints themselves.

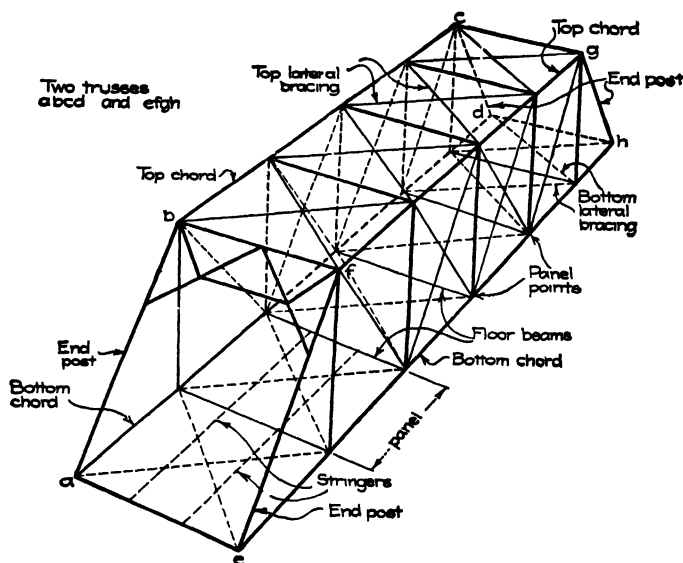



Fig. 8.

12b. Bridges.—The framing of a common form of steel highway bridge is shown in Fig. 7. In this type of bridge the floor is generally made of reinforced concrete supported on stringers. These stringers carry the load to the girders or floor beams which occur at every panel point of the lower chord. In this way any live load passing over the bridge floor is converted into joint loads acting at the panel points of the two trusses.

Lateral rigidity is provided by the top and bottom lateral bracing and the portal bracing.

The framing of a common form of steel railway bridge is shown in Fig. 8. A railroad bridge differs from a highway bridge only in the construction of the floor. In the open type floor system, the ties and rails take the place of the concrete floor in a highway bridge, the stringers being

placed below the ties and almost directly beneath the rails. The stringers are supported by the floor beams and the floor beams are connected with the trusses in the same manner as for highway bridges.

PRINCIPLES OF STATICS

13. Statics.—*Statics* is the science which treats of forces acting upon bodies at rest.[1]

14. Force.—A *force* is an action which, when applied to a body, changes its condition of rest or motion and also its form.

A force is either a push or a pull. If a horse is hitched to a log by means of a rope, and the horse drags the log, there is a pull in the rope. The pull in the rope is a force acting on the log. Likewise, if a man pushes an automobile, the push of the hand is the force which moves the car.

The effect of a force in changing the state of rest or motion of a body is apparent from everyday observation. When a croquet mallet strikes a ball, the ball changes from a state of rest to a state of motion. When a bat strikes a pitched ball, the ball changes from a motion in one direction to a motion in the opposite direction. In both cases the change in motion is a direct result of a force.

Bodies at rest, however, are also acted upon by forces. This is due to the fact that not one force, but a number of forces act upon the body, and these forces, as far as their effect upon the motion of the body is concerned, annul each other. This phase of the subject is discussed further in Art. **38.**

Likewise some examples of the effect of a force in changing the shape of a body are familiar. A rubber band is elongated. An eraser is compressed. A spring board is deflected. These particular changes in shape, or deformations, of a body are familiar to everyone because they are great enough to be seen. Similar deformations, however, take place in *all* bodies, when forces are applied to them. For example, a steel member of a truss subjected to tension elongates in the same manner as a rubber band, but because the elongation is small, it is not apparent. Nevertheless, it can be detected by the use of delicate instruments.

[1] Practically all engineering structures are bodies which remain at rest. It is true that as a train comes upon a bridge the bridge deflects slightly because of the weight of the train. But this movement is so small compared with the dimensions of the bridge, that the acceleration of the moving parts, and therefore the accelerating forces, are so small that they may, without appreciable error, be neglected in the analysis of the stresses in the structure. It is, in fact, the universal practice, in analyzing the stresses in ordinary engineering structures, to neglect the accelerating forces accompanying the deflection of the structure, and to treat the analysis of the stresses in the structure as a problem in pure statics.

15. Active and Passive Forces.—Some forces tend to move the bodies to which they are applied, whereas other forces prevent the bodies from moving. If a ball hangs by a string, gravity, acting on the ball, tends to move the ball and the tension in the string prevents it from moving. Likewise, some forces tend to change the form of a body, whereas, other forces prevent changes in form. Thus, in the case of a plank supported at the ends, if a man steps on the plank at the center it will deflect. But if a support is placed under the center of the plank the support will prevent the plank from deflecting.

A force which tends to move or to change the form of a body is known as an *active force.*

A force which prevents a body from moving or which prevents a change in form of a body is known as a *passive force.*

16. Elements of a Force.—Every force is composed of three elements; *i.e.*, *magnitude* (including sense or sign), *position* and *direction.*

The *magnitude* of a force is usually expressed in pounds. The *sense* or *sign* of a force is the direction along a line in which it acts, whether to the right or to the left, upward or downward. The sense of a force acting upward is opposite to the sense of a force acting downward. The sense of a force acting to the right is opposite to the sense of a force acting to the left. If a force acting upward is considered as positive, a force acting downward is negative.

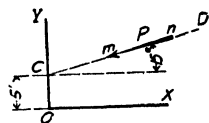
By *position* of a force is meant the location of the line along which the force acts. The position of a force is usually determined by locating one point in its line of action.

By *direction* of a force is meant the general direction of the line along which the force acts. It may act along a vertical line, along a horizontal line, or along a line making a known angle with the horizontal. The horizontal line may run east and west, or it may run north and south.

A force is not a *known force* unless all of its elements are known.

17. Vector.—All elements of a force can be represented graphically by a single line. The position of the line fixes the position of the force; the direction of the line fixes its direction; the length of the line represents its magnitude; and an arrowhead on the line determines its sense.

Fig. 9.

A line which represents the magnitude, direction, and sense of a force is known as a *vector.* Thus, in Fig. 9, the line CD passing through the point C, and making an angle of 15 deg. with a line parallel to OX, fixes the position and direction of the force P. The length of the line mn fixes its magnitude, and the arrow, indicating that the force acts downward and to the left, determines its sense.

In order to represent the magnitude of a force by the length of a line, it is necessary to establish a scale for converting distance into force—

that is, it is necessary to establish that a line 1 in. long represents a force of 10 lb., 100 lb., a ton, or any other convenient unit. If the scale decided upon is 1 in. = 1,000 lb., then a force of 4,500 lb. will be represented by a line 4.5 in. long. Likewise, with the same scale, a force-line 8.8 in. long represents a force of 8,800 lb.

The scale is determined arbitrarily but is influenced by the magnitude of the force and the size of the sheet of paper to be used.

18. Action and Reaction.—Every force is accompanied by an equal and opposite force acting in the same line. One of these forces is known as an *action,* the other as a *reaction.* Thus, in Fig. 10, a ball is suspended by a string. The ball exerts a downward force B upon the string; the string exerts an upward force S upon the ball. Furthermore, B and S are equal in magnitude, opposite in sense, and lie in the same line. Likewise, if we consider any point in the string, we find that the lower portion of the string pulls downward upon that point and the upper portion pulls upward with an equal force upon the same point.

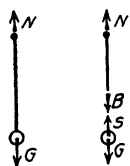



Fig. 10.

If a book lies on a table, the book presses downward upon the table and the table presses upward upon the book. An action is an active force and a reaction is a passive force. The pressure of the book on the table is an action; the pressure of the table on the book is a reaction.

19. External Force.—An *external force* upon a body is a force derived from some other body. In Fig. 10 the weight of the ball G and the pull of the string S are both external forces acting upon the ball. The reaction of the nail N and the pull of the ball B are both external forces acting upon the string. A man standing on a plank, a locomotive on a bridge, the weight of a structure itself, are all external forces. If A and B are two adjacent members of a bridge truss, the load delivered by A to B is one of the external forces acting on the member B.

An external force, if concentrated, is expressed as a force of a given number of pounds or tons. If the force is distributed, it may be expressed as a distributed force of a given number of pounds; as a force of a given number of pounds per square foot (or square inch) over a given area; or as a force of a given number of pounds per linear foot (or inch) over a given length.

20. Internal Force or Stress.—An *internal force* is the force which one particle of a body exerts upon an adjacent particle within the body. In Fig. 11 the equal and opposite forces, P and P, are applied to the opposite ends of the rod AB. The forces P and P are external forces acting upon the rod. There is tension in the rod from A to B which elongates the rod the same as a rubber band is elongated. A particle C just to the right of the section xx pulls to the right upon the particle

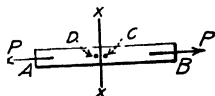



Fig. 11.

D just to the left of the section. Likewise the particle *D* exerts a pull to the left upon the particle *C*. This action, a push or a pull, of one particle upon adjacent particles is an internal force. It is also known as a stress, or an internal stress. The internal force or stress is caused by, but must not be confused with, the external forces.

An internal force or stress may be expressed as a total force of a given number of pounds; or it may be expressed as a stress of a given number of pounds per square inch or per square foot over a given area.

21. Concentrated Force.—A *concentrated force* is a force that is applied on a point. No forces exactly satisfy this requirement. However, forces which are distributed over very small areas are usually considered as concentrated forces. These include such forces as wheel loads from locomotives, the load from a column or girder, etc.

22. Distributed Forces.—*Distributed forces* are forces distributed over a considerable area. The load on a floor slab, hydraulic pressure on the sides of a water tank, and the load on a girder supporting a wall over its entire length, are examples of distributed forces.

23. Non-coplanar Forces.—*Non-coplanar forces* are forces that are not contained in a single plane.

24. Coplanar Forces.—*Coplanar forces* are forces that lie in a single plane.

25. Non-concurrent Forces.—*Non-concurrent forces* are forces that do not have a common point of intersection. In Fig. 12 the forces *A*, *B*, and *C* are non-concurrent since *C* does not pass through the point *O*, the intersection of *A* and *B*.

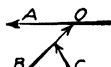



Fig. 12.

26. Concurrent Forces.—*Concurrent forces* are forces that do have a common point of intersection. In Fig. 13, *A*, *B*, *C*, and *D* are concurrent since, if they were extended, they would all pass through the point *O*.

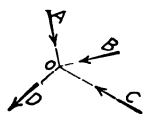



Fig. 13.

27. Equilibrium of Forces.—If a system of forces, when applied to a body at rest, does not cause the body to move, or when applied to a body in motion, does not change its rate of motion, then the system of forces is in *equilibrium*. If a system of forces is in equilibrium, any force of the system will balance all of the remaining forces of the system.

28. Resultant of a System of Forces.—A single force that has the same influence upon the state of rest or motion of a body as a given system of forces, is called the *resultant* of that system.

29. Equilibrant of a System of Forces.—Any force is completely annulled by an equal and opposite force acting in the same line. A force equal and opposite and acting in the same line as the resultant of a system will therefore annul the resultant and also annul the system itself—that is, it will hold the system in equilibrium.

A single force that holds a given system in equilibrium is called the *equilibrant* of the system. An equilibrant is always equal and oppo-

site and acts in the same line as the resultant of a system. For a system of forces to be in equilibrium each force must be the equilibrant of the remaining forces of the system.

30. Components of a Force.—Two or more forces that produce the same effect upon the state of rest or motion of a body as a single force, are called the *components* of the force.

31. Composition of Forces.—Finding the resultant of a system of forces is called the *composition* of forces.

32. Resolution of Forces.—Dividing a force into its components is called the *resolution* of forces.

33. Moment of a Force.—The *moment of a force* is the property by virtue of which the force tends to cause the body to which it is applied, to rotate about a given point known as the *center of moments.* The magnitude of the moment is the product of the force and its lever arm. The *lever arm* is the perpendicular distance from the center of moments to the line of action of the force. The force will tend to cause the body to rotate in either a clockwise or a counter-clockwise direction. The direction of the rotation indicates the sense of the moment. A clockwise moment is usually considered positive, and a counter-clockwise moment is usually considered negative. However, this designation of signs has been made arbitrarily and need not necessarily be followed.

34. Couple.—A *couple* consists of two equal and opposite parallel forces. The lever arm of a couple is the perpendicular distance between the forces.

35. Center of Gravity.—The *center of gravity of a mass* is the point of application of the resultant of the forces of gravity acting upon the individual particles of the mass. The center of gravity must satisfy this condition for all positions of the body. The resultant of the forces acting upon the individual particles is the weight of the mass.

Surfaces do not have either mass or weight and therefore cannot have centers of gravity in the strictest sense of the term. However, if weight proportional to area is assigned to surfaces, and if the above definition of the center of gravity is applied, we have defined a point whose location is of practical value to the structural engineer. Although not in its strictest sense a center of gravity, this point is known as the *center of gravity, or centroid, of an area.*

36. Moment of Inertia.—Both experimental data and rational equations establish the fact that if a mass moves about an axis with an accelerated (or retarded) motion of rotation, then the moment necessary to accelerate (or retard) each differential particle of the mass is proportional to the quantity $\rho^2 dM$, in which dM is the mass of the particle and ρ is the distance from the center of gravity of the differential particle to the axis of rotation. Furthermore, for a finite mass, the moment necessary to produce a given angular acceleration is found to be pro-

portional to the quantity $\Sigma\rho^2 dM$. This quantity has come to be known as the *moment of inertia of the mass* relative to the given axis. For algebraic convenience the quantity $\Sigma\rho^2 dM$, the moment of inertia, is usually represented by I.

The rational equations used to express the strength of structures frequently contain the expression $\Sigma x^2 dF$, $\Sigma y^2 dF$, or $\Sigma\rho^2 dF$, in which dF represents a differential element of a plane surface, x represents the distance of any element of the surface from the Y-axis lying in the plane of the surface, y represents the distance of any element of the surface from the X-axis lying in the plane of the surface, and ρ is the distance of any element of the surface measured from the Z-axis normal to the surface. Because of its similarity to the expression $\Sigma x^2 dM$ and for algebraic convenience, the quantity $\Sigma x^2 dF$ is known as the *moment of inertia of the area* relative to the Y-axis, and is represented by I_y. Likewise, the quantity $\Sigma y^2 dF$ is known as the moment of inertia of the area relative to the X-axis, and is represented by I_x; and the quantity $\Sigma\rho^2 dF$ is known as the moment of inertia of the area relative to the Z-axis, and is represented by I_z.

The quantities I_x and I_y are known as *rectangular moments of inertia* and I_z is known as the *polar moment of inertia* of the area.

Likewise, the quantity $\Sigma\rho^2 dS$, in which dS is a differential length of a line and ρ is the distance from any axis, is known as the moment of inertia of the line. This quantity is likewise represented by I.

The *moment of inertia of a force* is the product of the force and the square of the distance from the force to a given point or axis. The moment of inertia of a force is also known as the *second moment of the force*, inasmuch as it is the product of the moment and the lever arm.

The structural engineer deals almost exclusively with stationary masses, and, therefore, he is interested primarily in the moment of inertia of areas, and he has but little occasion to use the moment of inertia of masses or lines.

37. Radius of Gyration.—If all of a certain finite mass could be concentrated in one point, and if that point were so located that the moment of the inertia of the resulting mass relative to a given axis equals the moment of inertia of the original mass relative to the same axis, then the distance from the point where the mass is concentrated to the given axis, is known as the *radius of gyration* of the mass. The radius of gyration is usually represented by r. Algebraically

$$r = \sqrt{\frac{I}{M}}, \text{ or } I = Mr^2$$

Likewise, for an area, $r = \sqrt{\frac{I}{F}}$, and $I = Fr^2$

38. Conditions for Equilibrium of a System of Forces.—For a system of forces to be in equilibrium, the system when applied to a body at

rest will permit the body to remain at rest; there will be no motion in a horizontal direction, no motion in a vertical direction, and no motion of rotation. The conditions which must be fulfilled by a system of forces in equilibrium may therefore be expressed as follows: *In order that a system of forces may be in equilibrium, the sum of the horizontal forces must equal zero; the sum of the vertical forces must equal zero; and the sum of the moments about any point must equal zero.*

These three conditions for equilibrium are fundamental and may be expressed algebraically as follows.[1]

$$\Sigma X = 0 \qquad \Sigma Y = 0 \qquad \Sigma M = 0$$

Every body at rest is acted upon by a system of forces in equilibrium; if it were not so, the body would not remain at rest. We can therefore draw the conclusion that for every body at rest, roof truss, bridge truss, etc., $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$, in which ΣX is the sum of all of the horizontal forces acting upon the body, ΣY is the sum of all of the vertical forces acting upon the body, and ΣM is the sum of the moments about any point of all of the forces acting upon the body.

Since by definition the resultant of a system has the same effect upon the state of rest or motion of a body as the original system, the X-component of the resultant equals ΣX of the original system. Likewise, the Y-component and the moment of the resultant equal, respectively, ΣY and ΣM of the original system.

39. Resultant of Two Coplanar Forces.

39*a*. Graphical Method.—In Fig. 14, P and Q are two forces lying in the plane of the paper and intersecting at the point O. It is required to find the resultant of P and Q.

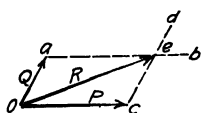



FIG. 14.

Tests in the laboratory have demonstrated that if a parallelogram is constructed such that two adjacent sides represent to scale any two forces, then the diagonal of the parallelogram formed upon the two forces as sides represents to the same scale the resultant of the forces in both magnitude and direction.

From a, the extremity of Q, draw the line ab parallel to P. Likewise, from c draw the line cd parallel to Q. The point e, the intersection of cd and ab, is the apex of the parallelogram, and the line Oe represents in magnitude, direction, and position, the resultant of P and Q. By applying the proper scale, the distance Oe can be converted into pounds. By scaling the drawing, the direction of the resultant can also be determined.

If the two forces do not intersect, extend the lines of action of the forces until the two lines do intersect. Then, since a force can be trans-

[1] X and Y need not be restricted to horizontal and vertical axes, but they may be allowed to represent axes in any two directions at right angles to each other and the equations will still hold.

ferred anywhere along its line of action, consider the forces to be applied at the point of intersection of their lines of action and find their resultant in the manner illustrated in Fig. 14. This construction is illustrated in Fig. 15.

In Fig. 15, P and Q are two coplanar forces. The point of intersection of their lines of action is e. Transfer P to P' and transfer Q to Q'. Then R, the diagonal of the parallelogram formed upon P' and Q' as sides, represents in magnitude, direction, and position the resultant of P and Q.

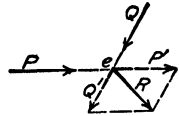



Fig. 15.

In Figs. 14 and 15, the sense of R is apparent when the sense of P and Q are known. In Fig. 14, the resultant of P and Q has been determined by constructing a parallelogram upon P and Q as sides. From an inspection of the figure it is apparent that Oae is a triangle whose sides are respectively equal and parallel to Q, P, and R. That is, instead of drawing the full parallelogram it is only necessary to draw the triangle that constitutes one-half of the parallelogram. This construction is illustrated in Fig. 16. The lines of action of P and Q intersect at e. Transfer P to P'. From the extremity of P' draw a line bg equal in length and parallel in direction to the line representing the force Q. From a similarity of the constructions in Figs. 15 and 16, eg of Fig. 16 is the resultant of P and Q.

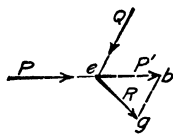



Fig 16.

39*b*. Algebraic Method.—As stated above, it has been demonstrated experimentally that, if Oa and Oc of Fig. 14, represent to scale the magnitude, direction, and position of two forces P and Q, then Oc represents to the same scale the magnitude, direction, and position of the resultant R.

Since the X-component of R equals ΣX for P and Q, $R \cos eOc = P + Q \cos aOc$. Also, from the law of sines, $\frac{R}{Q} = \frac{\sin Oae}{\sin acO}$. The value of R can be obtained algebraically from either of these equations.

40. Equilibrant of Two Coplanar Forces.—The equilibrant of a system of forces is always equal and opposite to the resultant, and always acts in the same line. This being true, if R of Fig. 16 is the resultant of P and Q, a force equal and opposite to R and acting in the same line as R is the equilibrant of P and Q.

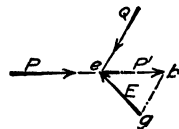



Fig. 17.

Figure 17 is identical with Fig. 16, except that R has been replaced by E, a force equal and opposite to R and acting in the same line. The equilibrant E acts upward and to the left, and, as evidenced by the figure, all of the arrows representing forces are point to butt around the triangle. This brings us to the

basic principle of the graphical composition and resolution of forces which is: If three coplanar forces form a system in equilibrium, the lines representing the forces form a triangle, a closed figure, and the arrows are point to butt around the figure.

41. Resolution of a Force into Two Components.

41*a*. Graphical Method.—Since the resultant of two coplanar forces is represented by the diagonal of a parallelogram formed upon the two forces as sides, the components of a force parallel to two given lines can be found by constructing a parallelogram whose diagonal represents the known force and whose sides are parallel, respectively, to the lines that determine the direction of the components.

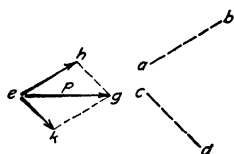



FIG. 18.

In Fig. 18, *P* represents any force. It is required to resolve *P* into two components, one parallel to *ab* and one parallel to *cd*.

Draw lines through *e* and *g* parallel to *ab*. Likewise draw lines through *e* and *g* parallel to *cd*. The two sets of lines intersect at *h* and *k*. Then *eh* and *ek* represent in magnitude, direction, and position the components of *P* parallel to *ab* and *cd*.

41*b*. Algebraic Method.—If the angles between the lines are known, the magnitudes of the two components can be determined algebraically as follows:

From the law of sines

$$eh = eg \frac{\sin egh}{\sin ehg}$$

and

$$ek = eg \frac{\sin egk}{ekg}$$

42. Resultant of Any Number of Concurrent Coplanar Forces.

42*a*. Graphical Method.—A system of coplanar forces

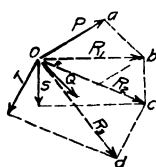
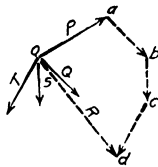
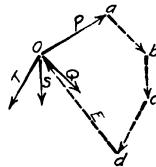



FIG. 19. FIG. 20. FIG. 21.

consists of a number of concurrent forces. It is required to find the resultant of the system.

First combine any two of the forces to find their resultant. Then combine this resultant with a third force. The resulting force will be

the resultant of the first three forces. Combine this second resultant with a fourth force, to get the resultant of the first four forces. Proceed in this manner until all the forces have been included and the last resultant will be the resultant of the whole system.

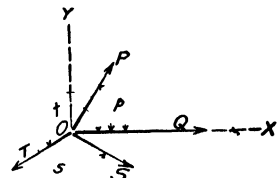
In Fig. 19, P, Q, S, and T are concurrent forces lying in a plane. It is required to find the resultant of the system.

Construct a parallelogram upon P and Q as sides. The diagonal of this parallelogram, R_1, is the resultant of P and Q. Likewise R_2, the diagonal of the parallelogram formed upon R_1 and S as sides, is the resultant of R_1 and S and, therefore, also the resultant of P, Q, and S. R_3, the diagonal of the parallelogram formed upon R_2 and T as sides, is the resultant of R_2 and T and, therefore, also the resultant of P, Q, S, and T. That is, R_3 represents in magnitude, direction, and position the resultant of the system.

From an inspection of the figure it is apparent that Oa coincides with P, ab is equal and parallel to Q, bc is equal and parallel to S, and cd is equal and parallel to T. The resultant R_3, which coincides with dO, could, therefore have been obtained by a simplified construction, as follows: From the extremity of P draw ab equal and parallel to Q, from b draw bc equal and parallel to S, and from c draw cd equal and parallel to T. Then the line Od, which closes the figure, represents, in magnitude, direction, and position, the resultant of P, Q, S, and T. This simplified construction is given in Fig. 20, in which the single force R is the resultant of the system.

It is important to note that, whereas the forces of the original system are point to butt around the figure, the resultant is point to point and butt to butt with the adjacent forces.

FIG 22

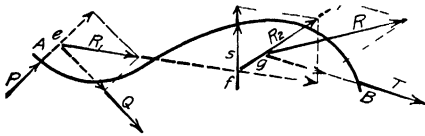
42b. Algebraic Method.—In Fig. 22, P, Q, S, and T represent the forces of any concurrent coplanar system. It is required to obtain the resultant of the system by the algebraic method.

This problem will be solved by making use of the fact that the X-component of the resultant equals ΣX for the system and the Y-component of the resultant equals ΣY for the system.

As the position of the axes may be fixed arbitrarily, the X-axis will be made to coincide with the line of action of the force Q and the Y-axis will be taken normal to the X-axis. The angles that the lines of action of the forces make with the X-axis will be represented by p, s, and t. The X-components of the forces are, therefore, $P \cos p$, Q, $S \cos s$, and $T \cos t$. The Y-components of the same forces are $P \sin p$, 0, $S \sin s$, and $T \sin t$. The X-component of the resultant, R_x is, therefore, $P \cos p + Q + S \cos s + T \cos t$, and the Y-component of the resultant R_y is P

$\sin p + 0 + S \sin s + T \sin t$. The resultant, R, equals $\sqrt{R_x^2 + R_y^2}$. The resultant passes through O and makes an angle with the X-axis whose tangent is $\frac{R_y}{R_x}$.

43. Equilibrant of Any Number of Concurrent Coplanar Forces.—If R of Fig. 20, acting from O to d, is the resultant of P, Q, S, and T, then an equal and opposite force acting from d to O is the equilibrant of P, Q, S, and T. Figure 21 is identical with Fig. 20 except that R has been replaced by E, an equal and opposite force acting in the same line. The system of Fig. 21, including the force E, is in equilibrium. E was determined by the following construction: From a, the extremity of P, draw a line ab equal and parallel to Q, from b draw bc equal and parallel to S, and from c, draw cd equal and parallel to T. Then dO, the line that closes the figure, is the force that holds the original system in equilibrium.

The diagram $OabcdO$ is known as a force polygon. It is a closed figure and the arrows are point to butt around the figure. This establishes the fundamental principle for the graphical determination of the equilibrant of a system of concurrent forces, which is: For a system of concurrent coplanar forces to be in equilibrium the force polygon must be a closed figure and the arrows must be point to butt around the figure; and conversely, if the force polygon for a system of concurrent coplanar forces is a closed figure, and if the arrows are point to butt around the figure, then the system is in equilibrium.

These statements relative to the graphical conditions which must be satisfied in order that a system of forces may be in equilibrium are equivalent to the algebraic conditions for static equilibrium given in Art. **38.** For if the force polygon is a closed figure, $\Sigma X = 0$ and $\Sigma Y = 0$ no matter what direction the X- and Y-axes may take.

44. Resultant of Any Number of Non-concurrent Coplanar Forces.

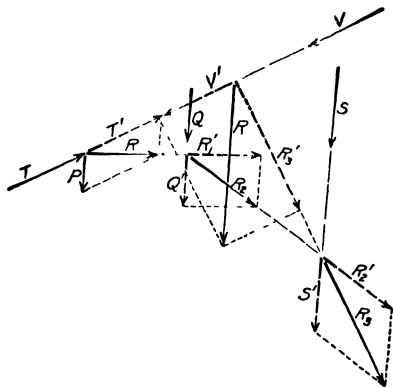
44a. Graphical Method (Special Case).—In Fig. 23 the body AB is acted upon by the system of forces, P, Q, S, and T. It is required to find the resultant of the system. Extend P and Q until they intersect at the point e. Their resultant is R_1. Extend R_1 and S until they intersect at f. Their resultant is R_2. The resultant of R_2 and T is R. The force R is therefore the resultant of the forces P, Q, S, and T, and, if R is applied to the body AB, it will have the same effect upon the motion of AB as the original system. A force equal and opposite to R is the equilibrant of the system and, when combined with the original system, forms a new system that is in equilibrium.

FIG. 23.

If the forces of a coplanar system are parallel, they will not intersect and the method outlined cannot be used. If, however, two equal and opposite forces acting in the same line are introduced into a system of parallel forces, the resulting system is equivalent to the original system, and the resultant of the new system can be determined by the method outlined in the preceding paragraph. This solution is used in the following problem.

In Fig. 24, P, Q, and S are parallel coplanar forces. It is required to find their resultant. The arrows T and V represent two equal and opposite forces acting in the same line that have been introduced into the system. The resultant of T, P, Q, S, and V is identical with the resultant of P, Q, and S, the original forces.

FIG. 24.

The resultant of T and P is R_1; the resultant of R_1 and Q is R_2; the resultant of R_2 and S is R_3; and the resultant of R_3 and V is R. Since R is the resultant of T, P, Q, S, and V, it is also the resultant of P, Q, and S, the original system. As a check, R must be parallel to P and the length of R must equal the algebraic sum of the lengths of P, Q, and S.

The constructions given in Figs. 23 and 24 are perfectly general, but they have the disadvantage that for many systems of forces the points of intersection are so widely scattered that they do not all fall on a sheet of drawing paper of ordinary size. For this reason a second construction has been devised which is more generally applicable. This construction is given in Art. **44c**.

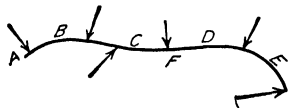



FIG. 25.

44b. Bow's Notation.—In Fig. 23 the forces are known as P, Q, S, and T. Obviously, as far as the effect of the forces is concerned, it is immaterial what system of nomenclature is used. However, because it lends itself so admirably to the graphical resolution and composition of forces, one system of nomenclature has come into almost universal use. This system will be explained in connection with Fig. 25.

In Fig. 25 the body AE is acted upon by a number of forces. Instead of having a letter representing a force, a letter is made to represent a space between two forces. Thus the letter B represents the space between two forces adjacent to each other at the left-hand end of the body, C represents the second space, D the third space, etc. The force itself is designated by the letters representing the spaces adjacent to the force.

Usually upper-case letters represent the spaces and the corresponding lower-case letters represent the forces. Under this system the first force at the left-hand end of the body is represented by *ab*, the second by *bc*, and the following by *cd, de, ef,* and *fa* respectively. The convenience of the system becomes apparent in connection with the determination of the resultant of a system of non-concurrent coplanar forces as set forth in the following article.

This system of nomenclature is known as Bow's notation.

44c. Graphical Method (General Case).—In Fig. 26 the body *AE* is acted upon by the system of forces consisting of *P, Q, S, T,* and *V*. It is required to find the resultant of the system.

From any point *n*, Fig. 27, draw the line *na* equal and parallel to *P*. From *a* draw *ab* equal and parallel to *Q*, from *b* draw *bc* equal and parallel to *S*, from *c* draw *cd* equal and parallel to *T*, and from *d* draw *de* equal and parallel to *V*. For a single force to be the resultant of the system, its X-component must equal the algebraic sum of the horizontal components of the forces of the system. Likewise the vertical component of the resultant must equal the algebraic sum of the vertical components of the forces of the system. The algebraic sum of the horizontal components of the system is represented in Fig. 27 by *x*, the horizontal distance from *n* to *e*. The algebraic sum of the vertical components of the system is represented in Fig. 27 by *y*, the vertical distance from *n* to *e*. The only force which has horizontal and vertical components equal respectively to *x* and *y* is the force *ne*. But *ne* together with *na, ab, bc, cd,* and *de* forms a closed figure. This figure is known as the force polygon.

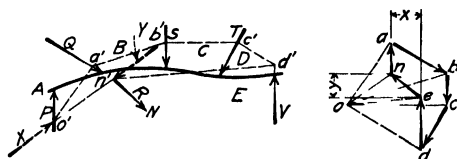
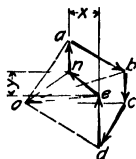



Fig. 26. Fig. 27.

A force equal and opposite to *ne* is the equilibrant of the system, and, together with the original system, forms a new system that is in equilibrium. Therefore we can say that: If a system of non-concurrent coplanar forces is in equilibrium, the force polygon will be a closed figure and the arrows will be point to butt around the figure.

The force polygon of Fig. 27 determines the magnitude, direction, and sense of the resultant of the system of forces in Fig. 26, but the line of action, or the position of the resultant, remains to be determined.

To determine the position in Fig. 26, of the resultant of the given system of forces, make the following construction:

From any point *o* in Fig. 27 draw the lines *oa, ob, oc, od, oe,* and *on*. From any point *o′* on the line of action of *P* in Fig. 26 draw a line parallel to *oa* of Fig. 27. This line cuts the line of action of *Q* at the point *a′*. From *a′* draw a line parallel to *ob* of Fig. 27. It intersects the line of

action of S at the point b'. From b' draw a line parallel to oc. It intersects the line of action of T at the point c'. From c' draw a line parallel to od. It cuts the line of action of V at the point d'. From d' draw a line parallel to oe and from o' draw a line parallel to on. These two lines intersect at n', and n' is a point on the line of action of the resultant ne. The magnitude, direction, and sense of the resultant is given by ne of Fig. 27, and its position is fixed by the location of the point n' in Fig. 26. The resultant R is therefore completely determined. A force equal and opposite to R, and acting in the same line as R, is the equilibrant of the system, and together with the original system forms a new system which is in equilibrium.

The statement is made above that n' in Fig. 26 is a point on the line of action of the resultant of the system. To prove that this statement is true, consider that the construction is made, and introduce at o' of Fig. 26 the force X equal and parallel to on of Fig. 27. Likewise, introduce at n' of Fig. 26 a force Y equal and parallel to no of Fig. 27. Since $o'n'$ was drawn parallel to on, both of these forces will lie in the line $o'n'$; and since they are equal and opposite they will completely annul each other and will not affect the original system. In other words, the resultant of the new system will also be the resultant of the original system.

In Fig. 27, the resultant of on and na, which represent X and P, is represented in magnitude and direction by the line oa. But $o'a'$ was drawn parallel to oa. Furthermore, X and P both pass through o'. Therefore, the resultant of X and P lies in the line $o'a'$, and its magnitude and sense are represented by oa. The resultant of oa and ab is represented in magnitude and direction by ob. But $a'b'$ was drawn parallel to ob. Also, both Q and the resultant oa of X and P, pass through a'. Therefore, the resultant of X, P, and Q lies in the line $a'b'$ and its magnitude and sense are represented by ob. Likewise, the resultant of X, P, Q and S lies in the line $b'c'$ and its magnitude and sense are represented by oc; the resultant of $X, P, Q, S,$ and T lies in the line $c'd$ and its magnitude and sense are represented by od; the resultant of X, P, Q, S, T and V lies in the line $d'n'$ and its magnitude and sense are represented by oe. This last resultant and Y are equivalent to the original system, and therefore their resultant will be the resultant of the original system. But the resultant of $X, P, Q, S, T,$ and V, as has just been demonstrated, lies in the line $d'n'$, and Y, as introduced, was made to lie in the line $o'n'$. Therefore, the resultant of the whole system must pass through the point n', the intersection of $d'n'$ and $o'n'$. Thus the intersection of these two lines determines one point on the line of action of the resultant whose location was to be determined.

Figure 27 is known as the *force polygon*. The point o, arbitrarily located, is known as the *pole* of the force polygon. The dotted lines oa, ob, oc, etc., are known as the rays of the force polygon. The force polygon

is used to determine the *magnitude, direction,* and *sense* of the resultant or of the equilibrant of any system of non-concurrent coplanar forces. It should be noted that the rays connecting the pole to the two extremities of a force in the force polygon may be considered as components of that force.

Figure 26 is known as the *space diagram.* The polygon $o'a'b'c'd'n'o'$ is known as the funicular polygon. The dotted lines $o'a'$, $a'b'$, etc., parallel to the corresponding rays of the force polygon, are known as the strings of the funicular polygon. The space diagram is used to determine the *line of action* of the resultant or of the equilibrant of any system of non-concurrent coplanar forces.

It is to be noted that although in drawing the force polygon the forces may be taken in any order, the forces must be considered in the same order in both the force polygon and the space diagram. In general, it is satisfactory to begin with the force at the left-hand end of the body on which the forces act and take the forces in order, going around the body in a clockwise direction. It should be remembered, however, that this is merely a convenient order, and need not be followed if, in any particular case, a more convenient order suggests itself.

It is also to be noted that although o can be located anywhere, and o' can be located anywhere on the first force considered, these points should be located so as to make the space diagram compact. To illustrate the difficulties that might arise, if o were located considerably below a, all rays would be inclined upward and to the right, and the space diagram would extend a considerable distance in a vertical direction.

It is further to be noted that the first string of the funicular polygon, $o'n'$, is parallel to the ray of the force polygon drawn to the *butt* of the first force considered; also that the last string of the funicular polygon, $d'n'$, is parallel to the ray of the force polygon drawn to the *point* of the last force.

The discussion of this section is summarized in the following instructions:

To determine the resultant of any system of non-concurrent coplanar forces:

(1) Draw a force polygon by drawing the forces point to butt around the figure, taking the forces in a clockwise order around the body on which the forces act. A line drawn from the butt of the first force to the point of the last force represents in magnitude, direction and sense the resultant of the given system.

(2) From any point selected as a pole, draw lines to the vertices of the force polygon. These are the rays of the force polygon.

(3) In the space diagram, from any point on the line of action of the first force considered, draw a line parallel to the ray extending from the pole to the butt of the first force in the force polygon. This is the first

string of the funicular polygon. From the same point on the first force in the space diagram, draw a second string parallel to the ray extending to the point of the first force in the force polygon. From the point where this string cuts the second force draw a third string parallel to the succeeding ray of the force polygon. Proceed in this manner until strings have been drawn in the space diagram corresponding to all rays in the force polygon. The last string will be parallel to the ray extending to the point of the last force in the force polygon. The intersection of the first and last strings of the funicular polygon is one point on the resultant of the system of forces.

(4) With one point on the resultant located in the space diagram, and with the magnitude, direction, and sense of the resultant given by the force polygon, the resultant is completely determined and can be represented by a line in the space diagram.

44*d*. Algebraic Method.—In determining the resultant of any system of non-concurrent coplanar forces by the algebraic method, the resultant will first be determined in magnitude, direction, and sense, and then its line of action will be located.

Consider a system of forces consisting of P, Q, S, T and V. Select any system of coordinate axes normal to each other. For convenience let the X-axis coincide in direction with the line of action of V. Let the angles which P, Q, S and T make with the X-axis be represented by p, q, s, and t. Then the X-components of the forces will be $P \cos p$, $Q \cos q$, $S \cos s$, $T \cos t$, and V. And the Y-components of the forces will be $P \sin p$, $Q \sin q$, $S \sin s$, $T \sin t$, and 0. The X-component of R equals ΣX, equals $P \cos p + Q \cos q + S \cos s + T \cos t + Q$. The Y-component of R equals ΣY, equals $P \sin p + Q \sin q + S \sin s + T \sin t + 0$. The magnitude of R is $\sqrt{\Sigma X^2 + \Sigma Y^2}$ and the tangent of the angle which R makes with the X-axis is $\frac{\Sigma Y}{\Sigma X}$. The sense of R will be determined from the sense of ΣX and of ΣY.

To determine the location of the resultant it will be necessary to use the equation $\Sigma M = 0$. The center of moments can be located at any convenient point. Since all elements of the original system of forces are known, the moments of the forces can be determined. Let p represent the lever arm of P, likewise let q, s, t, and v represent the lever arms of Q, S, T and V. Then $\Sigma M = Pp + Qq + Ss + Tt + Vv = Rr$, in which Rr is the moment of the resultant. Any one or all of the moments of the forces may be negative depending upon whether the indicated rotation is clockwise or counter-clockwise. The magnitude, direction and sense of R being known, the only unknown quantity in the equation $\Sigma M = 0$, is r, the lever arm of R. This distance can therefore be computed. With the lever arm of R known, one point on the line of action of R can be determined in the following manner: Through the center of moments draw

a line normal to the line of action of R. From the center of moments lay off on this line a distance equal to r. The point thus located will be one point on the line of action of the resultant. Since the direction of R is known, locating one point on its line of action completely determines its position.

If the forces are parallel, the resultant will be parallel to the forces and the magnitude of the resultant will be the algebraic sum of the component forces. The location of the resultant can be determined from the moment equation the same as for the general case.

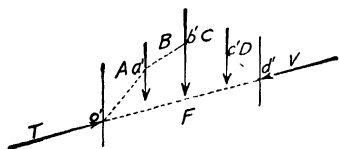
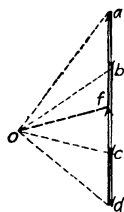
45. Equilibrants of Any Number of Parallel Coplanar Forces.—It frequently happens that it is necessary to determine the magnitude of two forces whose position and direction are known, which, acting together, will hold the given system of parallel forces in equilibrium. The problem of determining the reactions on a girder carrying a system of vertical loads is of this nature. The method of solving this problem is as follows:

The lines AB, BC and CD of Fig. 28 represent three parallel forces. It is required to determine two forces, one acting in the line DF and the other in the line FA, which will hold the given system in equilibrium.

Fig. 28. Fig. 29.

In Fig. 29, draw the load line consisting of ab, bc, and cd equal and parallel respectively to the forces ab, bc, and cd. (Bow's notation.)

The line ad represents the algebraic sum of the forces acting along the lines DF and FA. It remains to determine the magnitude and sense of each.

With any point o as a pole draw the rays of the force polygon oa, ob, oc, and od. Beginning at any point o' in the line FA, draw the strings of the funicular polygon $o'a'$, $a'b'$, $b'c'$, and $c'd'$ parallel respectively to the rays oa, ob, oc, and od of the force polygon. The line $o'd'$ is called the closing line of the funicular polygon. From o, the pole of the force diagram, draw a line parallel to the closing line $o'd'$. It cuts the load line ad in the point f, and the lines df and fa represent the magnitude of the forces that, acting along DF and FA, will hold the given system in equilibrium. The proof of this statement is as follows:

Consider that the force polygon and the space diagram have been constructed. Introduce into the system the two equal and opposite forces T and V. The line of action of these forces is the closing line of the space diagram and the magnitude of these forces is represented by of and fo, respectively, of the force polygon. Let fa represent a force that has been introduced in the line FA. Although we do not know that it is

the desired force, we do know that the resultant of this force and the force T acts along the line $o'a'$, for $o'a'$ by construction is parallel to oa. Also, the resultant of T, fa, and ab acts along the line $a'b'$; the resultant of T, fa, ab, and bc acts along the line $b'c'$; and the resultant of T, fa, ab, bc and cd acts along the line $c'd'$. The forces T, fa, ab, bc, and cd can, therefore, be held in equilibrium by a single force applied at d' equal in magnitude to do and acting along the line $d'c'$. If two forces are introduced at d', one equal and parallel to fo, and the other equal and parallel to df, their resultant will equal do and will lie along the line $d'c'$, since $c'd'$ was drawn parallel to od. Therefore, df and fo applied at d', together with T and fa applied at o', will hold ab, bc, and cd in equilibrium. But fo applied at d' completely annuls T applied at o', so that we have finally that df applied at d' and fa applied at o' balance ab, bc, and cd.

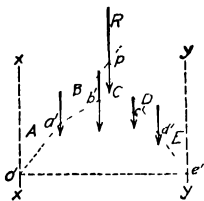



FIG. 30.

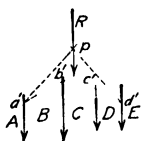
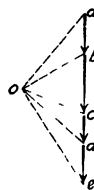
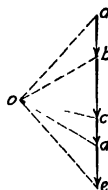



FIG. 31.

The construction which has just been presented for determining the magnitude of two forces that will balance a given system of parallel forces can also be used to determine the resultant or the equilibrant of a system of parallel forces.

In Fig. 30, ab, bc, cd and de are four parallel forces. It is required to determine the resultant of the system.

Construct the load line $abcde$ of the force polygon. Select any point o as the pole and draw the rays oa, ob, oc, od, and oe. Locate any two lines as xx and yy parallel to the forces. Beginning at any point o' in the line xx draw the funicular polygon $o'a'b'c'd'e'o'$ with strings parallel to oa, ob, oc, od and oe. Extend $o'a'$ and $e'd'$ until they intersect at p. The point p is one point on the line of action of R. The resultant R is, therefore, represented in magnitude by ae and it will pass through the point p. The proof of this construction is similar to the proof of the previous construction.

After the construction has once been established, the lines xx, yy, $o'a'$, $d'e'$ and $o'e'$ can be eliminated. The construction then becomes as follows:

In Fig. 31, draw the force polygon with the rays oa, ob, oc, od and oe. Beginning at a', any point on the line of action of ab, draw the strings of the funicular polygon $a'p$, $a'b'$, $b'c'$, $c'd'$ and $d'p$ parallel respectively to the rays oa, ob, oc, od and oe. The point p, the intersection of $a'p$ and $d'p$, is one point on the line of action of the resultant.

46. Moment of a Couple.—In Fig. 32, the two forces P and Q are equal, opposite, and parallel. Therefore, according to Art. **34,** P and Q form a couple. It is required to find the moment of the couple.

Let O represent any point in the plane of P and Q. The moment of Q about O is $-Qx$; and the moment of P about O is $+P(a + x)$. Therefore the moment of the couple about O is $(Pa + Px - Qx)$. But P and Q are equal in magnitude so that $-Qx$ is annulled by $+Px$ and the moment of the couple is Qa, or Pa. That is, the moment of a couple is independent of the location of the center of moments, and it equals in magnitude the product of one of the forces multiplied by the normal distance between the forces. This product is usually represented by Pa. Since the moment of a couple always equals Pa, a couple can be moved, either rotated or translated, to any place in its own plane without changing its moment.

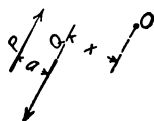



Fig. 32.

Both ΣX and ΣY of a couple equal zero for P_x is equal and opposite to Q_x, and P_y is equal and opposite to Q_y. Therefore a couple can be moved any place in its own plane without changing the values of ΣX and ΣY of the couple.

We have finally, therefore, that a couple can be moved any place in its own plane without changing ΣM, ΣX, or ΣY. Further, a couple can be replaced by any other couple in the same plane without affecting the system of forces of which it forms a part, so long as the moment of the new couple equals, in magnitude and sense, the moment of the original couple. That is, the couple can be translated and rotated, and the magnitude of P can be changed so long as there is a simultaneous change in a such that the product, Pa, remains constant.

47. Illustrative Problems in Statics.

47a. Summary of the Fundamental Conditions for Static Equilibrium.—The fundamental conditions for static equilibrium are as follows:

For a system of coplanar forces to be in equilibrium $\Sigma X = 0$, $\Sigma Y = 0$ *and* $\Sigma M = 0$.

For a system of coplanar forces to be in equilibrium the force polygon must be a closed figure with the arrows point to butt around the figure, and the first and last strings of the funicular polygon must intersect in a

point on the line of action of the one force which holds the balance of the system in equilibrium.

The statement that the force polygon must be a closed figure with the arrows point to butt around the figure is equivalent to saying that $\Sigma X = 0$ and $\Sigma Y = 0$. Similarly, the statement that the first and last strings of the funicular polygon must intersect in a point on the line of action of the force which holds the balance of the system in equilibrium, is equivalent to the statement that $\Sigma M = 0$.

47b. Determination of Unknown Elements of a System of Forces.—It has just been stated that for a system of forces to be in equilibrium three conditions must be satisfied. Therefore, if a system is in equilibrium three unknown elements in the system may be determined. These unknown elements may all relate to one force, to two forces, or to three forces. The three elements of a force are magnitude (including sense), direction, and position. The conditions for static equilibrium can therefore be used to determine various combinations of three of these unknown elements. Either the algebraic or the graphical method may be employed.

47c. Determination of the Magnitude, Direction, and Position of a Single Force.

Algebraic Solution.—Figure 33 represents a body acted upon by a force of 50 lb. at A, a force of 20 lb. at C, and a force of 10 lb. at D. It is required to find the magnitude, direction, and position of a single force which, combined with the given forces, will form a system in equilibrium.

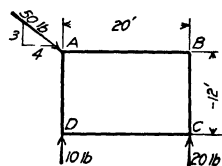



Fig. 33.

There are three unknown elements, and, as three equations are available, the unknowns can be determined. The equations are $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma M = 0$. The X-axis will be taken parallel to CD and the Y-axis will be taken parallel to AD. Moments will be taken about the point A.[1]

Let the force which is to be determined be represented by P and let the X-component of this force be P_x, and the Y-component P_y.

The X-component of the force at A is 40 lb. and the Y-component is 30 lb.

From $\Sigma X = 0$, we have $\Sigma X = 40 \text{ lb.} + P_x = 0$, or $P_x = -40$ lb. From $\Sigma Y = 0$ we have $\Sigma Y = 10 + 20 - 30 + P_y = 0$ or $P_y = 0$.

[1] The axes can be taken in any position and the moments can be taken about any point. However, we should choose the axes and the center of moments so as to make the algebraic work as simple as possible. With the axes as chosen, the forces at C and D are parallel to the Y-axis so that the Y-components equal the forces themselves, and the X-components are zero. Likewise, with the center of moments at A, the lever arms of the forces at A and D are zero and the forces do not enter the moment equation.

The force P, therefore, that balances the given system is a horizontal force of 40 lb. acting to the left.

From the equation $\Sigma M = 0$, we have $\Sigma M = P_x y - (20)(20) = 0$, in which y is the vertical distance of P from A. Solving this equation for y, we have $y = 10$ ft. The moment of P about A must balance the moment of the force at C about the same point. Since this latter moment is counter-clockwise, the moment of P must be clockwise. Therefore P is below and a distance of 10 ft. from A.

We have finally, then, that the force P, which balances the given system, is a horizontal force of 40 lb., acting to the left, at a distance 10 ft. below A. The force is therefore completely determined.

Graphical Solution.—The graphical solution of the above problem is as follows:

The body together with the forces acting upon it is reproduced in Fig. 34. The first step is to draw a force polygon. In this case we will use a scale of 1 in. = 20 lb.

To construct the polygon, from any point *1* draw the line *12* which, to the predetermined scale, is equal and parallel to the force D. Since D is 10 lb. it will be represented by a line ½ in. long. Likewise draw *23* parallel to A, 2½ in. long; and draw *34* parallel to C, 1 in. long. The line *41* which closes the figure is found to be parallel to CD and, when measured, is found to be 2 in. long, representing a force of 40 lb. The line *41* represents the force P which was to be determined. Since the arrows must be point to butt around the figure, P acts to the left.

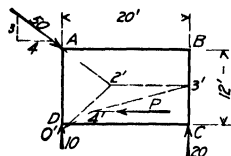



Fig. 34.

It will be necessary to draw the space diagram to locate the position of P. Select o as the pole of the force polygon and draw the rays $o1$, $o2$, $o3$, and $o4$. Beginning at o', any point on the line of action of D, draw the funicular polygon $o'2'3'4'o'$. The line $o'4'$ is parallel to $o1$ and the line $3'4'$ is parallel to $o4$. These two lines intersect at the point $4'$, one point on the line of action of P. By scaling the drawing it is found that the point $4'$ is 10 ft. below AB. The force P necessary to hold the given system in equilibrium is therefore a horizontal force of 40 lb. acting to the left and located a distance of 10 ft. below the line AB.

47*d*. Determination of the Magnitude of One Force, and the Direction and Position of a Second Force.

Algebraic Solution.—Figure 35 represents a body acted upon by the forces P, A, C, D, and E. The forces C, D, and E are completely determined; the force at A is known in position and direction but unknown in magnitude; and the force P is known to have a magnitude of 100 lb. but its direction and position are unknown. It is required to find the magnitude of the force at A and the direction and position of the force P.

As before, we will make use of the three equations, $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$. From $\Sigma X = 0$, we have $\Sigma X = P_x - 60$ (the X-component of the force at D) $= 0$, or $P_x = 60$ lb. acting to the right. Since $P_x =$ 60 lb., and since $P = 100$ lb., P_y must $= \pm\sqrt{100^2 - 60^2}$ or ± 80 lb.

The fact that the Y-component of P may be either plus or minus indicates that there are two sets of values that will satisfy the conditions for equilibrium. First, let us consider the case in which the Y-component of P is upward. Applying the equation $\Sigma Y = 0$, gives $\Sigma Y = 80 - 30 - 80 + 40 + A = 0$. In this equation it has been assumed that the force at A acts upward. Solving the equation we get $A = -10$, which shows that the force at A, instead of being an upward force as assumed, is a downward force of 10 lb. We do not know the position of P, but we do know that the line of action of P must cross the line AE or AE extended. Consider that P has been resolved into its X- and Y-components at the point where it crosses AE. The X-component of P will then have a moment about $A = 0$, and if we write $\Sigma M = 0$, we will get $\Sigma M = (30)(20) + (80)(30) - (60)(18) - (40)(30) - (80)(x) = 0$, in which x is the horizontal distance from A to the point in which P crosses AE. In this equation it has been assumed that the moment of P about A is counter-clockwise. Solving this equation for $80x$, we find that our assumption relative to the sign of the moment at P was correct; that is, the moment of P is actually negative and, inasmuch as the Y-component of P is upward, P will cross AE to the right of A. Solving the above equation for x, gives $x = 9$ ft., which shows that if the upward value of the Y-component of P is taken, then P crosses the line AE at a distance of 9 ft. to the right of A, it acts upward and to the right, and it makes a slope with AE of 60 horizontal to 80 vertical. The magnitude of A and the position and direction of P are therefore determined.

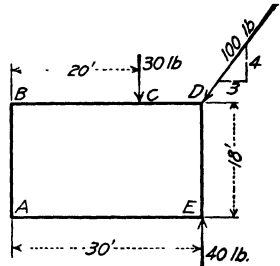



FIG. 35.

It remains to determine the corresponding values when the Y-component of P is taken as acting downward. Applying the equation $\Sigma Y = 0$, we have $\Sigma Y = -80 - 30 - 80 + 40 + A = 0$. Solving this equation for A gives $A = 150$ lb. As before, consider that P has been resolved into its X- and Y-components at the point where P crosses AE. Applying the equation $\Sigma M = 0$ with the point A as center of moments, we have $\Sigma M = (30)(20) + (80)(30) - (60)(18) - (40)(30) + (80)(x) = 0$. Solving this equation we find that $P_y x$ is negative, which indicates that our assumption as to the sign of the moment of P about A is wrong. Since the Y-component of P is downward, for the moment to be negative P must intersect AE extended to the left of A. Solving the equation for x gives $x = 9$ ft. We have, then, finally that if the Y-component of P

acts downward, the force P crosses AE extended at a point 9 ft. to the left of A, acts downward and to the right, making a slope with AE of 60 horizontal to 80 vertical, and the force at A is equal to 150 lb. acting upward.

Graphical Solution.—The force polygon is represented by Fig. 36. The lines *12*, *23* and *34* are equal and parallel respectively to the forces C, D, and E of Fig. 37. The force polygon from *4* to *1* is to be closed by two forces; one of these forces, being parallel to A, is vertical; the other is known in magnitude, having been given as a force P equal to 100 lb. The two remaining sides will therefore consist of one vertical line through the point *4* and one line passing through *1* whose length represents to scale a force of 100 lb. To construct these sides, draw a vertical line through *4*; with the point *1* as a center and with a radius whose length represents

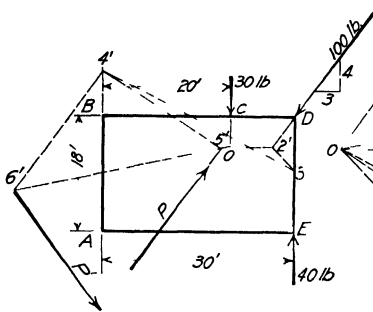
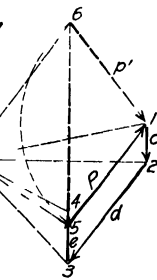



Fig. 37. Fig. 36.

to scale a force of 100 lb., describe an arc: this arc cuts the vertical line through *4* in the points *5* and *6*. Apparently, therefore, the force polygon can be closed with either the lines *45* and *51* or by the lines *46* and *61*. Scaling from the drawing, the line *45* represents a downward force of 10 lb., and the line *46* represents an upward force of 150 lb. The line *51* represents a force acting upward and to the right with a slope of 60 horizontal to 80 vertical and *61* represents a force acting downward and to the right with a slope of 60 horizontal to 80 vertical. The forces corresponding to either *45* and *51* or the forces corresponding to *46* and *61* will satisfy the conditions that $\Sigma X = 0$ and $\Sigma Y = 0$.

It will be necessary to draw a space diagram in order to locate the position of the force P. Locate a pole o and draw the rays of the force polygon as shown. Beginning at any point in the line of action of the force at C, as o', draw the lines $o'2'$, $2'3'$, $3'4'$, parallel respectively to the rays $o2$, $o3$, and $o4$ of the force polygon. The completion of the funicular polygon depends upon which pair of forces is to be considered.

If the forces represented by *45* and *51* are to be considered, then the funicular is completed by drawing a line through o' parallel to $o1$ and drawing a line through $4'$ parallel to $o5$. These lines intersect at the point

5', one point on the force P whose position is to be determined. The direction of P is given by the line *51* in the force polygon, and we find that P cuts the line AE at a point 9 ft. to the right of A. The force A which, combined with P, holds the system in equilibrium is represented by *45* and is a downward force of 10 lb.

If the forces represented by *46* and *61* of the force polygon are to be considered as the forces that hold the given system in equilibrium, then the funicular polygon is completed as follows: From o' draw a line parallel to $o\,1$, and from $4'$ draw a line parallel to $o6$. These lines intersect at $6'$ and the force P', equal and parallel to *61* of the force polygon and passing through *6* of the space diagram, is the force that was to be determined. Extending P' until it intersects AE, we find that P' cuts AE at a point 9 ft. to the left of A. All elements of P' therefore are known. The force A that, acting with P' holds the system in equilibrium, is represented by the line *46* and is an upward force of 150 lb.

47e. Determination of the Magnitude of One Force and the Magnitude and Direction of a Second Force.—Figure 38 represents a body in equilibrium under the action of the forces A, B, C, and D. All elements of B and C are known; the direction and position of D are known and the position of A is known. It is required to find the magnitude of D and the magnitude and direction of A.

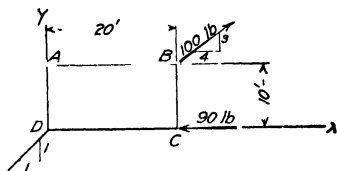



Fig. 38.

Algebraic Solution.—Introduce coordinate axes such that the X-axis coincides with DC and the Y-axis coincides with DA. The X-component of B is 80 lb., and the Y-component of B is 60 lb.

From $\Sigma M = 0$ with A as the center of moments, we have $\Sigma M = (90)(10) - (60)(20) + (D_x)(10) = 0$, or $D_x = \dfrac{1{,}200 - 900}{10} = 30$ lb.

The moment of D_x was assumed to be clockwise and the sign of $10D_x$ came out plus, indicating that the assumption was correct. Therefore the X-component of D is a horizontal force of 30 lb. acting to the left. Since D has a slope of 1 horizontal to 1 vertical, the Y-component of D is a downward force of 30 lb. The force D itself is, therefore, equal to $\sqrt{30^2 + 30^2} = 42.3$ lb. acting downward and to the left.

From $\Sigma X = 0$, we have $\Sigma X = A_x - 30 - 90 + 80 = 0$, or $A_x =$ 40 lb.

The component A_x was assumed to be plus and its value came out plus, indicating that the assumption was correct.

From $\Sigma Y = 0$, we have $\Sigma Y = A_y - 30 + 60 = 0$, or $A_y = -30$.

The component A_y was assumed to act upward but its value came out negative, indicating that the assumption was wrong. Therefore the

vertical component of A is a downward force of 30 lb. The magnitude of the force A is equal to $\sqrt{30^2 + 40^2} = 50$ lb. The force A, therefore, has a magnitude of 50 lb., passes through the point A, and acts downward and to the right with a slope of 30 vertical to 40 horizontal.

Graphical Method.—This problem cannot be solved graphically by the method given in Art. **44c**, for the force polygon cannot be drawn when the magnitudes of two forces and the direction of one are unknown. The following method will be used instead:

In Fig. 39, extend the lines of action of B and C until they intersect at e. Determine their resultant R. Extend the lines of action of R and D until they intersect at f. By construction, f is one point in the line of action of the resultant of B, C, and D. If A, B, C, and D form a system in equilibrium, A is the equilibrant of B, C, and D and, therefore, its line of action must coincide with the line of action of the resultant of B, C, and D. That is, A must pass through f. Since the force A must also pass through the point A, its line of action is Af.

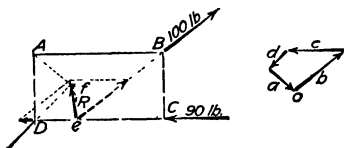



FIG. 39.

With the line of action of A known, the force polygon can be constructed as follows: Beginning at o, draw b and c equal and parallel, respectively, to the forces B and C. From the point of c draw a line parallel to the force D. Likewise, through the butt of b draw a line parallel to Af. The intersection of these two lines will close the force polygon and will determine the magnitudes of the forces A and D. The direction of A is determined by the line Af. The sense of A and D are determined by making the arrows point to butt around the force polygon. The magnitude of A and D can be obtained by scaling the drawing.

48. Location of the Center of Gravity of an Area.—If an area is symmetrical about two separate lines, the center of gravity is at the point of intersection of the two lines of symmetry. The center of gravity of a circular area is at the intersection of any two diameters; of a square at the intersection of the two diagonals; and of a rectangle at the intersection of the two diagonals. The center of gravity of a triangle is at the intersection of two lines each of which is drawn from a vertex to the middle of the opposite side.

If the outline of a surface is such that the center of gravity can not be determined by any of the above rules, then it is necessary to divide the total area into smaller areas, such that the center of gravity of each can be determined by inspection. Then each area is treated as a force proportional to the area, and applied at the center of gravity of the area. The resultant of these forces passes through the center of gravity of the area as a whole. By considering that the gravity forces act in one direc-

tion in the plane of the surface, a resultant in that direction can be determined. Then, by having the forces act in another direction, also in the plane of the surface, a second resultant can be determined. The intersection of the two resultants is the center of gravity of the area as a whole.

Illustrative Problem. *Algebraic Solution.*—Figure 40 represents the section of a T-beam. It is required to find the location of its center of gravity.

Since the area is symmetrical, the center of gravity will be somewhere on the center line. It remains to determine the distance of the center of gravity from the axis OY.

Consider the area to be so located that the gravity lines are parallel to OY. Divide the section into the two areas A and B. The area of A is 60 sq. in. and the area of B is 108 sq. in. The center of gravity of A is on the center line, and at a distance of 5 in from OY. The center of gravity of B is on the center line. and at a distance of 13

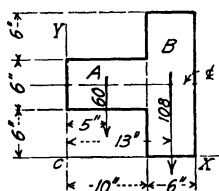
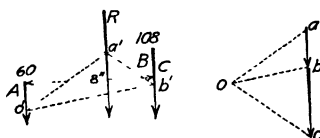



FIG. 40. FIG. 41.

in. from OY. The problem then reduces itself to determining the location of the resultant of the two parallel forces, one having a magnitude of 60, and the other a magnitude of 108. The magnitude of the resultant is 168. From $\Sigma M = 0$, with O as the center of moments, we have $\Sigma M = (168)(x) = (60)(5) + (108)(13)$, or $x =$ 10.14 ft. The center of gravity is therefore on the center line and at a distance of 10.14 in. to the right of OY.

Graphical Solution.—The forces A and B of Fig. 41 are proportional to the areas A and B of Fig. 40. The magnitude of A is 60 and of B is 108. The lines of action of the two forces are 8 in. apart. Draw the load line abc. From any point o, draw the rays oa, ob, and oc. From o', any point in the line of action of A, draw $o'a'$ parallel to oa. Also from o' draw $o'b'$ parallel to ob. From b', the intersection of $o'b'$ with B, draw $b'a'$ parallel to oc. The point a', the intersection of $o'a'$ and $b'a'$, is one point on the resultant of A and B. Therefore a line through a' parallel to A and B is a gravity line of the original area, and the intersection of this gravity line with the horizontal center line of the area is the center of gravity of the figure.

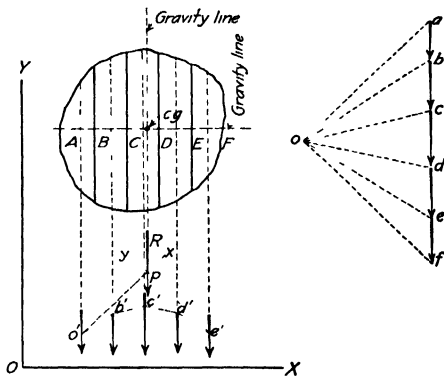



FIG. 42.

Illustrative Problem.—Figure 42 represents an area of irregular outline. It is required to determine the location of the center of gravity of the area.

First place the area in such a position that the axis OY (arbitrarily chosen) will be in a vertical position. The force of gravity will then act in a line parallel to OY.

Divide the area into a number of strips as shown. The number of strips necessary will depend upon the irregularities of the figure and upon the degree of accuracy desired. The area of each strip can be replaced by a force proportional to its area and acting through the center of gravity of the strip. These forces are represented by *AB*, *BC*, *CD*, *DE*, and *EF*. Draw the force polygon with the load line *abcdef*. The first string, *o'x*, of the funicular polygon is parallel to *oa* and the last string, *e'y*, is parallel to *of*. Then *p*, the intersection of *o'x* and *e'y*, is one point on a gravity line parallel to *OY*.

Now rotate the area until the axis *OX* is in a vertical position, and, by repeating the construction, locate a gravity line parallel to *OX*. The center of gravity of the area is the point of intersection of the two gravity lines.

To solve the same problem algebraically, divide the area into strips in two directions as before and locate the two gravity lines by writing the moment of the resultant equal to the sum of the moments of the forces representing the areas of the strips. This gives two gravity lines and their intersection is the center of gravity of the area.

49. Location of the Center of Gravity of a Solid.—If a homogeneous solid has three planes of symmetry, then the point in which one plane cuts the line of intersection of the other two is the center of gravity of the body. In the case of a sphere two vertical diametrical planes intersect in a line which is the vertical axis of the sphere. Any diametrical plane that is not vertical will cut this axis in a point that is the center of gravity of the sphere.

The center of gravity of a homogeneous prism is at the center of gravity of a transverse section and is at the mid-height of the prism.

To determine the location of the center of gravity of an irregular solid, divide the solid into a number of laminae, or parallel slices, and replace each lamina with a force equal to its weight and acting through its center of gravity. Determine the resultant of these forces. A plane containing the resultant and parallel to the lamina will be one gravity plane of the solid. By taking a second set of laminae intersecting the first, a second gravity plane can be located. These two planes will intersect in a line that will be a gravity line of the solid. By taking a third set of laminae intersecting the established gravity line, a third gravity plane can be located. The point in which this plane cuts the established gravity line is the center of gravity of the original solid. The number of laminations to be used in each case depends upon the irregularities of the body and upon the accuracy desired. The gravity planes can be located either algebraically or graphically.

50. Polar Moment of Inertia.—In Fig. 43, *X* and *Y* are coordinate axes lying in the plane of the surface *CD*. The quantity *dF* represents any differential area of the surface. Its distance from the X-axis is *y*, from the Y-axis is *x*, and from the Z-axis, normal to the plane of the paper at *O*, is ρ. By definition, the moments of inertia about the various axes are as follows, $I_x = dFy^2$; $I_y = dFx^2$; and $I_z = dF\rho^2$. But, by construction, $\rho^2 = x^2 + y^2$. Therefore $I_z = I_x + I_y$. That is, the polar moment of inertia of *dF* equals the sum of the two rectangular moments of inertia of the same differential area, if the polar axis and the

two rectangular axes intersect in a common point, as O. Since this statement is true for the moment of inertia of each differential element of the area it must also be true for the moment of inertia of the whole area.

Thus, the polar moment of inertia of an area is equal to the sum of the rectangular moments of inertia of the same area, providing the rectangular axes and the polar axis all meet in a common point and at right angles to each other.

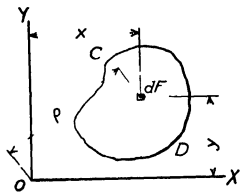



FIG. 43

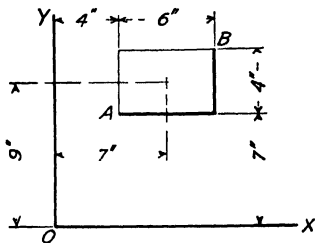



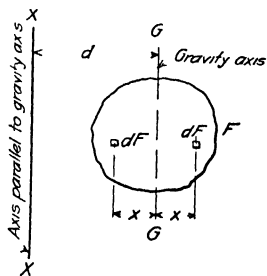
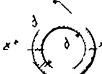
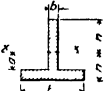
FIG. 44.

Illustrative Problem.—It is required to find the polar moment of inertia of the rectangle AB of Fig 44, about an axis normal to the plane of AB and passing through the origin O.

From Arts. **51 and 52,** $I_x = (\frac{1}{12})(6)(4)^3 + (4)(6)(9)^2 = 1{,}976$ in.^{4} $I_y = (\frac{1}{12})(4)(6)^3 + (4)(6)(7)^2 = 1{,}248$ in.^{4} From $I_p = I_x + I_y$, we get $I_p = 1{,}976 + 1{,}248 = 3{,}224$ in ^{4}

51. Moment of Inertia about Axes Parallel to a Gravity Axis.—In Fig. 45, F represents any area and GG represents a gravity axis of this area. The line XX is any line in the plane of F parallel to the gravity axis, and its distance from GG is represented by d. It is required to find the moment of inertia of the area F about XX in terms of the moment of inertia of the same area about the gravity axis GG.

FIG. 45.

The quantity dF represents any differential element of the area at a distance x from the gravity axis. The moment of inertia of a differential area about GG is x^2dF, and the moment of inertia I_g of the whole area about GG is Σx^2dF. The moment of inertia of a differential area about XX is $(d + x)^2dF$, or $d^2dF + x^2dF + 2dFxd$, in which x may be either a positive or a negative quantity. The moment of inertia for the whole area is equal to the summation of the moments of inertia of the differential elements of the area. That is, $I_x = \Sigma d^2dF + \Sigma x^2dF + \Sigma 2dFxd$. Since a and 2 are constants, this expression may be written in the form $I_x = d^2\Sigma dF + \Sigma x^2dF + 2d\Sigma dFx$. Since x is measured from a gravity axis, ΣdFx equals zero and the last term drops out of the equation. Moreover, Σx^2dF equals

I_g. Since $\Sigma dF = F$, the first term of the right-hand member of the equation becomes Fd^2. Making these changes in the above equation it reduces to the form $I_x = Fd^2 + I_g$.

Therefore, the moment of inertia of an area about an axis parallel to a gravity axis equals the moment of inertia about the gravity axis plus the product of the area and the square of the distance between the two axes.

52. Moment of Inertia of Simple Areas.—By definition, the moment of inertia of a differential element of an area is x^2dF, in which dF is the area of an element and x is the distance from the element to the axis. Also, the moment of inertia of a finite area is the summation of the moments of inertia of the differential areas, or $I = \Sigma x^2dF$. The value of the moment of inertia of any area about any axis may therefore be determined by obtaining the numerical value of the quantity Σx^2dF.

The moment of inertia of the sections of rolled steel shapes are given in the handbooks published by the various steel companies and need not be computed by the structural engineer.

The moment of inertia of the more common areas of geometry are given in the accompanying table.

Sections	I	Sections	I
	$I = \frac{bh^3}{12}$		$I = \frac{bh^3 - b'h'^3}{12}$.
	$I' = \frac{bh^3}{3}$.		$I = 0.0491(d^4 - d'^4)$
	$I = \frac{bh^3}{36}$		$I = \frac{b'n^3 + bn'^3 - (b - b')a^3}{3}$.
	$I' = \frac{bh^3}{12}$.		$I = \frac{bh^3 - 2b'h'^3}{12}$.
	$I = \frac{\pi d^4}{64} = 0.0491d^4$.		

The moment of inertia of an area about one axis, when its moment of inertia about a parallel is known, can be obtained by applying the equation $I_x = I_g + Fd^2$.

Illustrative Problem.—In Fig. 46, the area of the surface AB is 60 sq. in. and its moment of inertia about the gravity axis GG is 300 in.[4] It is required to find the moment of inertia of the area AB about the axis XX parallel to GG.

In the equation $I_x = I_g + Fd^2$, $I_g = 300$ in.[4] $F = 60$ sq. in. and $d = 9$ in. Substituting these values in the equation, gives $I_x = 300 + (60)(9)^2 = 5{,}160$ in.[4]

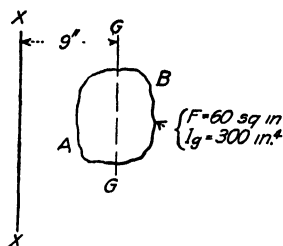
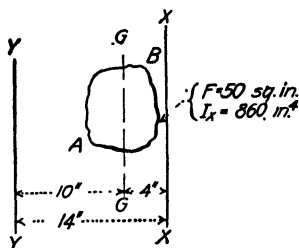



Fig. 46. Fig. 47.

Illustrative Problem.—In Fig. 47 the area of the surface AB is 50 sq. in. The moment of inertia about the axis XX is 860 in.[4] The gravity axis parallel to XX is 4 in. from XX. It is required to find the moment of inertia of the area about the axis YY parallel to XX and 14 in. from XX.

We will first obtain I_g from I_x, then, knowing I_g, obtain I_y. From Art. **51,** $I_g = I_x - Fd^2$ in which $I_x = 860$ in.[4] $F = 50$ sq. in., and $d = 4$ in. Substituting these quantities in the above equation gives $I_g = 860 - (50)(4)^2 = 60$ in.[4] In the equation $I_y = I_g + Fd^2$, $I_g = 60$ in.[4] $F = 50$ sq. in., and $d = 10$ in. Substituting these values in the equation gives $I_y = 60 + (50)(10)^2 = 5{,}060$ in.[4]

53. Moment of Inertia of Compound Areas.

—The moment of inertia of some irregular areas may be obtained by dividing the original area into a number of areas whose moments of inertia are known, and then equating the moment of inertia of the original area to the sum of the moments of inertia of the component parts.

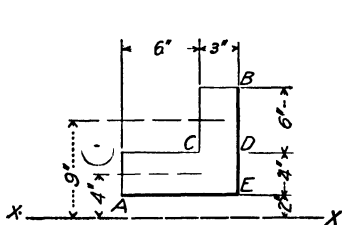
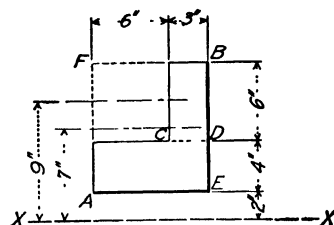



Fig. 48. Fig. 49.

Illustrative Problem.—It is required to find the moment of inertia of the area AB of Fig. 48 about the axis XX parallel to the base AE.

Divide the area into the two rectangles AD and CB. From Art. **52,** the moment of inertia of each of these areas about its own gravity axis parallel to XX is given by the expression $I_g = \frac{1}{12}bh^3$. For the area AD, $I_g = (\frac{1}{12})(9)(4)(4)(4) = 48$ in.[4]; for the area CB, $I_g = (\frac{1}{12})(3)(6)(6)(6) = 54$ in.[4] For the area AD, $I_x = 48 + (36)(4)^2 =$

624 in.^{4}; for the area CB, $I_x = 54 + (18)(9)^2 = 1{,}512$ in.^{4} For the entire area AB, $I_x = 624 + 1{,}512 = 2{,}136$ in.^{4}

The same problem can also be solved by considering the original area to be composed of a positive area EF, Fig. 49, and a negative area FC. Then, I_g for $EF = (\frac{1}{12})(9)(10)^3 = 750$ in.^{4} and I_g for $FC = -(\frac{1}{12})(6)(6)^3 = -108$ in.^{4} Also, I_x for $EF = 750 + (90)(7)^2 = 5{,}160$ in.^{4} and I_x for $FC = -[108 + (36)(9)^2] = -3{,}024$ in.^{4} Then I_x for the original area $= 5{,}160 - 3{,}024 = 2{,}136$ in.^{4}, the same value obtained by the first method.

Illustrative Problem.—The sketch in Fig. 50 represents the transverse section of a column From the Steel handbook, the properties of the angles and plates are as shown in the sketch It is required to find the moment of inertia of the section of the column about the axis XX. For the four angles $I_x = (4)(6.27) + (4)(4.75)(1.24)^2 = 54.29$. For the plate $I_x = I_g = 0.19$. For the entire section $I_x = 54.48$. Since the radius of gyration equals $\sqrt{\frac{I}{A}}$, $r = \sqrt{\frac{54.48}{28}} = 1.40$ in.

54. Moment of Inertia of an Irregular Area.

—If an area is bounded by a curved line for which the algebraic equation is known, the moment

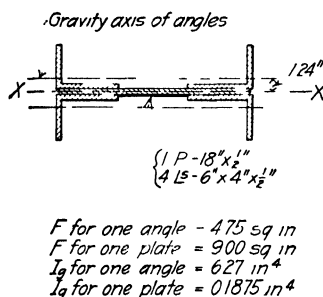
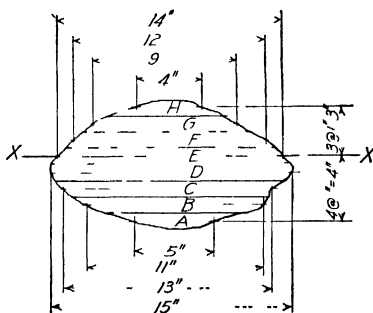



FIG. 50. FIG. 51

of inertia of the area can often be obtained by integrating the quantity Σx^2dF. If the algebraic equation of the enclosing line is not known, or if the equation is of such a nature that the integration is difficult or impossible, an approximate value for the moment of inertia can be obtained by dividing the area into a number of strips parallel to the axis and, considering each strip as a rectangle, determine the moment of inertia of each strip. The moment of inertia of the original area will be approximately equal to the sum of the moments of inertia of the strips. The number of strips into which the area is divided depends upon the irregularities of the outline and upon the degree of accuracy required.

Illustrative Problem.—It is required to find the moment of inertia of the area AH of Fig. 51 about the axis XX.

Divide the area into eight strips each parallel to XX. Each strip is found to be 1 in. wide. The lengths of these strips, measured along their center lines, are given in the figure. Assuming each strip to be a rectangle whose width is 1 in. and whose length equals the length of the strip measured on its center line, we have:

Strip	Length	Area F	d	Fd^2	I_g	$I_x = I_g = Fd^2$
A	5	5	4	80	0.4	80.4
B	11	11	3	99	0.9	99.9
C	13	13	2	52	1.1	53.1
D	15	15	1	15	1.3	16.3
E	14	14	0	0	1.2	1.2
F	12	12	1	12	1.0	13.0
G	9	9	2	36	0.8	36.8
H	4	4	3	36	0 3	36.3
Totals	..	..	...	330	7.0	337.0

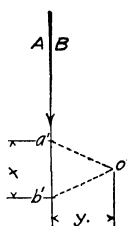
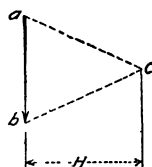
The quantity I_x is made up of the two parts, I_g and Fd^2. The relative values of these two quantities depend upon the number of strips and upon the location of the axis. In this case the axis is near the gravity axis and the number of strips is quite small. But, even so, I_g is only a little over 2 per cent of Fd^2 and I_g could have been omitted without causing appreciable error. If either the number of strips is increased or if the axis is taken at a greater distance from the center of gravity, the error due to neglecting I_g is decreased.

55. Graphical Determination of the Moment of a Single Force.—In Fig. 52, AB is a force and o' is any point in the plane of the force. It is required to determine graphically the moment of AB about o'.

In Fig. 53, draw the force polygon with a load line consisting of the single vector ab. Select as a pole any point o. The pole-distance is represented by H. From o' of Fig. 52, draw the strings of the funicular polygon $o'b'$ and $o'a'$ parallel to ob and oa respectively. The intercept of the line of action of the force between these strings is $a'b'$. Let this distance be represented by x.

Fig. 52. Fig. 53.

The moment of the force equals the magnitude of AB, represented by ab, multiplied by y, $= ab(y)$. By construction, the triangles oab and $o'a'b'$ are similar. Therefore $\frac{x}{y} = \frac{ab}{H}$ or $y(ab) = Hx$. That is, the moment of AB about o' is equal to the intercept of the line of action of the force between the strings of the funicular polygon, multiplied by the pole-distance of the force polygon.

It is to be observed that, in Fig. 53, the rays oa and ob are the components of the force ab, and that the strings of the funicular polygon that cut the intercept from the line of action of AB are the strings that are parallel to the components of the force. The product Hx is no more easily

obtained than the product resulting from the force multiplied by the distance from the center of moments to the line of action of the force. However, this proposition is of value because it embodies the principle upon which the following demonstrations are based.

56. Graphical Determination of the Moment of Any System of Non-concurrent Coplanar Forces.—In Fig. 54, *ab*, *bc* and *cd* are non-concurrent coplanar forces. It is required to find the moment of this system about any point *P* in the plane of the forces.

The moment of the forces of the system equals the moment of the resultant of the system. The resultant *R* is determined in magnitude, direction, and sense by means of the force polygon of Fig. 55. The location of the resultant is determined from the funicular polygon of Fig. 54. It is found to pass through *e'*, the intersection of *c'd'* and *o'a'* which are strings of the funicular polygon parallel respectively to the rays *oa* and *od* of the force polygon and which are components of the resultant *R*.

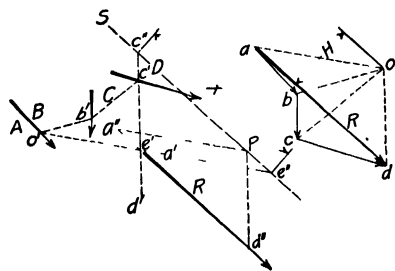



FIG. 54. FIG. 55.

From the preceding article the moment of *R* about *P* can be determined as follows: Through *P*, draw *Pa''* parallel to *oa* and *Pd''* parallel to *od*. The intercept of *R* between these two lines is represented by *a'' d''*. The moment of *R* about *P* is therefore equal to *H* multiplied by *a'' d''*.

A more simple construction will give the same results: Through *P* draw the line *PS* parallel to the resultant *R*. Extend *o'a'* until it intersects *PS* at *e''*. Likewise extend *d'c'* until it intersects *PS* at *c''*. By construction, the triangles *Pa''d''* and *e'e''c''* are identical. Therefore *c''e''* equals *a''d''* and, letting *x* represent *e''c''*, the moment of *R* about $P = Hx$. It is to be noted that the strings *o'e'* and *e'c'* of the funicular polygon are parallel to the rays of the force polygon that are, or may be considered as, components of the resultant *R*.

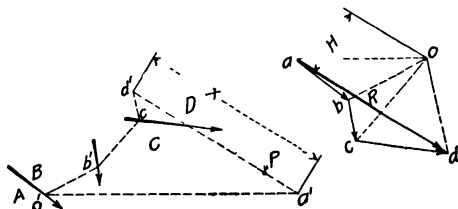



FIG. 56. FIG. 57.

Therefore, to determine the moment of a system of non-concurrent coplanar forces about any point in the plane of the forces, draw the force polygon and the funicular polygon thus determining the magnitude, sense, direction, and position of the resultant of the given system. Draw a line through the center of moments parallel to the resultant. Determine the intercept of this line between the extended strings of the funic-

ular polygon. (The strings to be extended are parallel to the rays of the force polygon which are components of the resultant.) The product of this intercept and the pole-distance of the resultant is the required moment. The pole-distance, being a line in the force polygon, represents force. The intercept, being a line in the space diagram, represents distance. The scales to be used in translating the results are the same as the scale used in constructing the force polygon and the space diagram. The sense of the moment is determined from the space diagram by inspection. The construction for this solution is given in Figs. 56 and 57.

57. Graphical Determination of the Moments of Parallel Coplanar Forces.—The graphical method of determining moments can be used to good advantage in determining the moment due to a system of parallel forces. It is especially applicable to the problem of determining the moments in a girder carrying a system of vertical loads.

In Fig. 58, the horizontal girder carries the vertical loads *ab*, *bc*, *cd* and *de*. The girder is supported by means of the vertical reactions *fa* and *ef*. It is required to determine the moment about *m*. the neutral axis of any section of the girder, as xx.

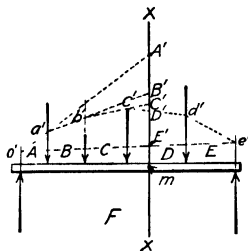
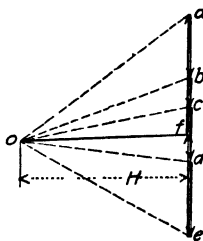



Fig. 58. Fig. 59.

Construct the force polygon represented by Fig. 59. Its pole-distance is H. Beginning at any point as o' in the line of action of *fa*, draw the corresponding funicular polygon. The closing line is $o'e'$. The reactions are represented by *ef* and *fa* of the force polygon.

From Art. **56**, the moment of the reaction *af* about m is the intercept $A'E'$ multiplied by the pole-distance H. Likewise the moment of *ab* is the intercept $A'B'$ multiplied by H, the moment of *bc* is $B'C'$ multiplied by H, and the moment of *cd* is $C'D'$ multiplied by H. The first moment is clockwise and the last three moments are counter-clockwise. Therefore, the resultant moment about m is equal to $H(A'E' - A'B' - B'C' - C'D')$, or the moment at m equals H multiplied by $D'E'$, in which $D'E'$ is the ordinate of the funicular polygon for the section at which the moment is to be determined.

In a similar manner it can be demonstrated that, for any other section of the girder, the moment about the neutral axis of the section due to a system of transverse loads equals the pole-distance of the force polygon multiplied by the ordinate of the funicular polygon corresponding to the section at which the moment is to be determined.

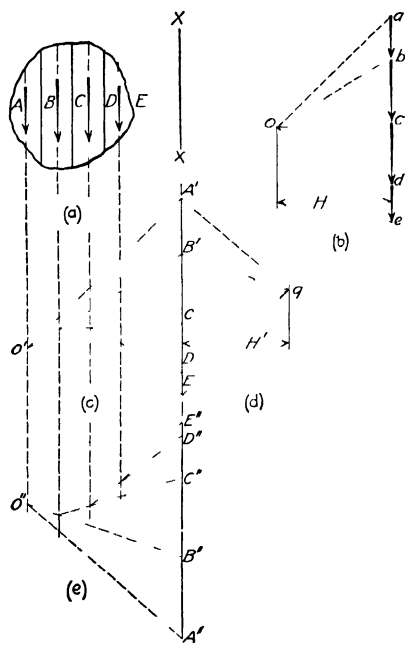
We have finally, therefore, that for a beam carrying a system of parallel loads, the funicular polygon can be used as a moment diagram; the moment equals the pole-distance of the force polygon multiplied by the ordinate of the funicular polygon, the pole-distance represents force to the same scale that was used in drawing the force diagonal; and the ordinate of the funicular polygon represents distance to the same scale that was used in drawing the space diagram.

58. Graphical Determination of Moments of Inertia.

58*a*. Approximate Method.—In Fig. 60*a*, it is required to find the moment of inertia of the area AE about the axis XX.

Fig. 60.

Divide the area into a number of strips parallel to the axis XX and replace each strip by a force parallel to XX and passing through the center of gravity of the strip. Let the magnitudes of these forces be equal numerically to the areas of the strips. Then the moment of inertia or the double moment of these forces will be approximately equal to the moment of inertia of the area.

In Fig. 60*a* the area is divided into four strips and the forces representing these strips are *ab*, *bc*, *cd* and *de*. Draw the force polygon, Fig. 60*b*, and the funicular polygon, Fig. 60*c*. In the funicular polygon, $A'B'$ multiplied by H is the moment of the strip A about XX, $B'C'$ multiplied by H is the moment of the strip B about XX, etc. The moment of the entire area about XX is equal to $A'E'$ multiplied by H.

The moments of *ab*, *bc*, *cd* and *de* are proportional to $A'B'$, $B'C'$, $C'D'$ anb $D'E'$, respectively. The line $A'E'$ can, therefore, be considered as the load line of a force polygon whose forces represent the moments of the forces *ab*, *bc*, etc. In Fig. 60*d*, with any pole q and any pole-distance H', construct the rays of a force polygon. In Fig. 60*e*, construct the corresponding funicular polygon. Then by construction, $(A''B'')(H')(H)$ equals the double moment of the force *ab* about XX. Likewise $(B''C'')$ $(H')(H)$ is the double moment of *bc* about XX, and $(A''E'')(H')(H)$ is the sum of the double moments of all of the forces *ab*, *bc*, etc. about XX. That is, $(A''E'')(H')(H)$ is the approximate moment of inertia of the area AE about XX.

As stated in the above discussion, the double moments of the forces only approximate the moments of inertia of the strips, for, as stated in Art. **51,** the moment of inertia about an axis parallel to a gravity axis is given by the equation $I_x = I_g + Fd^2$. The double moment obtained by the construction shown is the value of Fd^2 only, the quantity I_g has been neglected. As stated in Art. **54,** the relative value of I_g and Fd^2 depends upon the number of strips into which the area is divided and upon the location of the axis. If the number of strips is large, or if the axis is a considerable distance from the gravity axis, I_g can be omitted without appreciable error. If the value of I_g is to be included in obtaining the value of I_x, the construction given in Art. **58b** can be used.

58b. Exact Method. From the definition of the radius of gyration, $I = Fr^2$. If the radius of gyration of an area about its own gravity axis is represented by r_1, then, for an axis parallel to the gravity axis, $r = \sqrt{d^2 + \bar{r}_1^2}$.

The radius of gyration of a rectangle about its own gravity axis is known. The radius of gyration about any other axis parallel to the gravity axis is, therefore, the hypotenuse of a right triangle of which one side is the distance from the given axis to the parallel gravity axis and the other side is the radius of gyration about the gravity axis.

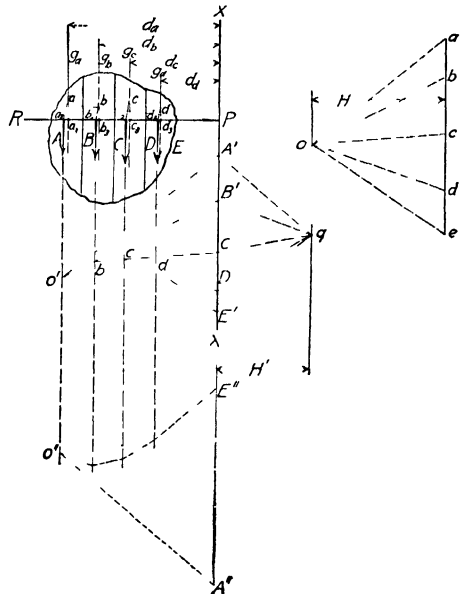



FIG. 61.

In Fig. 61, it is required to find the moment of inertia of the area AE about the axis XX, taking into account I_g for each strip.

The lines g_a, g_b, g_c, and g_d are gravity axes of the strips parallel to XX. Draw the line PR normal to XX. It cuts the gravity lines at the points a_3, b_3, c_3 and d_3. From a_3 lay off the distance a_3a_1 equal to the radius of gyration of the strip A about the axis g_a. Then Pa_1, the hypotenuse of the triangle formed upon Pa_3 and a_1a_3 as sides, equals the radius of gyration of the strip A about the axis XX. Pa_2 is made equal to Pa_1, Pb_2 is made equal to Pb_1, etc. If, therefore, a force ab, numerically equal to the area of the strip A and parallel to the axis XX, is made to pass through the point a_2, the double moment of this force equals the moment of inertia of the strip A about the axis XX. Likewise, the

double moments of the forces *bc*, *cd*, and *de*, passing through the points b_2, c_2 and d_2 equal the moment of inertia of the strips *B*, *C* and *D* about *XX*. By means of the construction given in Fig. 61, the moment of inertia of the area *AE* is found to be $(A''E'')$ $(H)(H')$.

REACTIONS

The function of all structures is to resist forces or to carry loads. If the structure is to carry loads, it, in turn, must be supported. The structure may be supported upon the earth, upon a masonry foundation, or upon some other fabricated structure, or part of a structure. It is very common for one part of a structure to be the support for another part. On a building, the roof trusses are supports for the purlins and columns are supports for the trusses and girders. On a through truss railroad bridge, the piers support the trusses, the trusses support the floorbeams, the floorbeams support the stringers, the stringers support the ties, the ties support the rails, and the rails support the wheels of the train. In the design of a structure, an engineer must trace the path of a force from its origin to the point where it is delivered to the earth.

If one body carrying loads, either its own weight or superimposed loads, is supported by another body, the body which is supported will exert a force upon the supporting body. Likewise the supporting body will exert a force upon the body supported. These two forces are equal and opposite and lie in the same line. The force exerted by the supported body is known as a *load* upon the supporting body. The equal and opposite force exerted by the supporting body is known as a *reaction* on the supported body.

59. Structures Statically Determinate with Respect to the Outer Forces.—The outer forces acting upon a structure comprise the reactions as well as the loads which the structure carries. These outer forces form a system of forces in equilibrium. If the loads are known, the reactions can be determined by applying the fundamental conditions for static equilibrium, providing, however, that there are not more than three elements of the reactions unknown (see Art. **47***b*). If the number of unknown force-elements is not greater than three, the reactions are said to be *statically determinate*. If the number is greater than three, the reactions are called *statically indeterminate*.

To say that the reactions on a structure are statically indeterminate is to say that the reactions cannot be determined by applying the fundamental conditions for static equilibrium alone, but that some other relation or relations must be established between the loads and the reactions. These additional relations involve the elastic properties of the structure.

60. Structures Whose Reactions are Statically Determinate.—Structures whose reactions are statically determinate are used wherever possi-

ble for two reasons: (1) Statically determinate structures are much easier to design than statically indeterminate structures, and (2) the actual stresses are much more likely to agree with the computed stresses in statically determinate than in statically indeterminate structures.

The reactions of the girder shown in Fig. 62 are statically determinate. The unknown elements are the magnitudes of the reactions at A and B. The directions of the reactions are known to be vertical.

The conditions that must be satisfied in order that the reactions of Fig. 62 may be statically determinate are as follows: (*a*) All loads must be vertical, (*b*) the girder must be straight, and (*c*) there must be no restraining moment at the ends of the girder to prevent the ends from rotating in the plane of the loads.

The beam represented by Fig. 63 differs from the beam represented by Fig. 62 only in that it is curved instead of straight. It is obvious

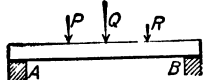
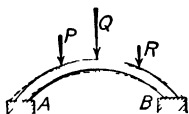
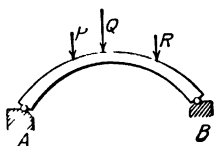



FIG. 62. FIG. 63. FIG. 64.

that if a piece of rubber having the shape of the beam shown in Fig. 63 is loaded as shown, the rubber will straighten and tend to slip at A or B. All structural material is flexible although its flexibility is not as apparent as the flexibility of rubber. Therefore, even though the beam of Fig. 63 is made of steel, the loads P, Q, and R will cause the member to take a form similar to the form represented by the dotted lines. But in order that the beam may take the form shown by the dotted lines, there must be slipping at A and B. Unless the surfaces of contact are frictionless, just as soon as there is a tendency for slipping to occur, the reactions at A and B cease to be vertical and their directions become unknown. The reactions of the structure represented by Fig. 63 are, therefore, statically indeterminate because both the magnitude and direction of each reaction are unknown.[1] If the surfaces at A and B are frictionless, then the reactions will be normal to the surface and their directions are known.

Two modifications of Fig. 63 are frequently encountered in structural engineering. One is the two-hinged arch of Fig. 64, which is similar to the structure of Fig. 63 except that the friction surfaces at A and B are replaced by hinges or pins that permit rotation but do not permit translation at A and B. The other modification is the no-hinge arch of

[1] It is equally proper to say that the magnitudes and directions of the reactions at A and B are unknown, or to consider that each reaction is made up of a horizontal and a vertical component and to say that the magnitudes of the two components of each reaction are unknown.

Fig. 65, in which both rotation and translation at A and B are prevented by having the material continuous.

The structure represented by Fig. 66 is the same as the structure represented by Fig. 62 except that the forces are inclined instead of

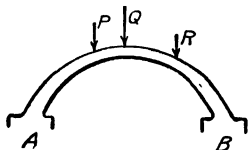
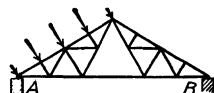



FIG. 65. FIG. 66. FIG. 67.

vertical. One or both of the reactions at A and B must have horizontal components, otherwise the beam would move horizontally on its supports. Since the horizontal components of these reactions are unknown, the unknown elements exceed three, and the reactions are statically indeterminate.

The most common example of a structure subjected to inclined loads that is encountered by the structural engineer is the roof truss subjected to wind pressure. This structure is illustrated by Fig. 67.[1]

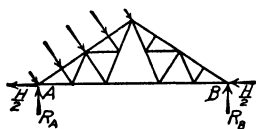
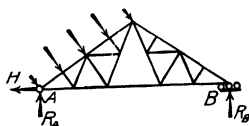
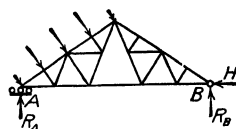



FIG. 68. FIG. 69. FIG. 70.

There is no practicable analysis by which the directions of the reactions at A and B can be accurately determined. In general the engineer is compelled to assume the division of the horizontal forces between the two supports. In the case of a truss supported upon masonry walls, the most common methods of procedure are: (1) Assume that the horizontal component of the load is equally divided between the two supports; or (2) assume first that all of the horizontal component of the load is taken at A and determine the stresses in all members of the truss; then assume that all of the horizontal component of the load is taken at B and again determine the stresses in all members of the truss. In this way two sets of stresses are determined for each member. Each member of the truss is designed so that it can resist either stress.

Under the first method of procedure the forces acting on the truss are assumed to be as shown in Fig. 68. The horizontal component of the

[1] Although only pressure forces on the windward side are shown, the possibility of external or internal pressure or suction forces often should be considered on either windward or leeward slopes, depending upon their slope and shape (see Sec. 2, Art. **5**, and Sec. 2, Art. **11*b***, case (*b*), for illustrative problems).

load is represented by H. The horizontal component of each reaction is then assumed to be $\frac{H}{2}$. The only two unknown elements left are the magnitudes of R_A and R_B. Therefore the reactions are statically determinate.

Under the second method of procedure the forces acting on the truss are first assumed to be as shown in Fig. 69[1] and later as shown in Fig. 70.[1] In both cases, with the magnitudes of R_A and R_B as the only unknown elements, the reactions are statically determinate.

In the case of a roof truss supported on columns, one or the other of the methods suggested for a truss supported on a wall may be used, or, and this is the better way, the truss and the supporting columns may be treated as a single unit. Figure 71[1] represents a typical bent of

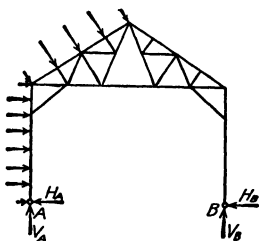
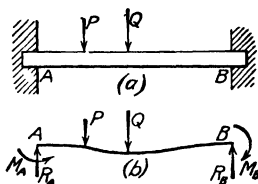



Fig. 71. Fig. 72.

a mill building. The customary procedure is to assume the relative magnitudes of H_A and H_B, and then with this assumption as a basis, to determine V_A and V_B. With V_A and V_B known, the whole structure, columns, truss, and knee braces, is analyzed as a unit.

The relation between the magnitudes of H_A and H_B of Fig. 71 for the bent of a mill building is discussed on p. 206.

We must bear in mind that the reactions on the structures represented by Figs. 68, 69, 70 and 71, are statically determinate only after certain assumptions have been made relative to the values of H_A and H_B.

Figure 72 represents a girder similar to the girder represented by Fig. 62 except that the ends of the girder are restrained. That is, the reaction at A consists not only of a vertical force R_A, but also of a couple whose moment is M_A. Likewise the reaction at B consists of the vertical force R_B and the couple whose moment is M_B. Utilizing the fundamental condition for equilibrium, $\Sigma M = 0$, we have an equation containing M_A and M_B, and therefore the equations cannot be solved. The reactions are therefore seen to be statically indeterminate.

Fortunately most girders and trusses meet the three requirements that must be satisfied in order that the reactions may be statically determinate—that is, the loads are vertical, the girders or trusses are

[1] See footnote on preceding page.

straight, and the ends are not restrained. In the case of beams for buildings, and girders, stringers, floorbeams, and trusses for bridges, these conditions are either satisfied or are so nearly satisfied that the reactions are assumed to be statically determinate. The exceptions are cases where special designs are used to restrain the ends of members.

61. Structures Whose Reactions Are Statically Indeterminate.—Under certain conditions a considerable saving of material can be effected by using structures whose reactions are statically indeterminate. Also the methods of construction are sometimes such that the resulting structures are statically indeterminate, although a statically determinate structure would be equally satisfactory if not, in fact, to be preferred.

The most common structures whose reactions are statically indeterminate are the two-hinged, one-hinged, and the no-hinged arches, continuous beams, and restrained beams.

The two-hinged arch and the no-hinged arch have already been illustrated in Figs. 64 and 65. The one-hinged arch is illustrated in

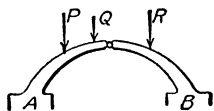
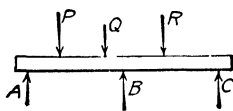



FIG. 73. FIG. 74.

Fig. 73. With this structure the magnitude and direction of the forces at A and B are unknown as well as the restraining moment at the same points.

A girder continuous over three supports is represented by Fig. 74. We might at first conclude that, since the only unknown elements are the three unknown magnitudes of the reactions, the reactions are statically determinate. But we must remember that we have utilized one condition for equilibrium, *i.e.*, $\Sigma X = 0$, in concluding that the reactions at A, B, and C are all vertical reactions. Therefore, only two static relations remain with which to determine the three unknown elements.

Girders may be continuous over any number of supports; in general the greater the number of supports, the harder it is to determine the reactions.

62. Reactions on Beams with Restrained Ends.—The reactions at the ends of beams with restrained ends may be determined from the three fundamental conditions of equilibrium providing the total number of unknown elements does not exceed three. That is, after one or more elements have been determined, either from the nature of the supports, or from the elastic properties of the structure, the reactions are then statically determinate. However, in writing the moment equations, we must include not only the moments of the loads and of the reactions, but we must also include the moment of the external couples that restrain

the beam. For example, Fig. 75a represents a beam that is not only supported vertically at A and B, but that is also restrained at the ends by a couple lying in the plane of the load. The forces and couples which act upon the beam are shown in Fig. 75b.

From $\Sigma Y = 0$, we have

$$R_A + R_B - P - Q = 0$$

From $\Sigma M = 0$, with A as the center of moments, we have

$$M_A + M_B + Pa + Qb - R_BL = 0$$

Since there are no horizontal forces, only two equations are available for determining the unknown elements of the reactions. Therefore two of the four quantities, R_A, R_B, M_A and M_B must be known in order that the other two may be determined. It is to be noted that M_A and M_B both enter the moment equation, and that they are not affected by the location of the center of moments.

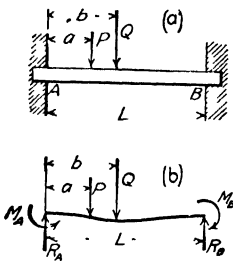



Fig. 75.

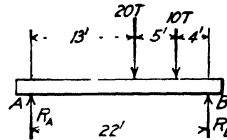



Fig. 76.

Illustrative Problem.—The simply supported beam represented by Fig. 76 is 22 ft. long, center to center of supports, and carries a load of 20 tons, 13 ft. from the left-hand support, and a load of 10 tons 4 ft. from the right-hand support. It is required to find the reactions at A and B.

By inspection the reactions are seen to be vertical and to pass through A and B. The only unknown elements are therefore the magnitudes of R_A and R_B.

From $\Sigma M = 0$, with A as the center of moments, we have

$$M_A = (20)(13) + (10)(18) - (R_B)(22) = 0$$

or

$$R_B = 440/22 = 20 \text{ tons}$$

From $\Sigma M = 0$, with B as the center of moments, we have

$$M_B = (R_A)(22) - (10)(4) - (20)(9) = 0$$

or

$$R_A = 220/22 = 10 \text{ tons}$$

As a check, from $\Sigma Y = 0$, we have

$$\Sigma Y = 10 + 20 - 20 - 10 = 0$$

The same problem can be solved graphically as follows: To any convenient scale lay off in Fig. 77 the load lines ab and bc. With any point o as a pole, draw the rays of the force polygon oa, ob and oc. In the space diagram beginning with o', any point on the left-hand reaction, draw $o'a'$, $a'b'$ and $b'c'$, parallel respectively to oa, ob and oc. The line $o'c'$ is the closing line of the space diagram. A line through o parallel to

$o'c'$ cuts the load line ac at d. The line cd represents to scale the reaction at the right-hand end of the beam and the line da represents to scale the reaction at the left-hand end. Scaling the distances cd and da, they are found to represent 20 tons and 10 tons, respectively.

Illustrative Problem.—The simply supported beam represented by Fig. 78 carries a uniformly distributed load of 1,000 lb. per ft. over a length of 12 ft. It is required to find the reactions at A and B.

In determining reactions a distributed load may be replaced by its resultant. The

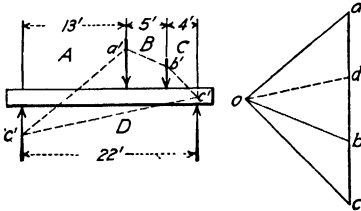



Fig. 77.

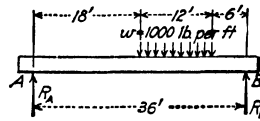



Fig. 78.

resultant of the distributed load in this case is a single force of 12,000 lb. applied at a distance of 24 ft. from the left-hand end of the beam.

From $\Sigma M = 0$, with A as the center of moments, we have

$$M_A = (12{,}000)(24) - (R_B)(36) = 0$$

or

$$R_B = 8{,}000 \text{ lb.}$$

Likewise with moments about B, we have

$$M_B = (R_A)(36) - (12{,}000)(12) = 0$$

or

$$R_A = 4{,}000 \text{ lb.}$$

Also from $\Sigma Y = 0$, with R_B known, we have as a check

$$\Sigma Y = R_A + 8{,}000 - 12{,}000 = 0$$

or

$$R_A = 4{,}000 \text{ lb.}$$

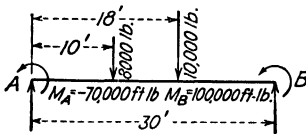



Fig. 79.

Illustrative Problem.—Figure 79 represents a beam restrained at the ends and carrying loads as shown. The magnitudes of the restraining couples are known. It is required to find the reactions at A and B.

From $\Sigma M = 0$, with A as the center of moments, we have

$$\Sigma M \text{ at } A = (8{,}000)(10) + (10{,}000)(18) - 100{,}000 - 70{,}000 - (R_B)(30) = 0$$

or

$$R_B = 3{,}000 \text{ lb.}$$

With B as the center of moments, we have

$$\Sigma M \text{ at } B = (30)(R_A) - 100{,}000 - 70{,}000 - (10{,}000)(12) - (8{,}000)(20) = 0$$

or

$$R_A = 15{,}000 \text{ lb.}$$

From $\Sigma Y = 0$, we have, as a check

$$\Sigma Y = 3{,}000 + 15{,}000 - 10{,}000 - 8{,}000 = 0$$

Illustrative Problem.[1]—Figure 80 represents a roof truss supported on masonry walls. The wind pressure is applied at the panel points and is normal to the top chord of the truss as shown in the figure. It is required to find the reactions at A and B.

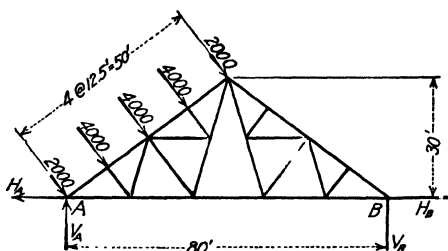



Fig. 80.

Since the two supports are on the same level, the vertical reactions are independent of the division of the horizontal component of the load between the two supports.

Taking moments about A to get V_B, we have

$$M_A = (2{,}000)(0) + 4{,}000(12.5 + 25 + 37.5) + (2{,}000)(50) - (V_B)(80) = 0$$

or

$$V_B = 5{,}000 \text{ lb.}$$

Taking moments about B to get V_A, we have

$$M_B = (V_A)(80) - 2{,}000(14 + 64) - 4{,}000(26.5 + 39 + 51.5) = 0$$

or

$$V_A = 7{,}800 \text{ lb.}$$

The vertical component of the wind load is $(2{,}000 + 4{,}000 + 4{,}000 + 4{,}000 + 2{,}000)\ {}^{40}\!/_{50} = 12{,}800$ lb.

The horizontal component of the wind load is $(2{,}000 + 4{,}000 + 4{,}000 + 4{,}000 + 2{,}000)\ {}^{30}\!/_{50} = 9{,}600$ lb.

From $\Sigma Y = 0$, we have, as a check

$$\Sigma Y = 5{,}000 + 7{,}800 - 12{,}800 = 0$$

If the horizontal component of the wind load is assumed to be equally divided between the two reactions, then

$$H_A = H_B = \frac{9{,}600}{2} = 4{,}800 \text{ lb.}$$

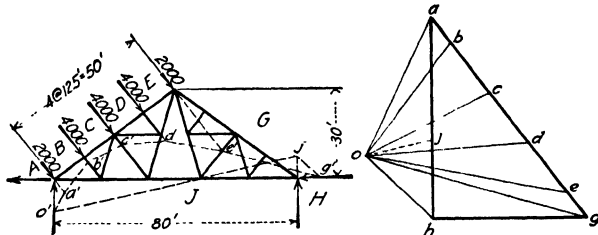



Fig. 81

If all of the horizontal component is assumed to be resisted by one support, then the horizontal component of that reaction is 9,600 lb. and the horizontal component of the other reaction is zero.

[1] See footnote to Fig. 67, Sec. 1, Art. 60.

The same problem is solved graphically in Fig. 81. In this figure *agha* is the force polygon. In the space diagram *o'j'* is the closing line. Then *oj*, drawn through *o* and parallel to *o'j'*, divides the vertical load line, *ha*, into two parts such that *hj* is the vertical component of the right-hand reaction and *ja* is the vertical component of the left-hand reaction. Scaling from the drawing the values obtained by the graphical method are found to check the values obtained by the algebraic method.

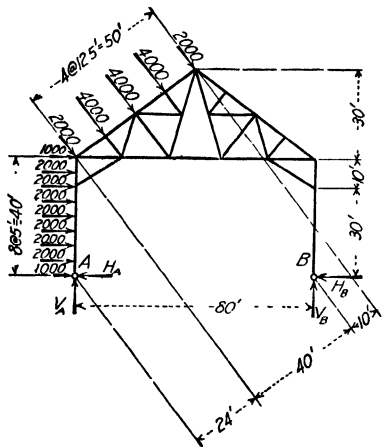



FIG. 82.

Illustrative Problem.[1]—Figure 82 represents a steel bent of a mill building. We will first consider the case in which the columns of the bent are hinged at their bases. It is required to find the reactions at the bases of the columns.

The reactions at A and B are statically determinate only after the division of the horizontal load between the supports at A and B has been assumed. A discussion of the relations between these horizontal reactions is given on p. 206. In this case we will assume that H_A and H_B are equal.

Taking moments about A to get V_B, gives

$$2{,}000(5 + 10 + 15 + 20 + 25 + 30 + 35) + (1{,}000)(40) + 2{,}000(24 + 74) + 4{,}000(36.5 + 49 + 61.5) - (V_B)(80) = 0$$

or

$$V_B = 13{,}800 \text{ lb.}$$

Taking moments about B to get V_A, gives

$$(V_A)(80) + 2{,}000(5 + 10 + 15 + 20 + 25 + 30 + 35) + (1{,}000)(40) + (2{,}000)(10) - 4{,}000(2.5 + 15 + 27.5) - (2{,}000)(40) = 0$$

or

$$V_A = -1{,}000 \text{ lb.}$$

The vertical component of the total wind load is $^{40}\!/_{50}$ $(2{,}000 + 4{,}000 + 4{,}000 + 4{,}000 + 2{,}000) = 12{,}800$ lb.

From $\Sigma Y = 0$, we have, as a check

$$\Sigma Y = 13{,}800 - 1{,}000 - 12{,}800 = 0$$

The horizontal component of the total wind load is $1{,}000 + 1{,}000 + (7)(2{,}000) + {}^{30}\!/_{50}(2{,}000 + 4{,}000 + 4{,}000 + 4{,}000 + 2{,}000) = 25{,}600$ lb.

If the two horizontal reactions are equal, then

$$H_A = H_B = \frac{25{,}600}{2} = 12{,}800 \text{ lb.}$$

A graphical solution of the same problem is given in Fig. 83. To simplify the work the horizontal pressure on the left-hand column and the normal pressure on the top chord have each been replaced by a single resultant.[2]

[1] See footnote to Fig. 67, Sec. 1, Art. **60**.

[2] This change can be made without affecting the reactions at the bases of the columns. But the substitution of the resultant for the distributed pressure cannot be made when the stresses in the truss and columns are being determined.

$o'd'$ is the closing line and the line oe, drawn through o and parallel to $o'd'$, cuts ad extended in the point e, so located that de represents the vertical component of the right-hand reaction and ea represents the vertical component of the left-hand reaction. Scaling from the drawing, we find that the values of de and ea determined by the graphical and algebraic methods agree.

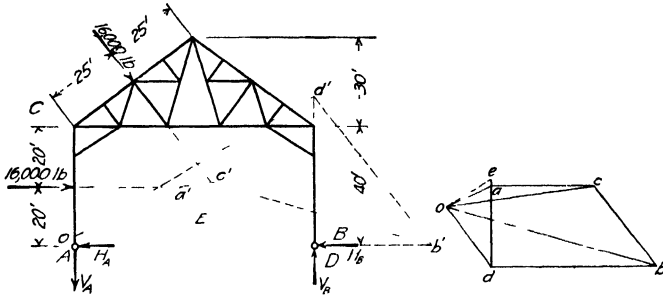



Fig. 83.

Illustrative Problem.[1]—Figure 84 represents a steel bent of a mill building. In this case it is assumed that the columns of the bent are fixed at the bases. It is required to find the reactions at the bases of the columns.

In order to determine the reactions at A and B, it will be necessary not only to assume the division of the horizontal load between the two supports, but it will also be necessary to assume the point of contraflexure[2] in the columns.

In this case we will assume that, of the horizontal component of the total load, 0.6 is delivered to A and 0.4 is delivered to B;[3] H_A will then equal 15,360 lb. and H_B will equal 10,240 lb. We will further assume that the point of contraflexure of the left-hand column is 18 ft. above A and that the point of contraflexure of the righthand column is 15 ft. above B.

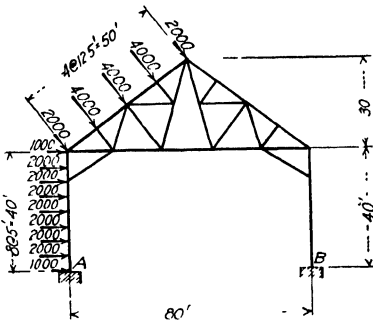



Fig. 84.

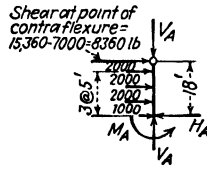
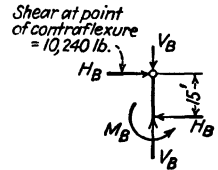



Fig. 85. Fig. 86.

The forces acting upon the portion of the left-hand column below the point of contraflexure are shown in Fig. 85. From $\Sigma M = 0$, we have

$$2{,}000(5 + 10 + 15) + (8{,}360)(18) + M_A = 0$$

or

$$M_A = -210{,}480 \text{ ft.-lb.}$$

[1] See footnote to Fig. 67, Sec. 1, Art. **60**.

[2] For definition of point of contraflexure, also known as point of inflection, see p. 207.

[3] For a discussion of the relation between the horizontal reactions at the bases of columns see p. 206. For a discussion of the location of the point of contraflexure in columns see p. 207.

The forces acting upon the portion of the right-hand column below the point of contraflexure are shown in Fig 86. From $\Sigma M = 0$ we have

$$(10,240)(15) + M_B = 0$$

or

$$M_B = -153,600 \text{ ft-lb.}$$

The moments M_A and M_B are the moments of couples acting at the bases of the columns. These couples represent the moment-restraining effect of the fixed bases of the columns.

The forces and couples acting upon the bent as a whole are represented by Fig. 87

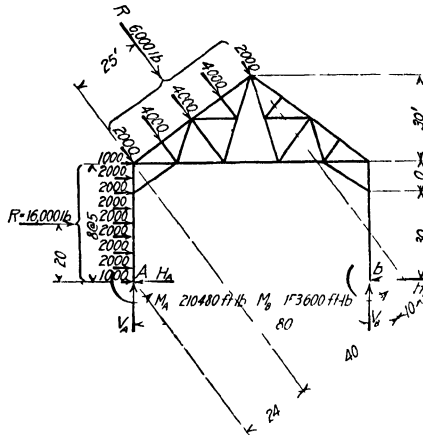



Fig 87.

From $\Sigma M = 0$, with A as the center of moments, we have

$$(16,000)(20) + (16,000)(49) - 210,480 - 153,600 - V_B(80) = 0$$

or

$$V_B = 9,249 \text{ lb.}$$

From $\Sigma M = 0$, with B as the center of moments, we have

$$(16,000)(20) - (16,000)(15) - 210,480 - 153,600 + V_A(80) = 0$$

or

$$V_A = 3,551 \text{ lb.}$$

The vertical component of the resultant load on the roof is $\frac{40}{50}(16,000) = 12,800$ lb.

From $\Sigma Y = 0$, we have, as a check,

$$\Sigma Y = 9,249 + 3,551 - 12,800 = 0$$

From $\Sigma X = 0$, we have, as a check,

$$\Sigma X = 25,600 - 15,360 - 10,240 = 0$$

MOMENTS AND SHEARS IN BEAMS AND TRUSSES

63. Bending Moment.—The bending moment (or moment) at any section of a beam or truss is equal to the algebraic sum of the moments

of the forces on either side of the section. In a beam, the moment is considered as acting about an axis through the center of gravity of the section, as shown in Fig. 88. In a truss, the moment is considered as acting about the moment center for the member in question, as shown in Fig. 89 (see chapter on "Stresses in Trusses").

This moment measures the tendency of the external forces to cause the portion of the beam or truss lying on one side of the section to rotate about the moment center, thereby causing bending of the beam or truss. Resistance to this tendency to bend the beam or truss is offered, in a beam,

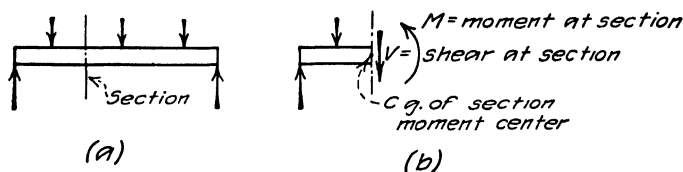



FIG. 88.

by internal fiber stresses of tension and compression, and in the truss by the stresses in the members.

When the resultant moment on the left of a given section is clockwise, the moment is called *positive*, and when it is counter-clockwise on the same side of the section, it is called *negative*. Since the structure is in equilibrium and at rest, $\Sigma M = 0$ when we consider the forces on both sides of the section. Hence the resultant moment of the forces on the left of the section is equal in magnitude but opposite in sense to the resultant moment on the right of the section. Thus it makes no differ-

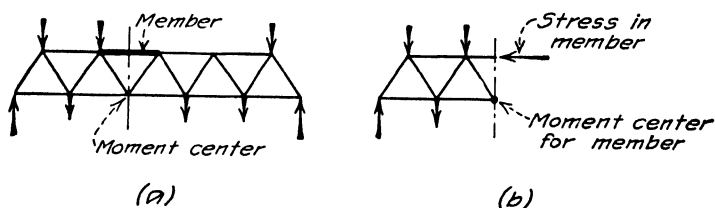



FIG. 89.

ence which side of the section we consider; the moment is *positive* when the resultant moment of forces on the left is clockwise and the resultant moment on the right is counter-clockwise. Also, the moment is *negative* when the resultant moment of the forces on the left is counter-clockwise and when the resultant moment of the forces on the right is clockwise.

64. Shear.—The shear on any section of a beam or truss is defined as the algebraic sum of the vertical forces on either side of the section. (If the beam or truss is not in a horizontal position, the summation of forces is to be taken in a direction perpendicular to the length of the structure under consideration). This shear is the tendency of the portion of

the structure on one side of the section to slide by the portion on the other side of the section. It is resisted in the case of a beam by the transverse shearing strength of the material composing the beam. The force V in Fig. 88*b* indicates this shearing resistance. In the case of a truss, the shear is resisted by the stress in the members (see chapter on "Stresses in Trusses").

When the resultant force on the left of the section acts upward, as shown in Fig. 90*a*, the shear is called *positive*. When the shear acts downward on the same side of the section, tending to cause the left-hand portion to move downward with respect to the right-hand portion, the shear is called *negative*.

Since the structure is in equilibrium and is at rest, $\Sigma V = 0$ when we consider the forces acting on both sides of the section. Hence the resultant of forces on the right of the section must be equal in magnitude but opposite in direction to the forces on the left of the section. Therefore,

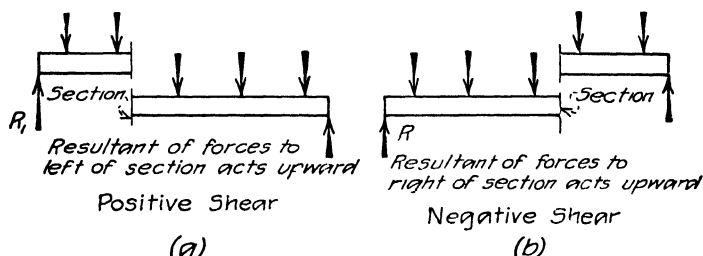



Fig. 90.

it makes no difference which side of the section we consider; the shear is *positive* when the resultant on the left is upward and the resultant on the right is downward. Also, the shear is *negative* when the resultant on the left is downward and the resultant on the right is upward.

65. Moments and Shears Due to Fixed Loads.

65*a*. Moment and Shear Diagrams.—The variation in bending moment from section to section of a beam or truss may well be represented by means of diagrams which are known as moment or shear diagrams. These diagrams are constructed by laying off a *base line* equal to the length of the beam or truss and marking off on this line the position of the loads and reactions. To assist in correlating these diagrams and the structure for which they are drawn, it is well to present also a loading diagram placed directly above the moment or shear diagrams, as shown in the following figures. In the diagrams which follow, positive moments or shears will be plotted above the base line, and negative values will be plotted below this line. Points representing the moment or shear at any section will be plotted on a vertical line through the section in question. The distance from these plotted points to the base line represents to some scale the magnitude of the moment or shear

at the given point on the structure. The lines joining these points form the required moment or shear diagram.

To illustrate, the moments and shears will be calculated and the moment and shear diagrams drawn for a single concentrated load P placed at a distance a from the left end of a simple beam of span l. Figure 91a shows the beam with the applied load and the supporting reactions in position. The reactions are calculated by the methods given in the chapter on "Reactions."

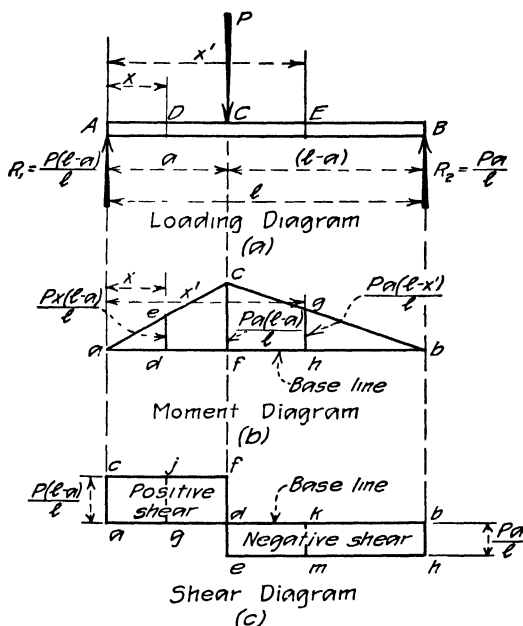



FIG. 91.

Consider the moment at point D, a distance x from the left end of the beam. From the definition of moment given in Art. **63,** we have

$$M_x = +R_1x = +P\frac{(l-a)}{l}x \tag{1}$$

where M_x is the moment at the point in question. To represent this value in the moment diagram of Fig. 91b, plot d-e to some convenient scale equal in magnitude to the moment given by Eq. (1). Since a is less than l in Eq. (1), this moment is positive. Therefore the ordinate d-e is to be plotted above the base line, in order to conform to the notation adopted above. Thus we locate e, a point on the moment line.

Other points on the moment line may be located in a similar manner. Thus for point A it will be found that the moment is zero, and at point C, the position of the applied load, it will be found that the moment is

$$M_c = R_1a = P\frac{a}{l}(l-a) \tag{2}$$

Plotting these values locates points a and c of the moment line. Connecting points a, e, and c will determine the moment line a-c, which proves to be a straight line.

It is also possible to draw the moment line by noting that Eq. (1) represents a straight line, since the variable x is of the first power. On substituting particular values of x in Eq. (1), points can be plotted and the moment line drawn as before. Convenient points for this purpose are the left end of the beam and the point at which the load is applied. Thus, at the end of the beam, where $x = 0$, we have from Eq. (1), $M = 0$. This moment is represented in Fig. 91*b* by point a. Again, at point C, the point of application of load P, $x = a$, and from Eq. (1), $M_c = P\frac{a}{l}(l - a)$. This moment is represented to scale in the moment diagram of Fig. 91*b* by the ordinate c-f. As before, the moment at C is positive, and is plotted above the base line. Connecting points a and c, we have the line a-c, which is the required moment line for the portion A-C of the beam. Note that the line a-c passes through point e, which was plotted from Eq. (1).

This discussion has considered only points to the left of the applied load. Consider now a point E to the right of the applied load and a distance x' from the left end of the beam. From the definition of moment given in Art **63**, the moment at E is

$$M_E = +R_1x' - P(x' - a) = P\frac{x'}{l}(l - a) - P(x' - a)$$

or

$$M_E = +P\frac{a}{l}(l - x') \qquad (3)$$

This moment is represented in the moment diagram of Fig. 91*b* by the ordinate h-g, plotted above the base line, since the moment is positive.

As before, Eq. (3) represents a straight line, since the variable x' is of the first power. For moment at C of Fig. 91*a*, where $x' = a$, we have from Eq. (3), $M_c = P\frac{a}{l}(l - a)$. This is the same as the value calculated above from Eq. (1) with $x = a$. By this means a check is obtained on Eqs. (1) and (3). Also, at point B, where $x' = l$, we have $M = 0$. The moment line for the right-hand portion of the beam is shown by the line c-b of Fig. 91*b*, and the complete moment diagram is shown by the triangular figure *acb*. It can be seen from the moment diagram of Fig. 91*b* that the greatest moment in the beam occurs under the load. At this point the value of the moment, as given by Eq. (2), is

$$M_c = R_1a = P\frac{a}{l}(l - a).$$

Since all points of the moment line are located above the base line of Fig. 91*b*, it can be seen that for the loading conditions shown in Fig. 91*a*, the moment is positive at all points on the beam. This statement may be verified by Eqs. (1) and (3) by noting that a and x' are always less than l. Therefore these equations represent positive moments. Based on this fact, the following important rule may be stated: *A downward vertical load placed at any point on a simple beam causes positive moment at every point on that beam.* This rule will be found useful in the work to follow.

The shear diagram for the beam under consideration is shown in Fig. 91*c*. To plot this diagram, we note from Fig. 91*a* and from the definition for shear given in Art. **64**, that the shear on the portion of the beam to the left of the load P is equal to the left reaction. Therefore

$$V = +R_1 = +P\frac{(l-a)}{l} \tag{4}$$

where V denotes the shear in question. Since R_1 acts upward, the shear given by Eq. (4) is a positive quantity. Also, as there are no loads between A and C, the shear is constant over this portion of the beam. In Fig. 91*c* this shear is represented by the shear line *c-f* plotted at a vertical distance $P\frac{(l-a)}{l}$ above the base line.

For points to the right of C, Fig. 91*a*, the shear is given by the expression

$$V = R_1 - P = P\frac{(l-a)}{l} - P$$

from which

$$V = -P\frac{a}{l} \tag{5}$$

Since the load P is greater than the reaction R_1, the resultant of the vertical forces acts downward, and hence the shear is negative. This is indicated by the minus sign in Eq. (5). It can be seen that this shear is constant over the portion of the beam to the right of the applied load.

To plot this shear in Fig. 91*c*, draw a line *e-h* at a distance $P\frac{a}{l}$ below the base line. The complete shear diagram is shown by the rectangle *acfd*, representing positive shear, and the rectangle *dehb* representing negative shear. From Fig. 91*c* it can be seen that the shear for the portion of the beam to the left of the load is positive and equal to the left reaction, and for the portion of the beam to the right of the load, the shear is negative and equal to the right reaction. At the section where the load is applied the shear is indeterminate, being somewhere between $+P\frac{(l-a)}{l}$

and $-P\frac{a}{l}$. It is generally stated, for the conditions under consideration, that the shear passes through zero at the point of application of the applied load.

Figure 92 shows moment and shear diagrams drawn for given fixed loads on a beam or a truss. The loads on the truss are assumed as applied at the joints, as shown in Fig. 92. Since the loads are all vertical and are applied in the same vertical lines for both the beam and the truss, it is evident, from the definitions for moment and shear given in Arts. **63** and **64**, that the same moment and shear diagrams answer for both structures. In plotting these diagrams, moments should be calculated at each load point, and shears should be calculated for the portions of the beam between each set of loads. For further explanation see the illustrative problems at the end of Art. **65**b.

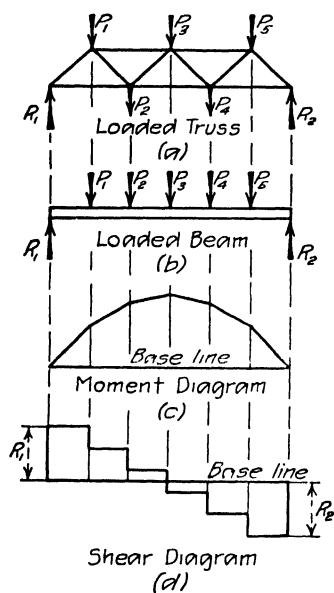
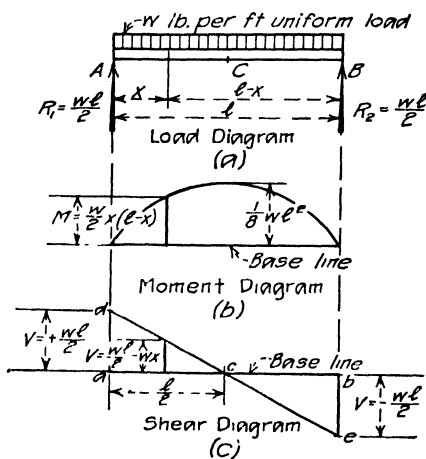



FIG. 92.

FIG. 93.

The moment and shear diagrams for a simple beam carrying a uniform fixed load which covers the entire beam are shown in Fig. 93. These diagrams are plotted from values of the moment and shear calculated for a number of points along the beam. The curves formed by connecting the plotted points give the required moment and shear diagrams. This procedure is necessary where the loading is variable or complicated. If the loading is uniform, as shown in Fig. 93, formulas are readily derived from which the required moments and shears may be determined directly by substitution. These calculated values may be plotted and the curves drawn as before.

To derive the general equation for moment at a point distant x from the left end of a beam loaded as shown in Fig. 93a, note that by

definition (see Art. 63), the bending moment at any point is equal to the resultant moment of all forces to the left of that point. Considering clockwise moments as positive, we have from Fig. 93a,

$$M_x = R_1 x - wx \cdot \frac{x}{2}$$

In this expression $wx \cdot \frac{x}{2}$ is the counter-clockwise moment of the uniform load to the left of the section. The term wx represents the total uniform load between the section and the left reaction, and $\frac{x}{2}$ represents the distance from the section to the center of gravity of the load wx. Since this load is uniformly distributed, its center of gravity is half-way across the load, or at a distance $\frac{x}{2}$ from the section. Substituting the value of R_1, the left reaction, which is $\frac{1}{2}wl$, and simplifying the resulting expression, we have

$$M_x = \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2$$

which may also be written,

$$M_x = \tfrac{1}{2}wx(l - x) \qquad (6)$$

From analytical geometry it can be shown that Eq. (6) is the equation of a parabola. This equation, when plotted, forms the moment diagram of Fig. 93b. It can be shown that the greatest moment occurs at the center of the beam, where $x = \frac{l}{2}$. Substituting this value of x in Eq. (6), we have

$$M_c = \tfrac{1}{8}wl^2 \qquad (7)$$

The fact that the greatest moment given by Eq. (6) occurs at the center of the beam may be proved by means of certain principles given in textbooks on Mathematics. It is there shown that when the sum of two terms is constant, their product is a maximum when the two terms are equal. Note that in Fig. 93a, x and $(l - x)$ are the two segments of the beam, and that the sum of these terms which is $x + (l - x) = l$, is a constant. Therefore $x = \frac{1}{2}l$ gives a maximum value for Eq. (6).

A very convenient rule for the determination of the moment at any point in a beam is founded on Eq. (6). This rule is as follows: *The moment at any point in a simple beam due to a uniform load which covers the entire beam is equal to one-half the load per foot times the product of the two segments into which the beam is divided by the moment center.* This rule may be verified by reference to Fig. 93a and Eq. (6).

The shear diagram for a beam subjected to a uniform load is given in Fig. 93*c*. A general expression for shear at any point may be derived from the definition for shear given in Art. **64**. Thus

$$V = R_1 - wx = +\frac{1}{2}wl - wx \tag{8}$$

This is the equation of a straight line. When $x = 0$, or at the left end of the beam, $V = +\frac{1}{2}wl$. In Fig. 93*c* this value is represented by the ordinate *a-d*, plotted above the base line, since the shear is positive. At the center of the beam, where $x = \frac{1}{2}l$, we have $V = 0$. This value is shown in Fig. 93*c* by the point *c*. At the right end of the beam, where $x = l$, we have $V = -\frac{1}{2}wl$. In Fig. 93*c* this shear is shown by the ordinate *b-e*, which is plotted below the base line, since the shear is nega-

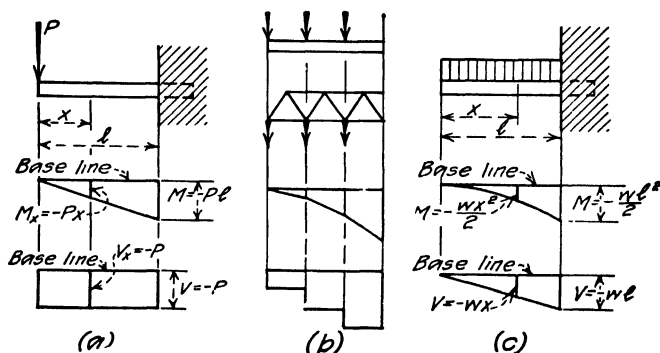



Fig. 94.

tive. The complete shear diagram is shown by the figure *adcbe*, where *dce* is a straight line.

The moment and shear diagrams for a cantilever beam are shown in Fig. 94. In Fig. 94*a* the beam carries a single concentrated load at the free end of the beam, and in Fig. 94*c* the beam carries a uniformly distributed load. The equations for the moment and shear lines for these two cases are shown on the diagrams. These equations were derived by the methods given above for the cases considered in Figs. 92 and 93. Figure 94*b* shows the moment and shear diagrams for a cantilever beam or truss carrying several concentrated loads.

In certain simple cases it is possible to determine moments and shears and construct the resulting moment and shear diagrams by graphical methods. These constructions are based on the methods given in the chapter on "Principles of Statics," Art. **57**, p. 47. To apply these methods to the beam shown in Fig. 95*a*, construct the force polygon of Fig. 95*b*. In Fig. 95*a* draw the equilibrium polygon and locate the closing line *o-e*. A line *O-E* in the force polygon of Fig. 95*b*, drawn parallel to the closing line *o-e*, locates the load divide point *E*, from which the values of the reactions R_1 and R_2 may be determined, as shown in Fig. 95*b*.

As stated in the chapter on "Principles of Statics," Art. 55, p. 45, the moment at any point X distance x from the left end of the beam, as shown in Fig. 95a, is equal to the pole distance H of the force polygon multiplied by y, the intercept on a vertical line through X, of the segments o-e and o-b of the equilibrium polygon of Fig. 95c. Hence if the pole distance H is chosen as some convenient unit, as 1,000, 10,000, or 100,000 lb., the ordinate y of Fig. 95c will give the moment directly to this scale. The equilibrium polygon of Fig. 95c is therefore a moment diagram for the forces under consideration. The closing line o-e may be considered as the base line for the moment diagram.

The shear diagram may be constructed by drawing horizontal lines from the several forces in the force polygon and projecting them to an

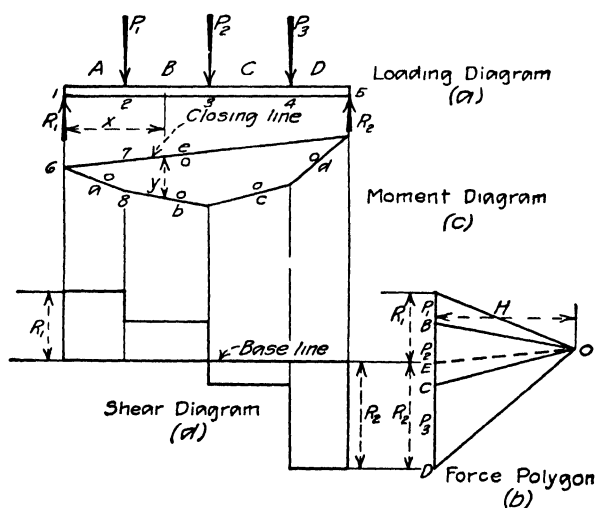



Fig. 95.

intersection with the verticals formed by projecting the lines of action of the corresponding forces. The base line for this diagram is formed by projecting a horizontal line through the load divide point E of Fig. 95b. Figure 95d shows the resulting shear diagram.

This method of construction may also be applied to a beam carrying a uniform load by dividing the uniform load into small lengths and replacing each load so formed by an equal concentrated load applied at the center of gravity of each small piece of uniform load. The construction of Fig. 95 is then applied to these loads. However, the time required for such a graphical construction will generally exceed that required by direct algebraic calculation. In general, it will be found best to use algebraic methods for the determination of moments and shears. Moment and shear diagrams can then be plotted from the calculated values. Illustrative problems will be found at the end of Art. 65b.

65b. Maximum Moments and Shears.—The maximum moment or shear in a beam or truss due to any given set of fixed loads may readily be determined by an inspection of the moment or shear diagrams drawn for these loads by the methods described in the preceding article. However, this procedure requires considerable work, and, moreover, in many cases only the maximum moment and shear due to a given set of loads is desired. It is possible in such cases to determine where the desired maximum values will occur and make the necessary calculations only at these points.

From a study of the moment diagrams for simple beams supported at the ends and loaded in any manner, it will be noted that the moment is zero at the left end of the beam. The moment increases in value from the end of the beam to a point near the beam center, where it reaches its greatest value. Beyond this point the moment decreases in value, finally becoming zero at the right end of the beam. The problem is to devise a method for locating the point at which the moment is a maximum.

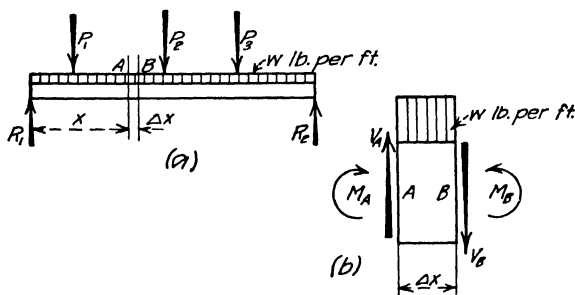



FIG. 96.

Consider a simple beam supported at the ends and loaded in any manner, as shown in Fig. 96a. At any point distance x from the left end of the beam remove a small portion of the beam Δx in length, together with all external forces, or loads, and all internal forces, or stresses, acting on the beam element, as shown in Fig. 96b. The external forces consist of the applied load, which is w lb. per ft., and the internal forces consist of moments and shears applied at the cut sections. From what has been said in Arts. **63** and **64,** it is evident that the moments and shears on the faces of the element act as shown in Fig. 96b.

Taking moments about point B, the center of the right-hand face of the element, we have

$$+ M_A - M_B + V_A\Delta x - w\frac{\Delta x^2}{2} = 0$$

This expression may be placed in the form

$$M_B - M_A = (V_A - \tfrac{1}{2}w\Delta x)\,\Delta x$$

Let ΔM represent the difference, $M_B - M_A$, between the moments on the two faces of the beam element. If ΔM be divided by Δx, the distance between the two faces, we have an expression for the change in moment per unit of length across the element, or

$$\frac{\Delta M}{\Delta x} = V_A - \frac{1}{2} w \Delta x$$

Assume now that the two faces of the element approach each other—that is, that Δx approaches a zero value. The limit of $\frac{\Delta M}{\Delta x}$ then becomes $\frac{dM}{dx}$, the rate of change of moment, and the limit of the right-hand term becomes V, the shear at the point where the moment is taken. The above expression then becomes

$$\frac{dM}{dx} = V \tag{9}$$

That is, the rate of change of moment is equal to the shear.[1]

Equation (9) shows that when the shear is positive, the rate of change of moment is positive, which indicates that the moment is increasing in value as x, the distance from the left end of the beam, increases in value. Also, if the shear is negative, the moment is decreasing in value. Therefore, it is evident that when the shear changes from a positive to a negative value, the moment ceases to increase in value and begins to decrease. Hence the moment at any point reaches its greatest value at the time the shear changes from a positive to a negative quantity. But the shear changes from a positive to a negative value at the point where the shear becomes zero. Therefore, the maximum moment in a simple beam occurs at the point where the shear is zero.

The above rule may be verified by referring to the moment and shear diagrams given in the preceding article. For a beam carrying a uniform load, Fig. 93, it can be seen that the shear line of Fig. 93c crosses the base line at the beam center. In Fig. 93b the moment diagram has its maximum ordinate at the beam center. Hence maximum moment and zero shear occur at the same point, which complies with the above rule. In Fig. 92 we note that for a beam carrying concentrated loads, the moment is a maximum at the point where the shear line crosses the base line. In this case the shear at the point of maximum moment is indeterminate. However, the shear *passes through a zero value.* Hence the above rule, stated in more general terms, may be written:

The moment in a beam is a maximum at the point where the shear is zero, or passes through a zero value.

[1] Those who are familiar with the calculus will recognize the left-hand term of Eq. (9) as the differential coefficient of the moment with respect to the variable x.

It is not possible to derive a similar relation for the determination of the position of maximum shear for the conditions shown in Fig. 96. However, the desired information may be obtained directly from the definition for shear given in Art. **64.** Since the shear at any point is equal to the reaction minus the sum of the loads between the end of the beam and the point at which the shear is desired, it is evident that the shear will be a maximum when the sum of the loads to be subtracted is as small as possible. Therefore, the shear is a maximum at a point taken so close to the reaction that no downward load need be considered. For the conditions shown in Fig. 96, the shear is a maximum at a section taken just to the right of R_1, or just to the left of R_2, depending upon which reaction has the greater value. These shears are evidently equal to the corresponding reaction. This statement may be verified by a study of Figs. 91, 92, and 93.

The relation between the shears at sections taken close together may be determined from a summation of forces taken for the beam element of Fig. 96*b*. Thus

$$V_A - V_B - w\Delta x = 0$$

Let $\Delta V = V_A - V_B$ represent the change in shear between the sections *A* and *B* of Fig. 96*a*. For sections taken an infinitesimal distance apart, the above expression may be written

$$\frac{dV}{dx} = w \tag{10}$$

That is, the rate of change of shear is equal to the load per unit of length of beam. For the conditions shown in Fig. 92, where the beam supports a series of concentrated loads, the load on any small element of the beam between two consecutive loads is equal to zero. Placing $w = 0$ in Eq. (10), we note that the rate of change of shear is zero, or, the shear is constant between loads. This fact is indicated by the shear lines drawn parallel to the base line of Fig. 92*d*. In the beam of Fig. 93, the load is uniform. Hence the load on any beam element is w, and Eq. (10) shows that the rate of change of shear is constant and equal to the load per unit of length of the beam. The sloping straight shear line of Fig. 93*c* indicates that the shear from section to section undergoes a constant change.

In cantilever beams, for which moment and shear diagrams are shown in Fig. 94, the conditions differ somewhat from those discussed above for simple beams. From the moment diagrams it can be seen that the moment increases from a zero value at the free end of the beam to a maximum value at the point where the beam enters the wall. Similar conditions exist for shear, which is also a maximum at the wall. Note that for cantilever beams, the moments and shears are a maximum at the same section.

Illustrative Problem.—Construct shear and moment diagrams for a simple beam, 20 ft. long, loaded as shown in Fig. 97.

Left reaction, R_1 = right reaction, $R_2 = \frac{(20)(80)}{2} = 800$ lb.

Shear at section just to right of $R_1 = +800$ lb.
Shear at $a = 800 - (4)(80) = +480$ lb.
Shear at $b = 800 - (8)(80) = +160$ lb.
Shear at $c = 800 - (12)(80) = -160$ lb.
Shear at $d = 800 - (16)(80) = -480$ lb.
Shear at section just to left of $R_2 = 800 - (20)(80) = -800$ lb.
Moment at $R_1 = 0$
Moment at $a = (800)(4) - (80)(4)(\frac{4}{2}) = +2{,}560$ ft.-lb.
Moment at $b = (800)(8) - (80)(8)(\frac{8}{2}) = +3{,}840$ ft.-lb.
Moment at $c = (800)(8) - (80)(8)(\frac{8}{2}) = +3{,}840$ ft.-lb.
Moment at $d = (800)(4) - (80)(4)(\frac{4}{2}) = +2{,}560$ ft.-lb.

Due to symmetrical conditions the shear is zero at the center. Therefore, the moment is a maximum at this point and is equal to

$$(800)(10) - (80)(10)(\tfrac{10}{2}) = 4{,}000 \text{ ft.-lb.}$$

In order that the moment curve may be completely determined, moments should be calculated at points 2 ft. apart, at the most. However, since the computations are

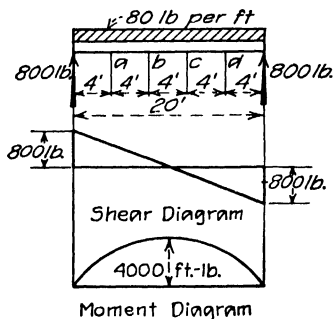



Fig. 97.

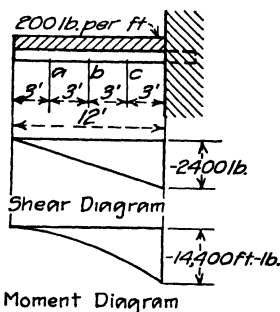



Fig. 98.

similar to those given above and since we know that the moment curve is parabolic in form, these computations will not be reproduced here.

Illustrative Problem.—Construct shear and moment diagrams for a cantilever beam 12 ft. long, loaded as shown in Fig. 98.

Shear at left end = 0
Shear at $a = -(200)(3) = -600$ lb.
Shear at $b = -(200)(6) = -1{,}200$ lb.
Shear at $c = -(200)(9) = -1{,}800$ lb.
Shear at support $= -(200)(12) = -2{,}400$ lb. (Maximum shear)
Moment at left end = 0
Moment at $a = -(200)(3)(\frac{3}{2}) = -900$ ft.-lb.
Moment at $b = -(200)(6)(\frac{6}{2}) = -3{,}600$ ft.-lb.
Moment at $c = -(200)(9)(\frac{9}{2}) = -8{,}100$ ft.-lb.
Moment at support $= -(200)(12)(\frac{12}{2}) = -14{,}400$ ft.-lb (Maximum moment)

Illustrative Problem.—Construct shear and moment diagram for a simple beam 40 ft. long, loaded as shown in Fig. 99.

To get the value of R_1, take moments about R_2

$$(R_1)(40) = (2,000)(8) + (2,000)(8 + 6) + (2,000)(8 + 6 + 6) + (2,000)(8 + 6 + 6 + 6)$$

$$R_1 = \frac{136,000}{40} = +3,400 \text{ lb.}$$

To get the value of R_2, take moments about R_1

$$(R_2)(40) = (2,000)(14) + (2,000)(14 + 6) + (2,000)(14 + 6 + 6) + (2,000)(14 + 6 + 6 + 6)$$

$$R_2 = \frac{184,000}{40} = +4,600 \text{ lb.}$$

$$3,400 + 4,600 = (4)(2,000) = 8,000 \text{ lb. (check)}$$

While the value of R_2 might have been obtained by subtracting R_1 from the sum of the loads, the method used above is better, since a check is afforded.

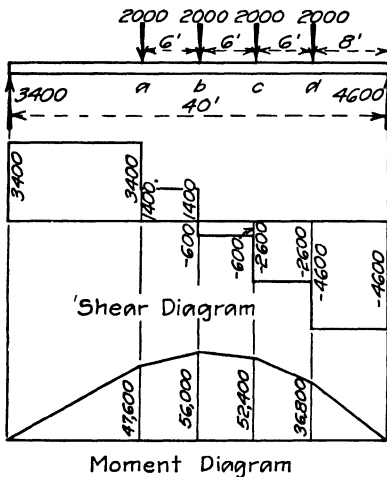
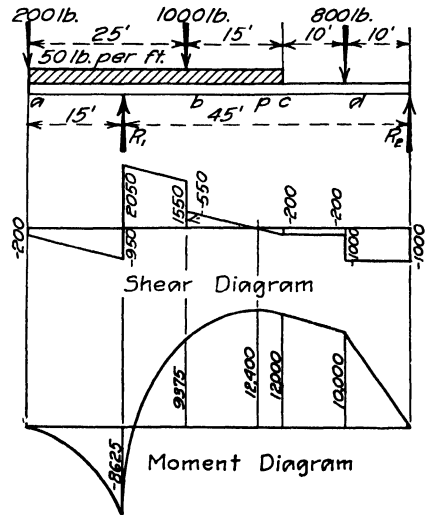



FIG. 99. FIG. 100.

Shear at section just to right of R_1 = +3,400 lb
Shear to the left of a = +3,400 lb.
Shear to the left of b = 3,400 − 2,000 = +1,400 lb.
Shear to the left of c = 3,400 − (2)(2,000) = −600 lb.
Shear to the left of d = 3,400 − (3)(2,000) = −2,600 lb.
Shear to the left of R_2 = 3,400 − (4)(2,000) = −4,600 lb.
Moment at R_1 = 0
Moment at a = (14)(3,400) = 47,600 ft.-lb.
Moment at b = (20)(3,400) − (6)(2,000) = 56,000 ft.-lb.
Moment at c = (14)(4,600) − (6)(2,000) = 52,400 ft.-lb.
Moment at d = (8)(4,600) = 36,800 ft -lb.

The shear and moment diagrams plotted from these calculated values are shown in Fig. 99.

Illustrative Problem.—Construct shear and moment diagrams for the beam shown in Fig. 100.

For the purpose of securing the values of the reactions, the uniform load of 50 lb. per lin. ft., distributed over a length of 40 ft. may be regarded as a concentrated load of (40)(50) = 2,000 lb., applied at the center of gravity of the load, or 5 ft. to the right of R_1.

Taking moments about R_2

$$R_1 = \frac{(60)(200) + (40)(2{,}000) + (35)(1{,}000) + (10)(800)}{45} = +3{,}000 \text{ lb.}$$

Taking moments about R_1

$$R_2 = \frac{(35)(800) + (10)(1{,}000) + (5)(2{,}000) - (15)(200)}{45} = +1{,}000 \text{ lb.}$$

Shear at section just to right of $a = -200$ lb.

Shear at $R_1 = \begin{cases} \text{to left} = -200 - (50)(15) = -950 \text{ lb.} \\ \text{to right} = -950 + 3{,}000 = +2{,}050 \text{ lb.} \end{cases}$

Shear at b $\begin{cases} \text{to left} = -200 - (50)(25) + 3{,}000 = +1{,}550 \text{ lb.} \\ \text{to right} = +1{,}550 - 1{,}000 = +550 \text{ lb.} \end{cases}$

Shear at $c = -200 + 3{,}000 - 2{,}000 - 1{,}000 = -200$ lb.

Shear at $d = \begin{cases} \text{to left} = -200 \text{ lb.} \\ \text{to right} = -200 - 800 = -1{,}000 \text{ lb.} \end{cases}$

Shear just to left of $R_2 = -1{,}000$ lb.

It is evident from these results that the shear passes through a zero value at some point between b and c. Since a point where the shear changes sign is a point of maximum moment, it will be necessary to locate this point. Let m represent the distance from the left end of the beam to this point, p, between b and c, where the shear is zero.

$$\text{Shear at } p = -200 - 50m - 1{,}000 + 3{,}000 = 0$$
$$50m = 1{,}800$$
$$m = 36$$

Moment at a = 0

Moment at $R_1 = -(15)(200) - (15)(^{15}\!/\!_2)(50) = -8{,}625$ ft.-lb.

Moment at $b = -(25)(200) - (25)(^{25}\!/\!_2)(50) + (10)(3{,}000) = +9{,}375$ ft.-lb.

Moment at $p = -(36)(200) - (36)(^{36}\!/\!_2)(50) - (11)(1{,}000) + (21)(3{,}000) = +12{,}400$ ft.-lb.

Moment at $c = +(20)(1{,}000) - (10)(800) = +12{,}000$ ft.-lb.

Moment at $d = +(10)(1{,}000) = +10{,}000$ ft.-lb.

The moment is a maximum at the point p where the shear is zero.

65c. Effect of Floorbeams.—The track and ties of railway bridges and the floors of highway bridges are supported by longitudinal beams called stringers. These stringers are supported by transverse beams called floorbeams, which carry the applied loads to the trusses or girders. As actually constructed, the stringers are riveted directly to the floorbeams, which in turn are riveted directly to the supporting trusses or girders. However, for clearness in presentation, it will be assumed, in the cases which follow, that the stringers rest upon the floorbeams, and that the floorbeams rest upon the trusses or girders. Figure 101 shows a girder bridge supporting a floor of the type described above. Any applied load carried by the floor is transferred to the stringers. As

soon as the type of floor is known, the amount of load carried by each stringer may be determined, for the stringer load is equal to the reaction for the floor considered as a simple beam supported by the stringers. In the same manner, the floorbeam loads are equal to the stringer reactions, and finally, the truss or girder loads are equal to the floorbeam reactions.

Figure 101 shows a girder with a floor supported by an arrangement of floorbeams and stringers of the type described above. This system supports a single concentrated load which will be assumed as placed mid-way between the stringers. Hence each stringer load will be equal

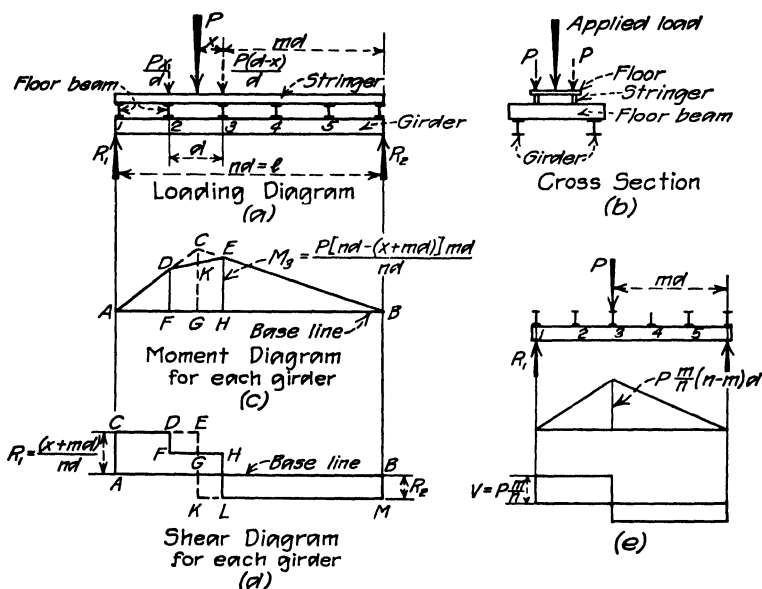



FIG. 101.

to one-half of the applied load. Let P of Fig. 101a represent the portion of the applied load which is carried by each stringer. The arrow indicates the position and direction of this stringer load. This load is transferred to the girder at panel points 2 and 3. Panel point 2 of each girder receives a load $P\frac{x}{d}$ and panel point 3 receives a load $P\frac{(d-x)}{d}$—that is, these panel points receive the reactions of a simple beam one panel in length, the stringers not being continuous over the floorbeams.

The effect of floorbeams and stringers on the reactions R_1 and R_2 may be determined by calculating these reactions for the structure with and without floorbeams and stringers, and comparing the calculated values. Consider the structure as a simple beam of span l carrying an

applied load P at a distance $(x + md)$ from the right end. The resulting reactions are

$$R_1 = P\frac{(x + md)}{nd} \text{ and } R_2 = P\frac{[nd - (x + md)]}{nd}$$

Now consider the structure to be provided with floorbeams and stringers, as shown in Fig. 101. Due to this arrangement, the load P is divided into the parts calculated above which form the concentrations shown at panel points 2 and 3 by the dotted arrows. Considering these concentrations as loads on the span, the reactions are found to be

$$R_1 = \frac{P\frac{(d - x)}{d}md + P\frac{x}{d}(md + d)}{nd}$$

from which

$$R_1 = P\frac{(x + md)}{nd}$$

Also

$$R_2 = \frac{P\frac{x}{d}[nd - (md + d)] + P\frac{d - x}{d}(nd - md)}{nd}$$

from which

$$R_2 = P\frac{[nd - (x + md)]}{nd}$$

Since these values are the same as those given above for the girder without floorbeams, it is evident that the value of the reactions is not affected by the presence of floorbeams and stringers.

The effect of floorbeams and stringers on the moment in a girder or truss is shown in Fig. 101*c*. To plot the moment diagram for the girder with floorbeams assume that load P is replaced by the partial loads shown at panel points 2 and 3 of Fig. 101*a*. Calculate and draw the moment diagram for these loads as explained in Art. **65***b*. This moment diagram is shown by $ADEB$ of Fig. 101*c*. Assume now that the floorbeams are removed and that the load is carried directly by the girder acting as a simple beam. By the methods used in Art. **65***b*, the moment under load P is

$$M = P\frac{(x + md)[nd - (md + x)]}{nd}$$

This moment is represented by the ordinate GC of Fig. 101*c*, and the complete moment diagram is shown by ACB. Note that at points D and E the two moment diagrams coincide, and that for points outside the panel containing the load P, the diagrams are identical. For points between panel points 2 and 3 the moments in the simple girder are greater than those in a girder with floorbeams.

The moment in a girder with floorbeams differs from the moment in a simple girder of the same span by the triangular area *DCE* of Fig. 101*c*, and the greatest difference is represented by the ordinate *CK*. To determine the value of *CK*, calculate the moments represented by the ordinates *GC* and *GK* and find their difference. As given above,

$$GC = P\frac{(x + md)[nd - (md + x)]}{nd}$$

Taking moments about the line of action of the load *P*, which is assumed to be replaced by the floorbeam loads at panel points 2 and 3, we have

$$M = GK = P\frac{(x + md)}{nd}[nd - (md + x)] - P\frac{x}{d}(d - x)$$

Subtracting the value of *GK* from that of *GC*, we have

$$CK = GC - GK = P\frac{x}{d}(d - x)$$

But $P\frac{x}{d}(d - x)$ is the moment in a beam of span *d* due to a load *P* placed at a distance *x* from one end of the span, which can be seen to be the loading condition for the stringer between panel points 2 and 3 of Fig. 101*a*. Therefore, the moment at any point between panel points in a girder with floorbeams is less than the moment at the same point in a simple beam of the same span by an amount equal to the moment in the stringer at the section in question.

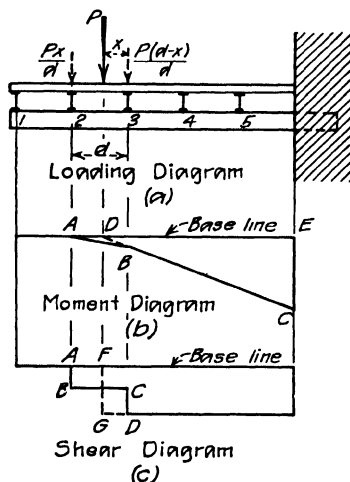



Fig. 102.

Figure 101*e* shows a girder supporting a concentrated load *P* placed at a panel point. Moment diagrams drawn for this structure with and without floorbeams are found to be identical. Hence, when the loads are placed at panel points, the presence of floorbeams does not affect the moments, which may be calculated as for a simple beam of the same span.

The shear diagram for a girder with floorbeams and stringers subjected to a single concentrated load in one panel is shown in Fig. 101*d* by the shear line *CDFHLM*. This diagram is calculated and drawn by the methods given in Art. **65***a*. For a simple beam without floorbeams, the shear diagram is shown by *CEKM*. In general, the shear in the panel containing the applied load is less than that at the same point in a simple beam of the same span. The difference in shear can be seen to be equal to the end reactions for the stringer

due to the load in the given panel. Note that in the panels on either side of the one under consideration, the shear is not affected by the presence of floorbeams.

A cantilever girder with floorbeams and stringers is shown in Fig. 102. For a load P at a distance x from the right end of panel 2–3, the panel concentrations are the same as for the case shown in Fig. 101. The vertical reaction at the wall for the conditions shown in Fig. 102a is the same for a girder with floorbeams as it is for a girder without floorbeams.

Figure 102b shows the moment diagram drawn for the conditions shown in Fig. 102a. This moment diagram is shown by the figure $ABCE$. For a girder without floorbeams, the moment diagram is shown by the triangle DCE. Hence, the moment in a girder with floorbeams is greater than the moment in a girder of the same span but without floorbeams.

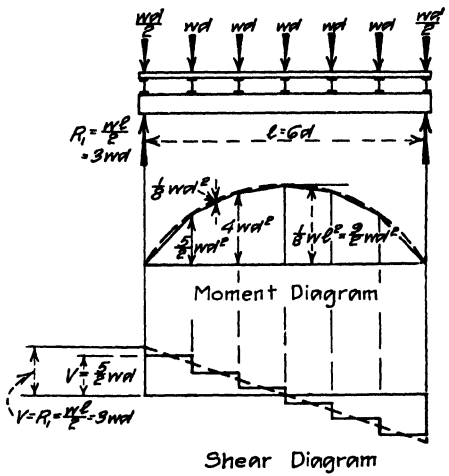
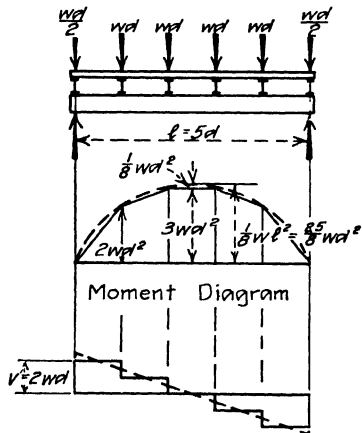



Fig. 103.

By an analysis similar to that given above, it can be shown that the moment in a girder with floorbeams exceeds the moment in a girder without floorbeams by an amount equal to the moment at that point in the stringer due to the applied load.

Fig. 104

The shear diagram of Fig. 102c shows that, for a girder with floorbeams, the shear in the panel containing the applied load is less than the shear at the same point in a girder without floorbeams. However, the portion of the girder subjected to shear is somewhat greater for the girder with floorbeams than for the girder without floorbeams. Note also that the shear is constant between adjacent floorbeams.

The variation in moment and shear for a girder with floorbeams and stringers supporting a uniform load of w pounds per foot is shown in Figs. 103 and 104. Since the load is uniform per foot, the reaction at

each panel point due to the load on a stringer is equal to one-half the total load on the stringer, or $\frac{1}{2}wd$. As shown on the figures, two stringer reactions are carried by each floorbeam. Hence the concentration at each panel point is equal to wd, a stringer load. These concentrations are shown in place on Figs. 103 and 104. Since the end of one stringer is carried at the ends of the girder, half panel loads are shown at these points. Generally these outside loads need not be considered in the determination of moments and shears at interior points on the girder, for it is evident that they have no effect on the value of the moment and shear.

Moment and shear diagrams for the concentrated loads shown in Figs. 103 and 104 can be plotted by the methods described in Art. **65***a*. The moment and shear diagrams for the girder with floorbeams are shown by the full lines. Calculated values of moments and shears are indicated on the diagrams.

As a means of comparing the moments and shears in girders with floorbeams with the corresponding values for girders without floorbeams, moments and shears have been calculated for simple beams of the same span as the girders shown in Figs. 103 and 104. These moments are shown by the dotted lines in the moment and shear diagrams of Figs. 103 and 104.

On comparing the moment and shear diagrams for girders with floorbeams with those for girders without floorbeams, it can be seen that the moments at panel points are the same for both cases. For points between panel points, the moment in the simple beam exceeds the moment in the girder with floorbeams. By an analysis similar to that used for Fig. 101, it can be shown that the moment in the simple beam exceeds that in the girder with floorbeams by an amount equal to the stringer moment at the same point. At the center of each panel this difference in moment is equal to $\frac{1}{8}wd^2$.

The shear diagrams for the two cases show that the end shear for the girder with floorbeams is less than that in a simple beam of the same span. It can also be seen that the shear at the center of each panel is equal to the shear at the same point in a simple beam. In fact, the shear line for a simple beam is an average line for the shear line for the girder with floorbeams. Note also that while the shear in a simple beam varies uniformly, the shear in a girder with floorbeams is constant across each panel.

Figure 103 shows the moment and shear diagrams for a girder with an even number of panels, while Fig. 104 is drawn for a girder with an odd number of panels. In Fig. 103 the center moments are equal for the simple beam and for the girder with floorbeams. In Fig. 104, where an odd number of panels is used, the center moment in the simple beam exceeds that in the girder with floorbeams by $\frac{1}{8}wd^2$. For this reason it is generally found best, if possible, to make use of an odd number

of panels. The advantage gained by the reduction of moment is considerable where the number of panels is small. For example, in the five-panel girder shown in Fig. 104, the reduction in moment is 12.5 per cent.

A cantilever beam with floorbeams, carrying a uniform load, is shown in Fig. 105. Moment and shear diagrams are drawn for the girder with floorbeams and for a simple cantilever beam of the same span. The full lines show the diagrams for the girder with floorbeams, and the dotted lines show the diagrams for the simple beam. Note that the simple beam values for moment exceed those for the girder with floorbeams. The shear diagrams differ in the same manner as for Figs. 103 and 104. Calculated values of moments and shears are indicated on Fig. 105.

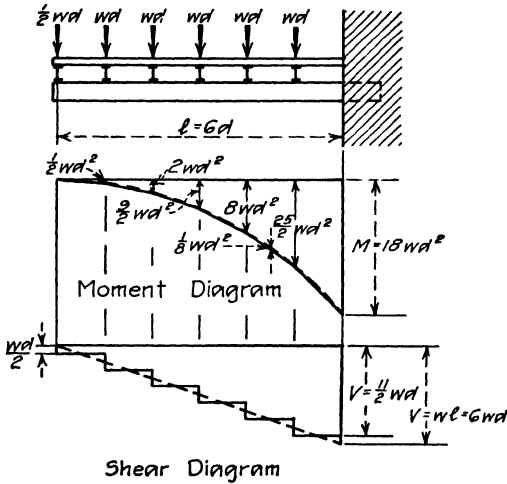



Fig. 105.

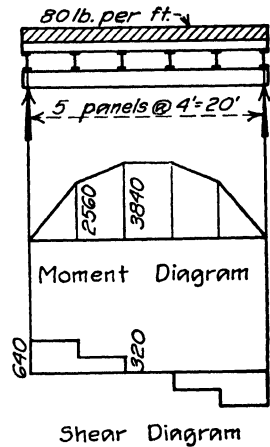



Fig. 106.

Illustrative Problem.—Construct shear and moment diagrams for the girder shown in Fig. 106.

The conditions shown in this figure are the same as those shown in Fig. 97 except for the addition of the floorbeams. In this case, we have a series of concentrated loads acting upon the girder instead of a uniformly distributed load. The load concentrated at each floorbeam is equal to $(4)(80) = 320$ lb. Since the calculations of the shears and moments are similar to those given in preceding problems, they will not be reproduced here. The values of the shears and moments are shown in the accompanying diagrams. It should be noted that the maximum moment, in this case, is less than that in the beam shown in Fig. 97. This illustrates the advantage of an odd number of panels. If there had been an even number of panels, a floorbeam would have been situated at the center of the girder, and the moment would have been 4,000 ft.-lb. as in the beam of Fig. 97.

Illustrative Problem.—Construct shear and moment diagrams for the cantilever girder shown in Fig. 107.

This problem is like that illustrated in Fig. 98 except that floorbeams have been added. The values of the shears and moments are shown in the accompanying dia-

grams. The maximum shear is less than that shown in Fig. 98 if, as assumed here, the half panel load adjacent to the wall is transferred to the wall directly. However, if a channel were introduced to transfer this load to the girder, the maximum shear in the girder would become 2,400 lb.

Illustrative Problem.—Construct shear and moment diagrams for a 40-ft. girder with floorbeams 8 ft. apart, loaded as shown in Fig. 108.

Load at a = $(\frac{1}{4})(2{,}000)$ = 500 lb.
Load at b = $(\frac{3}{4})(2{,}000) + (\frac{1}{2})(2{,}000)$ = 2,500 lb.
Load at c = $(\frac{1}{2})(2{,}000) + (\frac{3}{4})(2{,}000)$ = 2,500 lb.
Load at d = $(\frac{1}{4})(2{,}000) + 2{,}000$ = 2,500 lb.

The values of the shears and moments are given in the diagrams. Compare these diagrams with those shown in Fig. 99 for a beam without floorbeams.

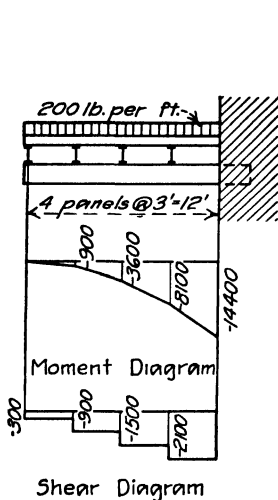



FIG. 107.

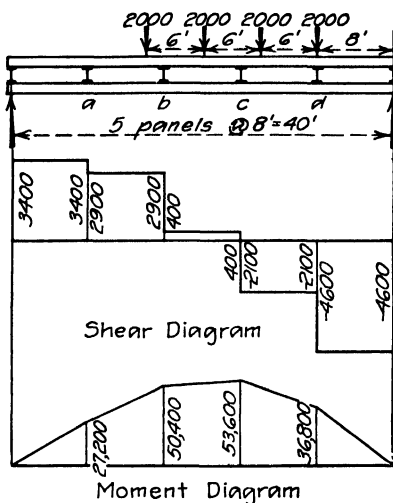



FIG 108.

Illustrative Problem.—Construct shear and moment diagrams for a 60-ft. girder with floorbeams 10 ft. apart, loaded as shown in Fig. 109.

Load at a = $(5)(50) + 200$ = 450 lb.
Load at b = $(10)(50)$ = 500 lb.
Load at c = $(10)(50) + 500$ = 1,000 lb.
Load at d = $(10)(50) + 500$ = 1,000 lb.
Load at e = $(5)(50)$ = 250 lb.
Load at f = 800 lb.

The variations in shear and moment are shown in the diagrams. Compare these diagrams with those shown in Fig. 100 for a beam without floorbeams.

66. Moments and Shears Due to Moving Loads.

66a. A Single Concentrated Load.—The moment and shear at any point in a beam or girder varies with the position of the applied load and with the arrangement of the floor system. However, there is some position of the load which will give the greatest, or maxi-

mum, moment or shear at any point. This position of a single concentrated load will be determined for beams with and without floorbeams.

Maximum Moment in a Girder without Floorbeams.—The maximum moment at any point C at a distance a from the left end of a beam of span l, Fig. 110, for a single concentrated load, occurs when the load is placed at the moment center. This maximum moment is

$$M_{\text{Max.}} = R_1 a = R_2(l - a)$$

That is,

$$M_{\text{Max.}} = P\frac{a}{l}(l - a) \tag{11}$$

To prove that the load should be placed at point C, note that if the load is placed to the right of point C, R_1 is reduced in value, and hence

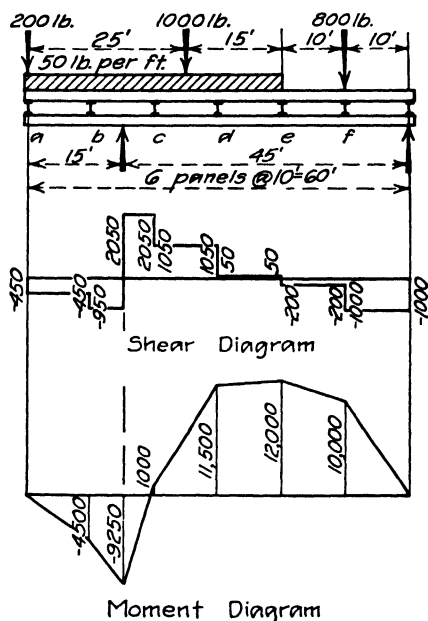
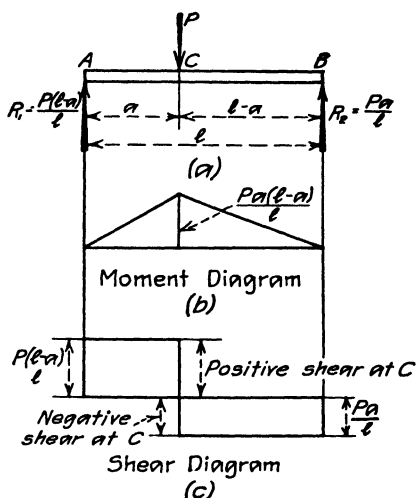



FIG. 109. FIG. 110.

the product R_1a, as given by Eq. (11) is also reduced. Also, if the load is placed to the left of point C, R_2 is reduced in value, and hence the product $R_2(l - a)$ of Eq. (11) is also reduced. Therefore, the maximum moment occurs when the load is placed at the moment center. The moment diagram for this position of the load is shown in Fig. 110*b*. It is drawn by the methods given in Art. 65*a*.

The moment given by Eq. (11) represents the maximum moment at any point on the girder. Evidently there is some point for which the moment is greater than for any other point. Since the moment at any

point on the beam can be stated by an expression of the form of Eq. (11), it is evident that the maximum moment occurs at the point for which the product $a(l - a)$ is a maximum. By an analysis similar to that used in the determination of the maximum value given by Eq. (6) of Art. **65,** it can be shown that the maximum value of Eq. (11) occurs at the beam center—that is, for $a = \frac{l}{2}$. Placing this value of a in Eq. (11), we have

$$M_{\text{Max.}} = \tfrac{1}{4}Pl \tag{12}$$

Illustrative Problems.—(*a*) Calculate the maximum moment at a point 10 ft. from the left end of a girder (without floorbeams) 30 ft. long supporting a single concentrated moving load of 60,000 lb.

As stated above, for maximum moment the moving load is to be placed at the point where the moment is desired. Figure 111 shows the load in the proper position. For the conditions shown, $R_1 = (60{,}000)(^{20}\!/_{30}) = 40{,}000$ lb. and $M_c = (R_1)(10) = (40{,}000)(10) = 400{,}000$ ft.-lb., which is the desired moment.

Fig. 111.

The problem may also be solved by substitution in Eq. (11). For the conditions shown in Fig. 111, $l = 30$ ft., $a = 10$ ft., $l - a = 20$ ft., and $P = 60{,}000$ lb. Substituting these values in Eq. (11), we have $M_c = (60{,}000)\ (^{10}\!/_{30})(20) = 400{,}000$ ft.-lb.

(*b*) Calculate the greatest moment which occurs in the beam of problem (*a*) for the given load.

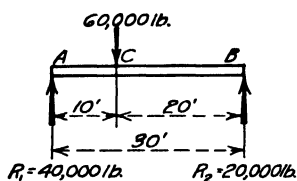
As stated above, the greatest moment in the beam occurs at the beam center. Placing the load at the beam center, $R_1 = (60{,}000)(^{15}\!/_{30}) = 30{,}000$ lb., and $M_{\text{Max}} = (R_1)(15) = 450{,}000$ ft.-lb. Again, substituting in Eq. (12) with $l = 30$ ft., and $P = 60{,}000$ lb., we have $M_c = (60{,}000)(^{30}\!/_{4}) = 450{,}000$ ft.-lb.

Maximum Shear in a Girder without Floorbeams.—The greatest positive shear at a point distance a from the left end of a simple beam or girder, Fig. 110, due to a single moving concentrated load, occurs when the load is placed at an infinitesimal distance to the right of the point. This maximum positive shear is

$$+V_c = +R_1 = P\frac{(l - a)}{l} \tag{13}$$

The maximum negative shear at the same point occurs when the load is placed at an infinitesimal distance to the left of the point C, and the shear is

$$-V_c = R_1 - P = -R_2 = -P\frac{a}{l} \tag{14}$$

To prove that these statements are correct, refer to the definition and notation for shear given in Art. **64.** It is there stated that positive shear occurs when the resultant of vertical forces to the left of a section or point is upward. Since the reaction R_1 due to any load on the beam is less than the load causing that reaction, an upward resultant force on the left of a

section in a beam carrying a single load can occur only when the load is to the right of the section. It is therefore evident from Fig. 110 that for the case under consideration, the upward force on the left of the section, which is R_1, can be made a maximum by causing the load P to approach point C from the right to a position as close to C as possible. This can be done by placing P an infinitesimal distance to the right of C, for then R_1 will have its maximum value subject to the condition that there shall be no downward loads to the left of C. A similar analysis for negative shear, considering forces to the right of the section, shows that the load is to be placed at an infinitesimal distance to the left of point C. However, in calculating reactions, the applied load may be considered as located at point C. It will be noted that this has been done in deriving Eqs. (13) and (14). In Fig. 110c the shear diagram is shown for the load in position for maximum shear at point C.

The greatest positive shear at any point in the beam occurs at the left end of the beam, and the greatest negative shear occurs at the right end of the beam. In each case the shear is equal to the applied load.

Illustrative Problems.—(*a*) Calculate the maximum positive and negative shear at a point 10 ft. from the left end of a girder (without floorbeams) 30 ft. long due to a single concentrated moving load of 60,000 lb.

For positive shear the load is to be considered as located at an infinitesimal distance to the right of the point in question. Figure 111 shows the load in position. For the conditions shown, $R_1 = (60,000)(^{20}\!/_{30}) = 40,000$ lb. Hence, positive shear at $C = R_1 = 40,000$ lb. The maximum negative shear at C occurs when the load is considered as located at an infinitestimal distance to the left of point C. Again Fig. 111 shows the load in position. Hence, negative shear at $C = R_1 - P = 40,000 - 60,000 = -20,000$ lb. Note that P is to the left of the section and must therefore be included in the forces to the left of the point.

The problem may also be solved by substitution in Eqs. (13) and (14). From Fig. 111, $P = 60,000$ lb., $l = 30$ ft., and $a = 10$ ft. With these values we have

$$+V_c = +R_1 = +60,000\frac{(30 - 10)}{30} = +40,000 \text{ lb.}$$

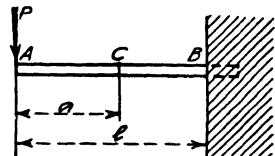
and

$$-V_c = -R_2 = -(60,000)(^{10}\!/_{30}) = -20,000 \text{ lb.}$$

(*b*) Calculate the greatest positive and negative shears for the beam of problem (*a*).

As stated above, the greatest positive and negative shears occur at the ends of the beam and are equal to the applied load. Therefore, the greatest positive shear occurs at point A, and it is equal to $+60,000$ lb. The greatest negative shear occurs at point B, and it is equal to $-60,000$ lb.

Fig. 112.

Maximum Moment and Shear in a Cantilever Beam.—At any point on a cantilever beam, such as point C of Fig. 112, the moment due to single moving concentrated load is a maximum when the load is placed at point A, the left end of the beam. This moment is $M_c = -Pa$. When the load is placed to the right of

C, $M_c = 0$. The point of greatest moment is at B, the right end of the beam. This moment occurs when the load is at A, and $M_B = -Pl$.

The shear at any point C is a maximum when the applied load is placed anywhere to the left of C, and the shear is negative and equal to the load. When the load is placed to the right of C, the shear at that point is zero.

Illustrative Problems.—(*a*) Calculate the maximum moment and shear at a point 5 ft. from the free end of a cantilever beam 15 ft. long, due to a single moving concentrated load of 20,000 lb.

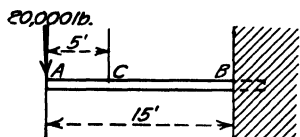



Fig. 113.

For maximum moment, and also for maximum shear, the applied load is to be placed at the free end of the beam. Figure 113 shows the load in position. The moment at C is $M_c = -(20{,}000)(5) = -100{,}000$ ft.-lb., and the shear at C is $V_c = -20{,}000$ lb.

(*b*) Calculate the greatest moment and shear in the beam of problem (*a*).

The greatest moment and shear occur at point B when the load is placed at point A. For this position of the load,

$$M_B = -(20{,}000)(15) = -300{,}000 \text{ ft.-lb.}$$
$$V_B = -20{,}000 \text{ lb.}$$

Maximum Moment and Shear in a Girder with Floorbeams.—In Art. **65c** it was shown that for girders with floorbeams, the moments for loads at panel points were the same as for the girder without floorbeams. When a load was placed at any point in a panel, as 2-3 of Fig. 101, it was shown that the resulting moment was less than for the beam without floorbeams. Since in designing structures only the maximum moments are desired, it is sufficient to calculate moments only at the panel points. This can be done by the methods given previously in this article.

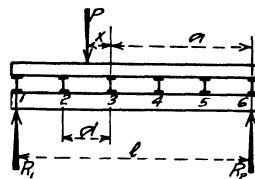



Fig. 114.

The shear in any panel of a girder with floorbeams, Fig. 114, due to a moving concentrated load P is

$$V = \text{Left reaction} - \text{Load at panel point 2.}$$

This equation is based on the definition for shear given in Art. **64.** As stated in Art. **65b**, R_1 is the same as for a girder without floorbeams—that is, $R = P\frac{(a + x)}{l}$. Since the load P is transferred to the girder by the stringers acting as beams, the load at panel point 2 is $P\frac{x}{d}$. Hence

$$V = P\frac{(a + x)}{l} - P\frac{x}{d} = P\left(\frac{a + x}{l} - \frac{x}{d}\right) \qquad (15)$$

The character of the shear depends upon the position of the load P in the panel. If the load is so placed that $\frac{a+x}{l} = \frac{x}{d}$, the shear is zero. This point of zero shear is called the *neutral point* in the panel. Solving the above expression for x, we have

$$x = \frac{ad}{l-d} \tag{16}$$

which is the distance from the right end of the panel to the neutral point. It can be seen from Eq. (16) that the position of the neutral point does not depend upon the magnitude of the applied load but simply upon the length of the panel and the position of the panel in the girder.

Equation (15) shows that the shear is positive when a load is placed to the right of the neutral point, for $\frac{a+x}{l}$ is then greater than $\frac{x}{d}$ and the resulting expression is positive When the load is placed to the left of the neutral point, $\frac{x}{d}$ is greater than $\frac{a+x}{l}$ and the resulting expression is negative.

To obtain the greatest positive shear in panel 2-3, the load must be placed at the right end of the panel, or for $x = 0$ in Fig. 114. For this position of P, the load at panel point 2 is zero and

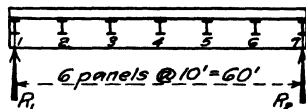
$$V = +R_1 \tag{17}$$

The greatest negative shear will occur when $x = d$, or when P is placed at the left end of the panel. For this position of P, the load at panel point 2 is equal to P, and

$$V = R_1 - P = -R_2 \tag{18}$$

Due to the presence of floorbeams, the shear in panel 2-3 is constant for any position of the applied load.

FIG. 115.

Illustrative Problem.—Calculate the maximum positive and negative shears in panel 2-3 of the girder of Fig. 115 due to a single moving concentrated load of 48,000 lb.

Maximum Positive Shear.—Place the load at panel point 3. Then from Eq. (17)

$$V = +R_1 = (^{40}\!/_{60})\,(48{,}000) = +32{,}000 \text{ lb.}$$

Maximum Negative Shear.—Place the load at point 2. From Eq. (18),

$$V = R_1 - P = (^{50}\!/_{60})\,(48{,}000) - 48{,}000 = -8{,}000 \text{ lb.}$$

Likewise,

$$V = -R_2 = -(^{10}\!/_{60})\,(48{,}000) = -8{,}000 \text{ lb.}$$

From Eq. (16) the neutral point in the panel is located $\frac{(40)(10)}{60-10} = 8$ ft. from point 3.

Maximum Moment and Shear in a Cantilever Beam with Floorbeams.—The loading conditions for maximum moment and shear in a cantilever beam with floorbeams are similar to those given for the beam without floorbeams. For maximum moment at any panel point, the load should be placed at the free end of the beam. For maximum shear in any panel the load may be placed anywhere between the free end of the beam and the panel in which the shear is to be determined.

66*b*. A Moving Uniform Load.—A uniform load is one in which the load per foot is the same at all points. This load may cover all or only a part of the structure under consideration. In the latter case, the load is uniform only for the covered portion of the structure. The positions of a moving uniform load will be determined which will produce maximum moments and shears in girders with and without floorbeams.

Maximum Moment in a Girder without Floorbeams.—It was shown in Art. **65***a* that a concentrated load placed at any point on a girder caused positive moment at all other points on that girder. Now a uniform load may be considered as made up of an infinite number of very small concentrated loads placed so close to each other that they form a continuous or uniform load. Since each and every one of these small concentrated loads produces positive moment at a given moment center, it is evident that for maximum moment at the point under consideration, all possible points should be occupied by these small concentrated loads. Therefore, to obtain maximum moment at any point in a girder due to a moving uniform load, the entire girder should be covered with the uniform load.

Since the beam is completely covered with the uniform load for maximum moment, the conditions are the same as given in Art. **65***a* and shown in Fig. 93. Therefore Eq. (6) gives the maximum moment at any point and Eq. (7) gives the maximum moment at the beam center, when w in these equations represents the uniform load per foot.

An interesting comparison between the moment at any point in a girder produced by a given load uniformly distributed over a girder and the moment due to the same load concentrated at the moment center may be had from Eq. (6) of Art. **65***a*, and Eq. (11) of Art. **66***a*. Let W be a given concentrated load, and let w be a load which is uniform per foot. From Eq. (6) the moment at a point distance a from the left end of a beam of span l is $M_u = \frac{1}{2} wa(l - a)$, and from Eq. (11) the moment due to a concentrated load at the same point is $M_c = W \frac{a}{l} (l - a)$. If the total loads in the two cases are equal, then $w = W/l$. Placing this value of w in the equation for M_u, we have $M_u = \frac{1}{2} W \frac{a}{l} (l - a)$. On comparing these values for M_u and M_c, it can be seen that $M_u = \frac{1}{2}M_c$—that is, the moment produced at any point in a simple beam by a given load

uniformly distributed is one-half the moment produced by that load when concentrated at the given moment center.

Illustrative Problem.—Calculate the moment at a point 10 ft. from the end of a 30-ft. beam due to a load of 60,000 lb. when concentrated at the point, and also when the load is uniformly distributed over the beam.

Load Concentrated.—From Eq. (11), the moment is

$$M = W\frac{a}{l}(l - a) = (60{,}000)\left(\frac{10}{30}\right)(30 - 10) = 400{,}000 \text{ ft.-lb.}$$

Load Uniformly Distributed.—The load per foot is $w = \frac{W}{l} = \frac{60{,}000}{30} = 2{,}000$ lb. per ft. From Eq. (6) the moment is $M = \frac{1}{2}wa(l - a) = (\frac{1}{2})(2{,}000)(10)(30 - 10) = 200{,}000$ ft.-lb. Note that the moment due to the load uniformly distributed is one-half that for the load concentrated at the moment center.

Maximum Shear in a Girder without Floorbeams.—It was shown in Art. **65a** that positive shear exists at any point in a simple beam for a load placed to the right of the point, and that a load to the left of a point causes negative shear at that point. Considering a uniform load to be made up of an infinite number of closely spaced concentrated loads, it is evident that for maximum positive shear at any section of a girder, the uniform load should cover that portion of the girder to the right of the section, as shown in Fig. 116*a*. Likewise, it is evident that Fig. 116*b* shows the position of the uniform load for maximum negative shear at point *C*.

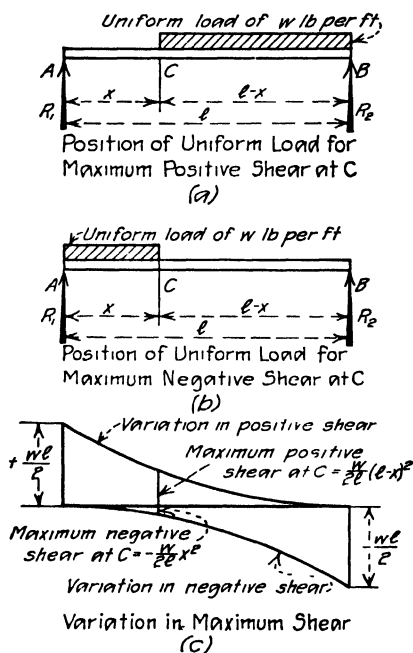



Fig. 116.

The maximum positive shear at point *C* of Fig. 116*a* is

$$+V_{\text{Max}} = +R_1 = \frac{w}{2l}(l - x)^2 \tag{19}$$

In Fig. 116*c* is shown a curve representing the variation in the maximum positive shear for all points on the girder. This curve is plotted from Eq. (19). From analytical geometry it can be shown that this curve **is a** parabola with the vertex at the right end of the girder.

The maximum negative shear at point C of Fig. 116b is

$$-V_{\text{Max}} = -R_2 = -\frac{w}{2l}x^2 \tag{20}$$

Figure 116c shows the variation in maximum negative shear for all points on the girder. This curve is also a parabola with the vertex at the left end of the girder. On Fig. 116c the values represented by Eqs. (19) and (20) are indicated at the ends of the load.

Since $(l - x)$ of Eq. (19) and x of Eq. (20) may be interchanged, it can be seen that the maximum positive shear at a given distance from the left end of a girder is numerically equal to the maximum negative shear at the same distance from the right end of the girder. Note, however, that the shear at the former point is positive while that at the latter point is negative. This relation will be found useful in the work to follow.

By an analysis similar to that given for moment in Art. **66**b, it can be

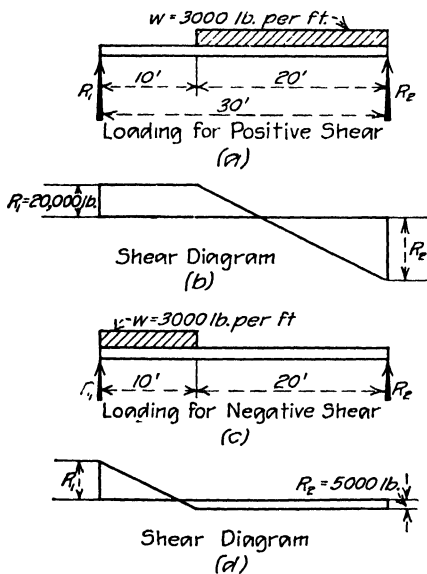



FIG. 117.

shown that the maximum shear due to a moving uniform load is one-half the shear for the same total load considered as a concentrated load.

Illustrative Problem.—Calculate the maximum positive and negative shear at a section 10 ft. from the left end of a 30-ft. beam due to a uniform load of 3,000 lb. per ft.

Maximum Positive Shear.—Loading conditions as shown in Fig. 117a. From Eq (19),

$$+V_{\text{Max.}} = +R_1 = \frac{3{,}000}{(2)(30)}(20)^2 = +20{,}000 \text{ lb.}$$

Figure 117b shows the shear diagram for this position of the load. This shear diagram is not to be confused with the curves of Fig. 116, which represent the variation in maximum shear.

Maximum Negative Shear.—Position of load shown in Fig. 117c. From Eq. (20)

$$-V_{\text{Max.}} = -R_2 = -\frac{3{,}000}{(2)(30)}(10)^2 = -5{,}000 \text{ lb.}$$

The shear diagram for this position of the load is shown in Fig. 117d.

Maximum Moment in a Girder with Floorbeams.—In a girder with floorbeams, as shown in Fig. 118, the uniform load is divided into panel loads of wd each, where w is the uniform load per foot and d is the length of panel. For maximum moment at any point these loads are to be placed at every panel point, for, as shown in Art. 65a, a load placed at any point on a beam causes positive moment at every other point.

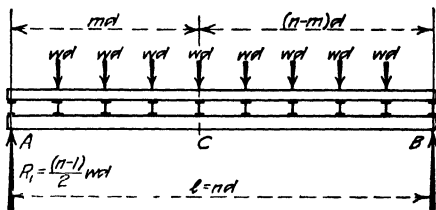



Fig. 118.

An expression for moment at point C of Fig. 118 will now be derived. Assume that the girder of Fig. 118 consists of n panels of length d, and let it be assumed that point C is m panels distant from the left end of the girder. For the loads shown in Fig. 118 the reaction at A is $\frac{1}{2}(n - 1)wd$ and hence the moment at C is

$$M_c = R_1md - wd[1 + 2 + \ . \ . \ . \ (m - 1)]d$$

The expression to the right of the first minus sign represents the negative moment of the loads to the left of the moment center. Substituting the above value of R_1 and noting that the expression in brackets is an arithmetical series,[1] the above equation becomes

$$M_c = \tfrac{1}{2}(n - 1)mwd^2 - \tfrac{1}{2}(m - 1)mwd^2$$

from which we have finally

$$M_c = \tfrac{1}{2}wd^2m(n - m) \qquad (21)$$

which is a general expression for moment at any point in the girder of Fig. 118.

[1] The sum of the terms of an arithmetical series is equal to the first term plus the last term times half the number of terms. That is,

$$[1 + 2 + \ . \ . \ (m - 1)] = [1 + (m - 1)]\frac{(m - 1)}{2} = \tfrac{1}{2}m(m - 1)$$

Illustrative Problem.—Calculate the maximum moment at panel point C of the girder shown in Fig. 119 due to a uniform load of 2,000 lb. per ft.

From Eq. (21) with w = 2,000 lb. per ft., d = 10 ft., n = 8, and m = 3, we have

$$M_c = (\tfrac{1}{2})(2{,}000)(10^2)(3)(8-3) = 1{,}500{,}000 \text{ ft.-lb.}$$

This problem may also be solved directly from Fig. 119. For a uniform load of 2,000 lb. per ft. each panel load is (2,000)(10) = 20,000 lb. These loads are shown in position on Fig. 119. For the conditions shown, $R_1 = (\tfrac{7}{2})(20{,}000) = 70{,}000$ lb. and $M_c = (70{,}000)(3)(10) - 20{,}000\ (1+2)10 = 1{,}500{,}000$ ft.-lb.

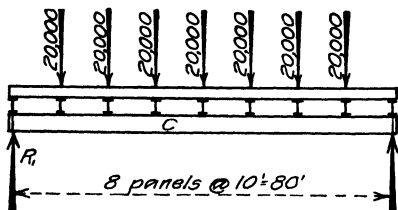
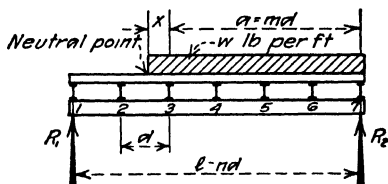



FIG. 119. FIG. 120.

Maximum Shear in a Girder with Floorbeams (Exact Method of Calculation).—The maximum positive shear in any panel of a girder with floorbeams will occur when the portion of the girder to the right of the neutral point (Art. **66**a) is covered with the uniform load, for, as shown in Art. **66**a, any load placed to the right of this point causes positive shear in the panel in question. Figure 120 shows a uniform load in position for maximum shear in panel 2–3. This shear is

$$+V_{\text{Max.}} = R_1 - \text{Floorbeam load at panel point 2.}$$

From Fig. 120, $R_1 = \frac{w}{2l}(a+x)^2$ and floorbeam load at 2 $= \frac{w}{2d}x^2$. Hence

$$+V_{\text{Max.}} = \frac{w}{2l}(a+x)^2 - \frac{w}{2d}x^2$$

The distance from panel point 3 to the neutral point, as given by Eq. (16) Art. **66**a, is $x = \frac{ad}{l-d}$. Substituting this value of x in the above equation, we have finally,

$$+V_{\text{Max.}} = +\frac{1}{2}w\frac{a^2}{(l-d)} \qquad (22)$$

In some cases it is convenient to express the dimensions shown in Fig. 120 in terms of the panel length d as a unit. Thus, let n be the number of panels in the structure, and let m be the number of panels from the right end of the panel under consideration to the right end of the girder—that is, $l = nd$, and $a = md$. Substituting these values in Eq. (22), we have

$$+V_{\text{Max.}} = \frac{1}{2}wd\frac{m^2}{(n-1)} \qquad (23)$$

Note that wd is the unit panel load for the girder under consideration.

Maximum negative shear may also be determined by means of Eq. (23). In this case m is to be taken as the number of panels from the left end of the panel under consideration to the left end of the girder.

Illustrative Problem.—Calculate the maximum positive and negative live load shear in the third panel from the left end of the girder shown in Fig. 121 due to a uniform live load of 2,400 lb. per ft.

Maximum Positive Shear.—Substituting in Eq. (23) with $n = 6$, $m = 3$, and $w = 2{,}400$, we have

$$+V_{\text{Max.}} = +\left(\frac{1}{2}\right)(2{,}400)(10)\left(\frac{3^2}{6-1}\right) = +21{,}600 \text{ lb.}$$

Maximum Negative Shear.—Substituting in Eq. (23) with $n = 6$, $m = 2$, and $w = 2{,}400$, we have

$$-V_{\text{Max.}} = -\left(\frac{1}{2}\right)(2{,}400)(10)\left(\frac{2^2}{6-1}\right) = -9{,}600 \text{ lb.}$$

Maximum Shear in a Girder with Floorbeams (Approximate Method of Calculation).—In practice it is generally assumed that in calculating

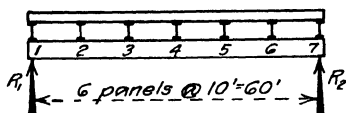
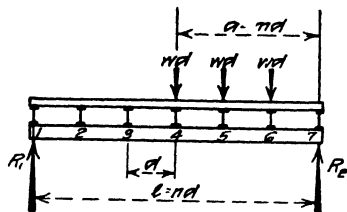



FIG. 121. FIG. 122.

shears due to a uniform load on a girder with floorbeams, the applied load may be divided into panel loads acting at the floorbeams. It is also assumed that, for positive shear, all panel points to the right of the panel under consideration carry full panel loads, as shown in Fig. 122. This condition is evidently not attainable, for in order to have a full panel load at point 4, Fig. 122, panel 3-4 must be completely loaded. When panel 3-4 is fully loaded there must be a half panel load, $\frac{1}{2}wd$, at point 3. Therefore, by assuming that no load exists at point 3, the resulting shear in panel 3-4 is slightly greater than the true shear, as given by Eq. (23).

However, since the loading conditions shown in Fig. 122 give shears somewhat on the side of safety, and since the desired shears can readily and conveniently be calculated for the conditions shown in Fig. 122, this method of loading is generally used in preference to the loading conditions shown in Fig. 120. The method of loading shown in Fig. 122 is known as the *Conventional Method of Loading.*

The maximum positive shear in panel 3-4 of Fig. 122 for the loading shown is

$$+V_{\text{Max}} = +R_1 = +\frac{wd}{nd}(1 + 2 + \ . \ . \ . \ m)d$$

Noting that the expression in parenthesis is an arithmetical series, the above equation may be written

$$V_{\text{Max.}} = \frac{wd}{2n}(m + 1)m \qquad (24)$$

Equation (24) gives the maximum positive shear in any panel of a girder with floorbeams for the above described conventional method of loading.

On comparing the exact value of the shear, as given by Eq. (23), and the approximate value, as given by Eq. (24), it will be noted that the term $\frac{m}{n-1}$ of Eq. (23) is replaced by the term $\frac{m+1}{n}$ of Eq. (24). To estimate the amount of approximation involved in the use of Eq. (24), subtract Eq. (23) from Eq. (24), letting D denote the difference between the two equations. We then have, after reducing the resulting expression to its simplest form,

$$D = \frac{1}{2}wd\frac{m}{n}\frac{(n - m - 1)}{(n - 1)}$$

It can readily be shown by the calculus that this expression has its maximum value when $m = \frac{1}{2}(n - 1)$, or for panels at or near the center of the girder. Substituting this value of m in the above equation for D, we have

$$D_{\text{Max.}} = \frac{1}{8}wd\frac{(n-1)}{n}$$

That is, the difference in the shears determined by the approximate and exact methods is a maximum in panels near the center of the girder, and the maximum difference in the shears calculated by the two methods is about one-eighth of a panel load.

Negative live load shears may also be determined by the conventional method of loading. For this case all panel points to the left of the panel in question are fully loaded, no load at points to the right. Equation (24) may be used for the calculation of negative shears, noting that the number of panels, m, is to be counted from the left end of the girder to the left end of the panel under consideration.

Since the same equation may be used for the determination of positive and negative shears, it is evident that the positive shear in a panel near the left end of the girder is numerically equal to the negative shear in the corresponding panel near the right end of the girder. A similar relation was noted in Art. **66b** for the girder without floorbeams.

Illustrative Problem.—Calculate the maximum positive and negative live load shears in panel 3-4 of the girder shown in Fig. 121 for a uniform live load of 2,400 lb. per ft. Use the conventional method of calculation.

Maximum Positive Shear.—Panel loads of $wd = (2,400)(10) = 24,000$ lb. are to be placed at points 4, 5, and 6. From Eq. (24) with $n = 6$ and $m = 3$,

$$+V = +\frac{24,000}{(2)(6)}(3 + 1)3 = 24,000 \text{ lb.}$$

Maximum Negative Shear.—Panel loads are to be placed at points 2 and 3 From Eq. (24) with $n = 6$ and $m = 2$, we have

$$-V = -\frac{24,000}{(2)(6)}(2 + 1)2 = -12,000 \text{ lb.}$$

Illustrative Problem.—Compare the maximum shears obtained by the exact and approximate methods of loading as applied to panel 3-4 of the girder of Fig. 121.

The shears determined in the above problems are given in the following table.

	Conventional Method	Exact Method	Difference in Shear
Positive shear (lb.).......	24,000	21,600	2,400
Negative shear (lb.)......	12,000	9,600	2,400

The values given by the Conventional Method exceed those given by the Exact Method by about 11 per cent of the latter in the case of positive shear and by about 25 per cent in the case of negative shear. These percentages vary with the position of the panel under consideration, being small at the ends of the girder and relatively large at the center.

Maximum Moments and Shears in a Cantilever Girder with Floorbeams. Figure 123 shows a cantilever girder with floorbeams supporting a uniform live load which is represented by the panel loads wd. Note that a half panel load is placed at the free end of the girder. Maximum moment at point C at a distance of m panels from the free end of the girder is caused by the panel loads at all points, or by the loading shown in Fig. 123. The value of the maximum moment at C, which is a negative moment, is

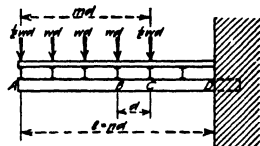



Fig. 123.

$$M_c = -wd[1 + 2 + \ . \ . \ . \ (m - 1)]d - \tfrac{1}{2}wd \cdot md$$

from which finally,

$$M = -\tfrac{1}{2}wm^2d^2 \tag{25}$$

Maximum negative shear in panel BC occurs under the loading shown on Fig. 123. The value of the shear is

$$V = -wd[\tfrac{1}{2} - (m - 1)] = -wd(m - \tfrac{1}{2}) \tag{26}$$

Positive moment or shear does not exist in a cantilever girder for any condition of loading involving vertical downward loads.

Illustrative Problem.—Calculate the maximum moment at point 5 of the cantilever girder of Fig. 124 due to a uniform live load of 2,000 lb. per ft.

Substituting in Eq. (25) with $w = 2{,}000$, $m = 4$, and $d = 5$, we have

$$M = -(\tfrac{1}{2})(2{,}000)(4)^2(5)^2 = -400{,}000 \text{ ft.-lb.}$$

Illustrative Problem.—Calculate the maximum shear in panel 4-5 of the cantilever girder of Fig. 124 due to a uniform load of 2,000 lb. per ft.

Substituting in Eq. (26) with $w = 2{,}000$, $m = 4$, and $d = 5$, we have

$$V = -(2{,}000)(5)(4 - \tfrac{1}{2}) = -35{,}000 \text{ lb.}$$

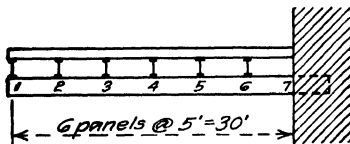



Fig. 124.

67. Moving Concentrated Load Systems.

67a. Conventional Methods of Representing Live Load Concentrations.—Live loads for bridges are represented either by a uniform load, or by the actual train or vehicle loads carried by the structure. As these loading systems are subject to considerable variation, it is generally found convenient to assume certain typical load systems

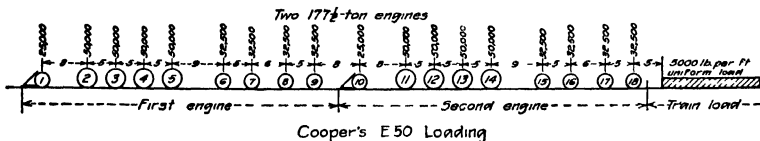



Fig. 125.

which experience has shown will represent the loading conditions encountered in practice. For highway bridges these typical loading systems are provided by the A.A.S.H.O.[1]

Train loads are generally represented by a set of typical loadings devised by Theodore Cooper or D. B. Steinman.[2] Cooper's E-50,

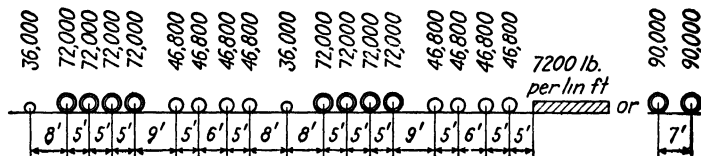



Fig. 126.

shown in Fig. 125, represents two typical consolidation freight locomotives followed by a train load which is considered as uniform per foot of track. The weight on each driving axle, wheels 2 to 5 and 11 to 14, is 50,000 lb.; that on the pilot axle, wheels 1 and 10, is 25,000 lb.; the weight on each tender axle, wheels 6 to 9 and 15 to 18, is 32,500 lb.; and

[1] American Association of State Highway Officials, "Standard Specifications for Highway Bridges," 1941.

[2] "Locomotive Loadings for Railroad Bridges," *Trans. Am. Soc. Civil Eng.*, Vol. LXXXVI, p. 606.

the train load is taken as 5,000 lb. per ft. of track. For reference, the system shown in Fig. 125 is known as an E-50 loading, taking its name from the amount of the driving axle load. Other loads are also used, but to facilitate calculation, the distances between the wheels are kept the same for all loadings, and the loads are all changed by the same ratio. Thus for an E-40 loading system, all loads are $^{40}/_{50}$ or 0.8 of those shown in Fig. 125. Cooper's E-72 or two axle loads of 90,000 lb. each, whichever produces the maximum stress, is recommended in the 1940 A.R.E.A. Specifications for Steel Railway Bridges. This loading is shown in Fig. 126.

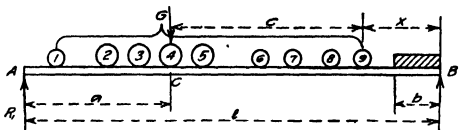



FIG. 127.

67b. Moments and Reactions for Concentrated Load Systems.—Let it be required to determine the reaction R_1 and the moment at point C of Fig. 127 for the loading conditions shown. From the definition of moment given in Art **63,** the moment at C is equal to R_1 times the distance from the moment center to the left support minus the moments of wheels 1, 2, 3, and 4 about point C. That is

$$M = R_1 a - \sum_1^4 M$$

where $\sum_1^4 M$ indicates the summation of the moments of wheels 1 to 4 about wheel 4.

Now the value of R_1 is determined by dividing the moment of all loads about B, the right end of the beam, by the span length. If we let Wz be the moment about B of any wheel of weight W at a distance z from B, and let $\sum_1^9 Wz$ be the sum of the moments of wheels 1 to 9 about B, we have, including the short piece of uniform load,

$$R_1 = \frac{M_B}{l} = \frac{1}{l}\left(\sum_1^9 Wz + \frac{1}{2}wb^2\right)$$

Substituting this value of R_1 in the expression for M_c, we have

$$M_c = \left(\sum_1^9 Wz + \frac{1}{2}wb^2\right)\frac{a}{l} - \sum_1^4 M$$

The term $\sum_1^9 Wz$ of the above equation may be placed in a form somewhat more convenient for calculation by the following process: Let G be

the total of all wheel loads, and let G be assumed as located at the center of gravity of all of the loads, which will be assumed as at a distance c from the last wheel load, as shown in Fig. 127. Then

$$\sum_1^9 Wz = G(c + x) = Gc + Gx$$

But $Gc = \sum_1^9 M$ = moment of loads 1 to 9 about wheel 9. Therefore, for any case involving n loads, we may write the general expression

$$\sum_1^n Wz = \sum_1^n M + \left(\sum_1^n W\right) x \tag{27}$$

where $\sum_1^n M$ = moment of all loads about the last load or the nth wheel;

$\sum_1^n W$ = sum of all loads up to and including the last or nth wheel; and

x = the distance from the last load or the nth wheel to the moment center. Substituting this expression for $\sum_1^9 Wz$ in the equations for R_1 and M_c, we have for the conditions shown in Fig. 127

$$R_1 = \frac{1}{l}\left[\sum_1^9 M + \left(\sum_1^9 W\right) x + \frac{1}{2} wb^2\right] \tag{28}$$

and

$$M_c = \left[\sum_1^9 M + \left(\sum_1^9 W\right) x + \frac{1}{2} wb^2\right] \frac{a}{l} - \sum_1^4 M \tag{29}$$

A study of Eqs. (28) and (29) shows that, in order to calculate moments at any point in a beam due to a system of concentrated loads, there is required the sum of the moments of all loads about the last wheel on the beam, as indicated by $\sum_1^n M$; the sum of all loads up to and including the last load on the beam, as indicated by $\sum_1^n W$; and the negative moment of all loads in front of the given moment center, as indicated by $\sum_1^4 M$. It is evident that such calculations are likely to be long and tedious, especially for the locomotive loading of Fig. 126. However, by means of certain tables which will be given in the following article, the work of calculation may be greatly simplified.

67c. Moment Tables.—Tabular forms suitable for the moment and reaction calculations described in the preceding article are known as *Moment Tables.* Many forms for these tables have been devised and have proved very satisfactory. Two such tables are given in Tables 1 and 2, pp. 103 and 104. In Table 1 the train load is taken as a uniform load, while in Table 2 the train load is assumed to consist of a series of concentrated loads 10 ft. apart, each load being equal to the total load on 10 ft. of track. Other differences in arrangement will be brought out in the discussion given below.

The principal items of Table 1 are shown in condensed form in Fig. 128 in order to simplify the explanation of the arrangement of the table. In horizontal line *a* of Fig. 128 the load on each wheel is given. Line *b* contains a summation of all loads from wheel 1 to any given

a	Loads	12 5	25	25	25	25	16 25	16 25	16 25	16 25	2500 lb per ft
b	Σ Loads	12 5	37 5	62 5	87 5	112 5	128 75	145	161 25	177 5	
c	Spacing	⊢ 8 –	5	5	5 –	9	5	– 6	– 5	– – 5 →	Uniform Train Load
d	Wheel No	①	②	③	④	⑤	⑥	⑦	⑧	⑨	
e	Σ Distance	0	8	13	18	23	32	37	43	48	53
f	Σ Distance	53	45	40	35	30	21	16	10	5	0
g	Σ Loads	177 5	165	140	115	90	65	48 75	32 5	16 25	0
h	Σ Moments	5257 5	4595	3470	2470	1595	845	503 75	243 75	81 25	0
i	Σ Distance	48	40	35	30	25	16	11	5	0	
j	Σ Loads	177 5	165	140	115	90	65	48 75	32 5	16 25 ⑨	
k	Σ Moments	4370	3770	2770	1895	1145	520	260	81 25	0	

l	Σ Distance	32	24	19	14	9	0 | 0	5	11	16
m	Σ Loads	128 75	116 25	91 25	66 25	41 25	16 25 ⑥ 16 25	32 5	48 75	65
n	Σ Moments	2050	1650	1050	575	225	0 | 0	81 25	260	520

FIG. 128.

wheel. Thus in line *b* above wheel 5 we find 112.5, which is the sum of the loads on wheels 1 to 5 inclusive. According to the notation of Art. 67*b*, this value is denoted by $\sum_1^5 W = 112.5$. The spacing of the wheels in feet is given in line *c*. In line *d* the numbers in circles are used as a convenient means of designating the wheels. The distance from wheel 1 to any wheel is given in line *e*. Thus under wheel 6 in line *e* we find 32, which is the distance from wheel 1 to wheel 6. In lines *f*, *i*, and *l*, the distances are measured from right to left. Thus line *f* gives distances from the head of the uniform load to any wheel; line *i* gives distances from wheel 9 to any other wheel; and line *l* gives distances from wheel 6 to any other wheel. For example: To find the distance from the head of the uniform load to wheel 4, refer to line *f* and under wheel 4 read 35, which is the required distance. Again, to find the distance from wheel 6 to wheel 2, refer to line *l*, and directly under wheel 2 read 24, which is the required distance

Load summations for several groups of wheels are given in lines g, j, and m. In line g the load summations are given from right to left. Thus in line g under wheel 3 we find 140, which is the sum of wheels 9 to 3 inclusive. According to the notation of Art. **67***b*, we have $\sum_{3}^{9} W = 140$. Similar information is given in line j. In line m, the summations start at wheel 6 and move to the left. Thus in line m under wheel 3 we find 91.25, which is the sum of the loads on wheels 6 to 3 inclusive, or $\sum_{3}^{6} W = 91.25$.

Moment summations, proceeding from right to left, are given in lines h, k, and n. These summations are made by taking the moment about any wheel of all loads to the left of that wheel. In line h, moments are taken about the head of the uniform load. Thus wheel 9, a load of 16.25 is 5 ft. from the head of the uniform load. Hence the required moment is $(16.25)(5) = 81.25$, the value which appears in line h under wheel 9. The moment of wheels 9 to 5 about the head of the uniform load is given in line h under wheel 5 as 1,595. This moment is determined as follows:

Wheel	Load	Distance to head of uniform load	Moment	Moment
9	16.25	5	81 25	81 25
8	16 25	10	162 50	243 75
7	16 25	16	260 00	503 75
6	16 25	21	341 25	845 00
5	25 00	30	750 00	1,595 00
Total moment	.....	..	1,595 00	

Note that the sum of the moments for wheels 9 and 8 is $81.25 + 162.5 = 243.75$, the value which appears under wheel 8; the sum for 9, 8, and 7 is $81.25 + 162.5 + 260 = 503.75$, the value which appears under wheel 7. The last column of the above tabulation gives these successive summations. By an extension of this tabular form to include all wheels of the locomotive, the table of Fig. 128, or Table 1 can readily be calculated.

Lines k and n of Fig. 128 give moment summations beginning with wheels 9 and 6 respectively. Thus the moment of wheels 2 to 6 about wheel 6 is given in line n under wheel 2 as 1,650. To express this moment in terms of the notation of Art. **67***b*, we write $\sum_{2}^{6} M = 1{,}650$.

Table 1 is more complete than the condensed form shown in Fig. 128. All wheels of both locomotives are shown. The table is so arranged that

values for one or both engines can be determined. Problems in the use of Table 1 are given at the end of this article.

Since it is sometimes convenient to have moments for loads to the right of any wheel, as for example in calculating floorbeam loads, these moments have also been given in Table 1. Thus lines l, m, and n of Fig. 128 give respectively, summations of distances, loads, and moments for the several wheels to the right of wheel 6. Table 1 contains similar values for all wheels.

Table 2 contains the same information as given in Table 1, although the arrangement of the two tables is somewhat different. Figure 129 gives in condensed form the principal items of Table 2. Lines a, b, and c contain, respectively, the amounts of the several loads; the spacing

a	Loads	12.5	25	25	25	25	16 25	16 25	16 25	16.25	25	25
b	Spacing	8′	5′	5	5′	9′	5′	6′	5′	10′	10′	
c	Wheel No	1	2	3	4	5	6	7	8	9		
d	ΣDistance	0	8	13	18	23	32	37	43	48	58	68
e	ΣLoads	12 5	37 5	62 5	87 5	112 5	128 75	145	161.25	177.5	202 5	227.5
f	ΣMoments	0	100	287 5	600	1037 5	2050	2693 75	3563 75	4370	6145	8170
g	ΣDistance		0	5	10	15	24	29	35	40	50	60
h	ΣLoads		(2) 25	50	75	100	116 25	132 5	148 75	165	190	215
i	ΣMoments		0	125	375	750	1650	2231.25	3026.25	3770	5420	7320

Fig. 129.

between loads; and the numbers to be used in designating the several wheels. Line d gives the summation of distances from the first wheel to any given wheel. Thus in line d under wheel 7 read 37, which is the distance from wheel 1 to wheel 7. Line e gives summations of loads from wheel 1 up to and including any given wheel. Thus under wheel 7 in line e read 145, which is the sum of loads 1 to 7 inclusive. Line f gives the summation of moments for all wheels to the left of a given wheel. Thus, wheel 1, a load of 12.5, is 8 ft. from wheel 2. The moment of wheel 1 about wheel 2 is then $(12.5)(8) = 100$. This value is given in line f under wheel 2. Again, the moment of wheels 1, 2, 3 and 4 about wheel 5 is as follows:

Wheel	Load	Distance to wheel 5	Moment
1	12.5	23	287.5
2	25.0	15	375 0
3	25.0	10	250.0
4	25.0	5	125.0
Total moment........	....	..	1,037 5

This value is given in line f under wheel 5.

Moment values given in Table 2 may be calculated by the method given above, adding on the moments for the successive wheels. However, a somewhat better method of calculation is offered by Eq. (27). The following tabulation shows how the work may readily be carried out.

a Wheel	*b* Total load (W)	*c* Distance to preceding wheel (x)	*d* Increment (Wx)	*e* Moment $M = M + Wx$
1	0 0	0	0 0	0.0
2	12 5	8	100 0	100 0
3	37 5	5	187 5	287 5
4	62 5	5	312 5	600 0
5	87 5	5	437 5	1,037 5

Moments in column *e* are obtained by adding the moments of column *d* to the total of the preceding moments. Thus for wheel 4, add the increment 312.5 of column *d* to 287.5, the moment for wheel 3, as given in column *e*. The total is 600, the desired moment for wheel 4.

Since in certain loading conditions wheels may pass off the structure as the loads are moved to the left, summations of loads, distances, and moments are given omitting the leading wheel. Thus in lines *g*, *h*, and *i* values are given considering wheel 2 as the leading wheel. In Table 2 successive wheels have been omitted as far as wheel 12 of the second locomotive.

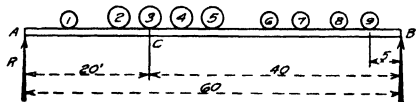



FIG 130

The choice of moment tables depends largely on the personal opinion of the calculator. Of the two tables given here, Table 1 is used to a greater extent than Table 2, although in many ways, Table 2 is the more convenient. Illustrative problems follow showing the use of the moment tables.

Illustrative Problem.—Calculate the left reaction and the moment under wheel 3 for the loads shown in Fig. 130.

The left reaction is $R_1 = \frac{M_B}{60}$ where M_B is the moment of all loads about the right end of the beam. From Eq. (27) with values of n for the 9th wheel and $x = 5$ ft., as shown in Fig. 130, the general form of the value of M is

$$M_B = \sum_1^9 M + \left(\sum_1^9 W\right) x$$

From the moment tables, $\sum_1^9 M = 4{,}370$ and $\sum_1^9 W = 177.5$. To find these values in Table 1, follow vertically downward from wheel 9 of the locomotive shown in the upper right-hand corner of the diagram until the circle containing the number 9 is reached. Then follow horizontally to the left to the column vertically under wheel 1 of the same locomotive. In the line Σ moments read 4,370, and in the line Σ loads read 177.5. In Table 2, these values will be found under wheel 9 for the lines which give the summations for wheels 1 to 9.

With these values of $\sum_1^9 M$ and $\sum_1^9 W$ we have,

$$M_B = \sum_1^9 M + \left(\sum_1^9 W\right)x = 4{,}370 + (177.5)(5) = 5{,}257.5$$

or

$$M_B = 5{,}257{,}500 \text{ ft.-lb.}$$

(Note that the values given in the tables are in thousand pound units.) Hence

$$R_1 = \frac{5{,}257{,}500}{60} = 87{,}625 \text{ lb.}$$

The moment at C is $M_c = (R_1)(20) - \sum_1^3 M$, where $\sum_1^3 M$ is the moment of wheels 1 and 2 about 3, which from the tables, is 287,500 ft.-lb. This value is found as described above for $\sum_1^9 M$. We then have

$$M_c = (87{,}625)(20) - 287{,}500 = 1{,}465{,}000 \text{ ft.-lb.}$$

In some cases the value of the reaction is not required. The calculation of moment can then be made without first calculating R_1 as follows:

$$M_c = \frac{20}{60} M_B - \sum_1^3 M = \frac{(5{,}257{,}500)(20)}{60} - 287{,}500 = 1{,}465{,}000 \text{ ft.-lb.}$$

Illustrative Problem.—Calculate the left reaction and the moment under wheel 6 for the loads shown in Fig. 131.

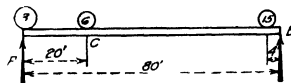



Fig. 131.

Note that summations of loads and moments are required for wheels 3 to 15 inclusive. Hence $\sum_3^{15} W = 268.75$, and $\sum_3^{15} M = 10{,}420$. Also $x = 4$ ft. Then from Eq. (27),

$$M_B = 10{,}420 + (268.75)(4) = 11{,}495$$

Therefore,

$$R_1 = \frac{M_B}{80} = \frac{11{,}495{,}000}{80} = 143{,}700 \text{ lb.}$$

Now

$$M_c = \frac{(M_B)(20)}{80} - \sum_3^6 M$$

From the moment tables

$$\sum_{3}^{6} M = 1{,}050{,}000$$

Therefore

$$M = \frac{(11{,}495{,}000)(20)}{80} - 1{,}050{,}000 = 1{,}823{,}750 \text{ ft.-lb.}$$

Illustrative Problem.—Calculate the moment under wheel 6 for the loads shown in Fig. 132.

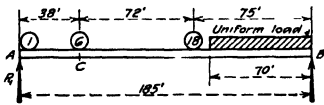



FIG. 132.

In problems of this kind the results given by Tables 1 and 2 may differ somewhat because of the assumptions made regarding the distribution of the uniform load.

Solution from Table 1.—For the conditions shown in Fig. 132, Eq. (29) may be written in the form

$$M_c = \left[\sum_{1}^{18} M + \left(\sum_{1}^{18} W\right) 75 + \tfrac{1}{2} \times 2.5 \times 70^2\right] {}^{38}\!/_{185} - \sum_{1}^{6} M$$

From Table 1 these terms have the values given in the following equation:

$$M_c = [18{,}680 + (355)(75) + (\tfrac{1}{2})(2.5)(70)^2]\ {}^{38}\!/_{185} - 2{,}050$$

From which finally

$$M_c = 8{,}514{,}000 \text{ ft.-lb.}$$

Solution from Table 2.—In using Table 2, where the uniform load is divided into concentrated loads, we must locate the last load on the structure. From Fig. 133 we note that wheel 18 is 75 ft. from the right end of the beam. The distance from wheel 1 to the right end of the span is therefore 104 + 75 = 179 ft. Referring to the line in Table 2 which gives the distance summations for all loads, we find that the right end of the span is 179 − 174 = 5 ft. to the right of load 25, which is shown in its proper position in Fig. 133.

FIG. 133.

Then from Eq. (29)

$$M_c = \left[\sum_{1}^{25} M + \left(\sum_{1}^{25} W\right) 5\right] {}^{38}\!/_{185} - \sum_{1}^{6} M$$

Selecting the required values from Table 2, we have

$$M_c = [48{,}780 + (530)(5)]\ {}^{38}\!/_{185} - 2{,}050$$

from which

$$M_c = 8{,}514{,}000 \text{ ft.-lb.}$$

TABLE 1.

WHEEL LOAD TABULATION — COOPER'S E50 LOADING

TWO 177½ TON CONSOLIDATION LOCOMOTIVES — UNIFORM TRAIN LOAD 5000 POUNDS PER FOOT

ALL LOADS AND MOMENTS ARE GIVEN FOR ONE RAIL.
DISTANCES ARE GIVEN IN FEET
LOADS ARE GIVEN IN THOUSANDS OF POUNDS
MOMENTS ARE GIVEN IN THOUSANDS OF FOOT POUNDS

Loads	12.5	25	25	25	25	16.25	16.25	16.25	16.25	2,500 Pounds per Ft
ΣLoads	12.5	37.5	62.5	87.5	112.5	128.75	145	161.25	177.5	
Spacing	← 8 →	5	5	5	9	5'	6	5	5'	→ Uniform Train Load
Wheel No.	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	
ΣDistance	0	8	13	18	23	32	37	43	48	53

Loads	12.5	25	25	25	25	16.25	16.25	16.25	16.25	12.5	25	25	25	25	16.25	16.25	16.25	16.25	2,500 Pounds per Ft	
ΣLoads	12.5	37.5	62.5	87.5	112.5	128.75	145	161.25	177.5	190	215	240	265	290	306.25	322.5	338.75	355		
Spacing	← 8 →	5	5	5	9	5	6	5	8	8	5	5	5	9	5'	6	5	5'	→ Uniform Train Load	
Wheel No	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)		
ΣDistance	0	8	13	18	23	32	37	43	48	56	64	69	74	79	88	93	99	104	109 →	ΣDistance
ΣDistance	109	101	96	91	86	77	72	66	61	53	45	40	35	30	21	16	10	5	0 ←	ΣDistance
ΣLoads	355	342.5	317.5	292.5	267.5	242.5	226.25	210	193.75	177.5	165	140	115	90	65	48.75	32.5	16.25	0	ΣLoads
ΣMoments	20,455	19,092.5	16,567.5	14,167.5	11,892.5	9742.5	8491.25	7321.25	6248.75	5257.5	4595	3470	2470	1,595	845	503.75	243.75	81.25	0	ΣMoments
ΣDistance	104	96	91	86	81	72	67	61	56	48	40	35	30	25	16	11	5	0 (18)	ΣDistance	
ΣLoads	355	342.5	317.5	292.5	267.5	242.5	226.25	210	193.75	177.5	165	140	115	90	65	48.75	32.5	16.25	ΣLoads	
ΣMoments	18,680	17,380	14,980	12,705	10,555	8530	7,360	6271.25	5280	4370	3770	2,770	1895	1,145	520	260	81.25	0 (9)	ΣMoments	
ΣDistance	99	91	86	81	76	67	62	56	51	43	35	30	25	20	11	6	0 (17) 0	5	ΣDistance	
ΣLoads	338.75	326.25	301.25	276.25	251.25	226.25	210	193.75	177.5	161.25	148.75	123.75	98.75	73.75	48.75	32.5	16.25 16.25	32.5	ΣLoads	
ΣMoments	16,986.25	15,748.75	13,473.75	11,323.75	9,298.75	7398.75	6,310	5,302.5	4392.5	3563.75	3,026.25	2,151.25	1,401.25	776.25	276.25	97.5	0 (8) 0	81.25	ΣMoments	
ΣDistance	93	85	80	75	70	61	56	50	45	37	29	24	19	14	5	0 (16) 0	6	11	ΣDistance	
ΣLoads	322.5	310	285	260	235	210	193.75	177.5	161.25	145	132.5	107.5	82.5	57.5	32.5	16.25 16.25	32.5	48.75	ΣLoads	
ΣMoments	15,051.25	13,888.75	11,763.75	9763.75	7,888.75	6,138.75	5,147.5	4,237.5	3425	2,693.75	2,231.25	1506.25	906.25	431.25	81.25	0 (7) 0	97.5	276.25	ΣMoments	
ΣDistance	88	80	75	70	65	56	51	45	40	32	24	19	14	9	0 (15) 0	5	11	16	ΣDistance	
ΣLoads	306.25	293.75	268.75	243.75	218.75	193.75	177.5	161.25	145	128.75	116.25	91.25	66.25	41.25	16.25 16.25	32.5	48.75	65	ΣLoads	
ΣMoments	13,520	12,420	10420	8545	6,795	5,170	4260	3,431.25	2700	2050	1650	1050	575	225	0 (6) 0	81.25	260	520	ΣMoments	
ΣDistance	79	71	66	61	56	47	42	36	31	23	15	10	5	0 (14) 0	9	14	20	25	ΣDistance	
ΣLoads	290	277.5	252.5	227.5	202.5	177.5	161.25	145	128.75	112.5	100	75	50	25 25	41.25	57.5	73.75	90	ΣLoads	
ΣMoments	10,910	9922.5	8147.5	6,497.5	4972.5	3572.5	2,808.75	2,126.25	1,541.25	1037.5	750	375	125	0 (5) 0	146.25	373.75	698.75	1,105	ΣMoments	
ΣDistance	74	66	61	56	51	42	37	31	26	18	10	5	0 (13) 0	5	14	19	25	30	ΣDistance	
ΣLoads	265	252.5	227.5	202.5	177.5	152.5	136.25	120	103.75	87.5	75	50	25 25	50	66.25	82.5	98.75	115	ΣLoads	
ΣMoments	9,585	8660	7,010	5485	4085	2810	2,127.5	1526.25	1022.5	600	375	125	0 (4) 0	125	352.5	661.25	1067.5	1555	ΣMoments	
ΣDistance	69	61	56	51	46	37	32	26	21	13	5	0 (12) 0	5	10	19	24	30	35	ΣDistance	
ΣLoads	240	227.5	202.5	177.5	152.5	127.5	111.25	95	78.75	62.5	50	25 25	50	75	91.25	107.5	123.75	140	ΣLoads	
ΣMoments	8,385	7,522.5	5997.5	4597.5	3322.5	2172.5	1,571.25	1051.25	628.75	287.5	125	0 (3) 0	125	375	683.75	1073.75	1561.25	2130	ΣMoments	
ΣDistance	64	56	51	46	41	32	27	21	16	8	0 (11) 0	5	10	15	24	29	35	40	ΣDistance	
ΣLoads	215	202.5	177.5	152.5	127.5	102.5	86.25	70	53.75	37.5	25 25	50	75	100	116.25	132.5	148.75	165	ΣLoads	
ΣMoments	7,310	6510	5110	3835	2685	1660	1,140	701.25	360	100	0 (2) 0	125	375	750	1,140	1611.25	2180	2830	ΣMoments	
ΣDistance	56	48	43	38	33	24	19	13	8	0 (10) 0	8	13	18	23	32	37	43	48	ΣDistance	
ΣLoads	190	177.5	152.5	127.5	102.5	77.5	61.25	45	28.75	12.5 12.5	37.5	62.5	87.5	112.5	128.75	145	161.25	177.5	ΣLoads	
ΣMoments	5,790	5090	3890	2815	1865	1040	650	341.25	130	0 (1) 0	200	525	975	1,550	2070	2671.25	3370	4150	ΣMoments	

68. Moments and Shears Due to Moving Concentrated Load Systems.

68a. Position of Loads for Maximum Effect.—Every possible position of a system of concentrated loads will cause stress, moment, or shear in the members of a truss or at the sections of a beam or girder. However, there is some position of the applied loads which will cause a stress, moment, or shear which is greater than that for any other position of the loads. It is this maximum value which is required in the design of a structure.

The required maximum value of stress, moment, or shear is determined by successive approximations—that is, many positions of the applied loads are assumed and the desired value calculated for each load position. By comparing these calculated values the desired maximum may be determined. This process is generally long and tedious, for all possible combinations of loading must be tried in order to make certain that the true maximum value has been determined.

To avoid the excessive work required by the method of successive approximations, certain formulas are readily derived in terms of the applied loads and the dimensions of the structure by means of which the correct position of loads for maximum stress, moment, or shear is readily determined. Such a formula is known as a criterion for position of loads for maximum moment, shear, or stress, as the case may be. In the following articles these criteria will be derived for the more important cases in general use.

68b. Position of Loads for Maximum Moment at Any Point in a Simple Beam without Floorbeams.—Maximum moment at any point in a simple beam due to a set of moving concentrated loads occurs when some load is placed at the given moment center. This statement may be verified by a study of moment and shear diagrams for systems of concentrated loads, as for example, Fig. 92 of Art. **65**a. Note that the moment is a maximum under one of the loads, and note also that the shear passes through zero at the point of application of one of the loads. Again, from the discussion given in Art. **65**b, the moment is a maximum at a point of zero shear. Hence, the moment is a maximum at some load, and therefore to obtain maximum moment at a given point, a load must be placed at that point.

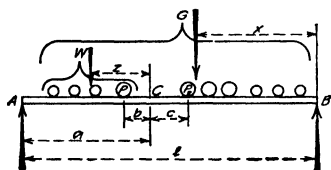



Fig. 134.

The above statement may also be proved by the following analysis. In Fig. 134, assume that a set of concentrated loads is placed so that the given moment center is located between two wheels, P_1 and P_2. Let G be the sum of the loads, and let x be the distance from the center of gravity of the loads to the right end of the beam. Also let W be the sum

of the loads in front of the moment center, and let z be the distance from the moment center to the center of gravity of these loads. For the conditions shown in Fig. 134,

$$M_c = G\frac{x}{l}a - Wz$$

Now assume that the loads are moved to the right until load P_1 reaches point C, assuming that the total load G is not changed during the movement. The distance x now becomes $x-b$ and distance z becomes $z-b$. Taking moments about C, we have

$$M_c = G\frac{(x-b)}{l}a - W(z-b)$$

Again, assume that the loads are moved to the left until P_2 reaches point C. The moment at C is then

$$M_c = G\frac{(x+c)}{l}a - W(z+c)$$

To determine the change in moment due to these movements of the loads, subtract the first of the above equations from each of the others. Letting D be the difference resulting from this subtraction, we have

$$D\begin{pmatrix}\text{movement to right}\\ P_1 \text{ at } C\end{pmatrix} = -\left(G\frac{a}{l} - W\right)b$$

and

$$D\begin{pmatrix}\text{movement to left}\\ P_2 \text{ at } C\end{pmatrix} = +\left(G\frac{a}{l} - W\right)c$$

From these equations we note that the character of the change in moment depends upon the relative values of the terms $G\frac{a}{l}$ and W. Assuming that the first term is greater than the second, it is evident that the moment at C is greatest when P_2 is at the point. If W is greater than $G\frac{a}{l}$, then P_1 at C gives the greatest moment. In any case, the moment at C is greater when there is a load at that point than it is when the loads are located one on each side of point C, as shown in Fig. 134. The statement is therefore proved.

To determine the proper position for a given system of concentrated loads in order that a maximum or minimum moment at a point will result, consider the conditions shown in Fig. 135. Let G represent the total load on the beam, and let W be the load to the left of the moment center, which is point C, at a distance a from the left end of the beam. Also let x represent the distance from the right end of the beam to the center of gravity of all the loads,

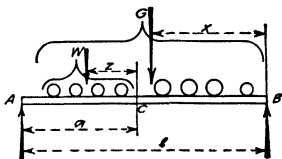



Fig. 135.

and let z be the distance from the moment center to the center of gravity of the loads W. For the conditions shown in Fig. 135

$$M_c = G\frac{x}{l}a - Wz \tag{30}$$

Now assume that the entire group of loads is moved a distance Δ to the left. During this movement of the loads let it be assumed that the load groups G and W are not changed due to loads passing points A, B, or C. For this new position of the loads,

$$M_c' = G\frac{(x+\Delta)}{l}a - W(z+\Delta) \tag{31}$$

The difference in moment caused by the change in position of the loads, which will be denoted by m, is found by subtracting Eq. (30) from Eq. (31), from which

$$m = G\frac{a}{l}\Delta - W\Delta = \left(G\frac{a}{l} - W\right)\Delta \tag{32}$$

The expression $\left(G\frac{a}{l} - W\right)$ of Eq. (32) gives the change in moment due to a movement of the loads in terms of the loading conditions, the distance from the moment center to the left end of the beam, and the total span length.

If the term $\left(G\frac{a}{l} - W\right)$ of Eq. (32) is a positive quantity for any given position of the loads, the moment at point C, Fig. 135, is increased as the loads are moved to the left. Suppose the loads are moved to a new position and the proper values substituted in Eq. (32). If $\left(G\frac{a}{l} - W\right)$ is found to be a negative quantity, the moment at C is decreasing in value for the new position of the loads. It is evident, therefore, that if a movement of the loads to the left causes an increase in moment, and if further movement of the loads finally causes a decrease in moment, there must be some position of the loads for which the moment is neither increasing or decreasing—that is, the moment has reached a maximum value. This condition may be indicated by placing $m = 0$ in Eq. (32). Hence the relation between loads and distances for a maximum moment is given by the expression

$$G\frac{a}{l} - W = 0 \tag{33}$$

Equation (33) is known as the *criterion* for position of loads for maximum moment at a point distance a from the left end of a beam of span l. Equation (33) may also be written in the form

$$\frac{G}{l} = \frac{W}{a} \tag{34}$$

Since G is the total load on the span and l is the span length, $\frac{G}{l}$ is the average load on the span per unit of length, and since W is the load to the left of the moment center, and a is the distance to the left end of the beam, $\frac{W}{a}$ is the average load per unit of length to the left of the moment center. Hence for maximum moment, *the average load per unit of length on the whole span must be equal to the average load per unit of length to the left of the moment center.*

The conditions stated in Eqs. (33) and (34) can be satisfied definitely by a uniform load. It is quite evident that, for a beam carrying a uniform load, the average load on the whole beam and the average load to the left of the moment center are equal when the beam is completely loaded. This has already been shown (Art. **66**b, p. 88) to be the required position for maximum moment due to a uniform load.

In the case of a system of concentrated loads an exact equality cannot in general be obtained by substitution in Eq. (33). However, it is possible to find some position of the loads for which $G\frac{a}{l}$ is greater than W at the beginning of the movement, and less than W at the end of the movement. While $G\frac{a}{l} - W$ is not equal to zero, it has passed through the zero value during the movement of the loads.

The discussion given in the first part of this article proved that for maximum moment due to a system of concentrated loads, one of the loads must be placed at the moment center. Suppose a system of loads has been placed on a beam with one of the loads, which we will call P, placed a very small distance to the right of the moment center. For this position of the loads assume that $G\frac{a}{l} - W$ of Eq. (33) is a positive quantity—that is, $G\frac{a}{l} > W$. In substituting in Eq. (33) note that load P is not to be included in the load W, for P is located to the right of the moment center. If the loads are moved a short distance to the left so that load P crosses the moment center, the value of W has been increased by the load P, while the total load G remains unchanged, assuming that during the forward movement no loads have passed off the left end of the beam or no loads have come on the beam from the right. It is evident that by this means, W may become greater than $G\frac{a}{l}$ or, the sign of Eq. (33) may be changed from positive to negative as load P crosses the moment center. As stated above, this is the required condition for a maximum moment.

To show that Eq. (33) may be made to pass from a positive to a negative value only when a load passes the moment center, assume a load is located just to the right of point B, Fig. 135, and assume that for this position of the loads $G\frac{a}{l}$ is greater than W. Now assume that the load in question passes point B, no loads passing points A and C. The only effect of this movement on the loads will be to increase G. Since $G\frac{a}{l}$ is greater than W at the beginning of the movement it can readily be seen that it is also greater at the end of the movement, and that $G\frac{a}{l} - W$ will not change sign. Now assume that a load is located just to the right of point A, Fig. 135, and assume also that for the given loads, $G\frac{a}{l}$ is greater than W. Assume further that the loads are moved to the left so that the load in question crosses point A. Both G and W are changed by this movement, but it is evident that $G\frac{a}{l}$ is not changed by the full amount of the load P (the load which crosses point A), while W is changed by the full amount of P. Hence the sign of $G\frac{a}{l} - W$ will remain unchanged. Therefore, a maximum moment will not occur for a load at points A or B of Fig. 135.

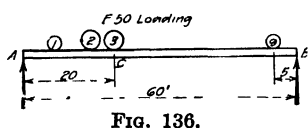
The above discussion has been confined to the determination of the position of loads for positive moment at a given point. By an analysis similar to that given above, it can be shown that for a minimum moment at a given point, a load must be placed at A or B, of Fig. 135, or loads may be placed at both points, no load at the moment center. If loads placed in this position satisfy Eq. (33), a minimum moment will result. However, the calculator is generally interested only in the maximum moment at a given point, and minimum moments are not in question. Therefore, no further attention will be paid to minimum moments.

Fig. 136.

Illustrative Problem.—Determine by means of Eq. (33) whether or not the loads shown in Fig. 136 produce a condition for maximum moment at point C.

For the conditions shown in Fig. 136, where $a = 20$ ft., and $l = 60$ ft., the criterion of Eq. (33) becomes

$$G\frac{a}{l} - W = \frac{G}{3} - W = 0$$

The total load on the span, as shown in Fig. 136, is composed of wheel loads 1 to 9, or $G = \sum_1^9 W$. From Tables 1 or 2, $\sum_1^9 W = 177.5$. Therefore, $G\frac{a}{l} = \frac{G}{3} = 58.2$

The value of W depends upon the position of wheel 3. If wheel 3 is placed to the right of point C, W = sum of loads 1 and 2 $= \sum_1^2 W = 37.5$; and, when W is placed to the left of C, W = sum of loads 1, 2, and 3 $= \sum_1^3 W = 62.5$. Two substitutions must be made in the criterion, as follows:

Wheel 3 to right of C,

$$G\frac{a}{l} - W = 58.2 - 37.5 = +$$

Wheel 3 to left of C,

$$G\frac{a}{l} - W = 58.2 - 62.5 = -$$

Note that the actual values of the remainders from the subtractions are not given, but that only the *sign* of the result is indicated. This is done because the sign of the remainder is important while the amount of the remainder is of no particular significance.

Since the sign of the remainder in the above substitution is positive when wheel 3 is placed to the right of point C and negative when wheel 3 is placed to the left of C, wheel 3 satisfies the condition for maximum moment at point C of Fig. 136.

FIG. 137.

Illustrative Problem.—Does wheel 4 of the loads shown in Fig. 137 answer the conditions for maximum moment at point C?

For the conditions shown in Fig. 137, the criterion is written

$$G\frac{a}{l} - W = \frac{(G)(20)}{50} - W = 0.4G - W = 0$$

In this case the total load G will depend upon the position of wheel 4. If wheel 4 is assumed as placed to the right of point C, then also will wheel 9 be to the right of point B, and therefore is not to be included in the total load G. When wheel 4 is placed to the left of point C, then also will wheel 9 be to the left of point B, and therefore is a part of the total load G.

The substitutions in the criterion are as follows:

Wheel 4 to right of C,

$$G = \sum_1^8 W = 161.25$$

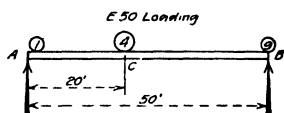
$$W = \sum_1^3 W = 62.5$$

$$0.4G - W = (0.4)(161.25) - 62.5 = 64.5 - 62.5 = +$$

Wheel 4 to left of C,

$$G = \sum_1^9 W = 177.5$$

$$W = \sum_1^4 W = 87.5$$

$$0.4G - W = (0.4)(177.5) - 87.5 = 71.0 - 87.5 = -$$

Since a change of sign occurred as wheel 4 passed point C, the condition for a maximum moment is satisfied by wheel 4.

Illustrative Problem.—Does wheel 5 of the loads shown in Fig. 138 answer the conditions for maximum moment at point C?

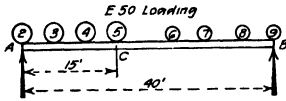



FIG. 138.

Note that the conditions at the right end of the beam are the same as in Prob. 2. At the left end of the beam, note that when wheel 5 is to the right of C, wheel 2 is on the beam, and when wheel 5 is to the left of C, wheel 2 is not on the beam. With these hints the reader can readily verify the following calculations.

The criterion for the case shown in Fig. 138 is $\frac{3}{8}G - W = 0$.

Wheel 5 to right of C,

$$G = \sum_{2}^{8} W = 148.75$$

$$W = \sum_{2}^{4} W = 75$$

$$\tfrac{3}{8}G - W = 55.8 - 75 = -$$

Wheel 5 to left of C,

$$G = \sum_{3}^{9} W = 140$$

$$W = \sum_{3}^{5} W = 75$$

$$\tfrac{3}{8}G - W = 52.5 - 75 = -$$

Wheel 5 does not satisfy the conditions for maximum moment.

Illustrative Problem.—Does the loading condition shown in Fig. 139 satisfy the conditions for maximum moment at point C?

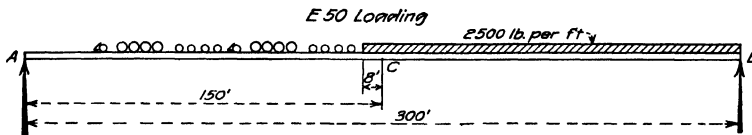



FIG. 139.

The form of the criterion for this case is $\frac{1}{2}G - W = 0$. For the conditions shown in Fig. 139, $G = 355 + (158)(2.5) = 750$, and $W = 355 + (8)(2.5) = 375$. Hence

$$\tfrac{1}{2}G - W = (\tfrac{1}{2})(750) - 375 = 0$$

The loading conditions shown in Fig. 139 satisfy the conditions for maximum moment.

68c. Determination of Maximum Moment at Any Point in a Beam.—The determination of the maximum moment at any point in a beam may be divided into two parts: (1) The determination of the wheel or wheels which satisfy the criterion for maximum moment at the given point, and (2) the determination of the maximum moment.

The wheel which answers the condition for maximum moment must be determined by trial. For engine loadings, such as Cooper's loadings,

the maximum moment at any point usually occurs under one of the heavy drive wheels. Also, it is found that practically the whole beam must be covered by the loads. In the case of two equal loads, such as the other loading shown in the typical diagram of Fig. 126, p. 96, it will be found that one of the loads is to be placed at the moment center while the other load is to be placed on the longer of the two segments into which the beam is divided by the moment center. For two unequal loads—as, for example, a road roller—a similar position of loads is to be used, except that the heavier wheel is to be placed at the moment

Note For greatest bending moment under uniform load, the uniform load should begin 142 feet from left end of span

The greatest bending moment in thousands of foot pounds for load per rail $=k[x(l-x)+3800(\frac{l-x}{l})]$, x being greater than 142 feet For E-40 loading, k=1, for E-50, k=1.25; and for E-60, k=1.5.

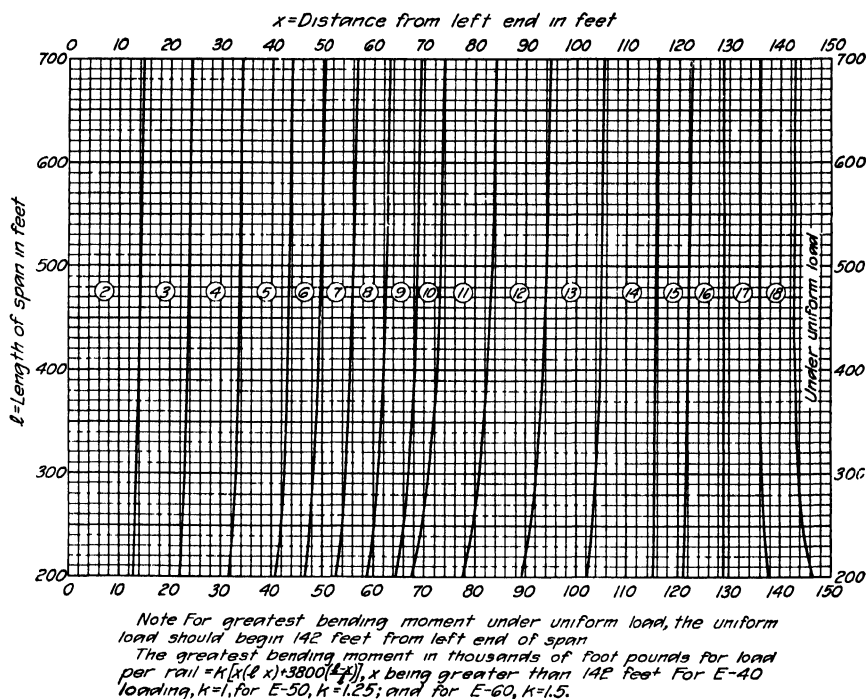



Fig 140.

center. For these simple cases, it is generally not necessary to have recourse to the criterion.

To determine which of a group of wheels answers for moment at a given point, place the loads on the structure so that it is practically covered by the loads, subject to the condition that one of the heavy loads shall be at the moment center. On substituting the proper values in the criterion of Eq. (33), it is possible to determine whether or not this wheel answers for maximum moment, as illustrated by the problems at the end of the preceding article. If the substitutions in the criterion result in two positive signs, the average load to the left of the moment center is less than the average load on the whole span. To equalize

Note: The greatest bending moment at any point in the shaded areas is less than that at the point on the opposite side of the center of the span, equi-distant from the center.
Distance from left end of span in feet
Length of span in feet
Locus of ends of spans
Locus of centers of spans

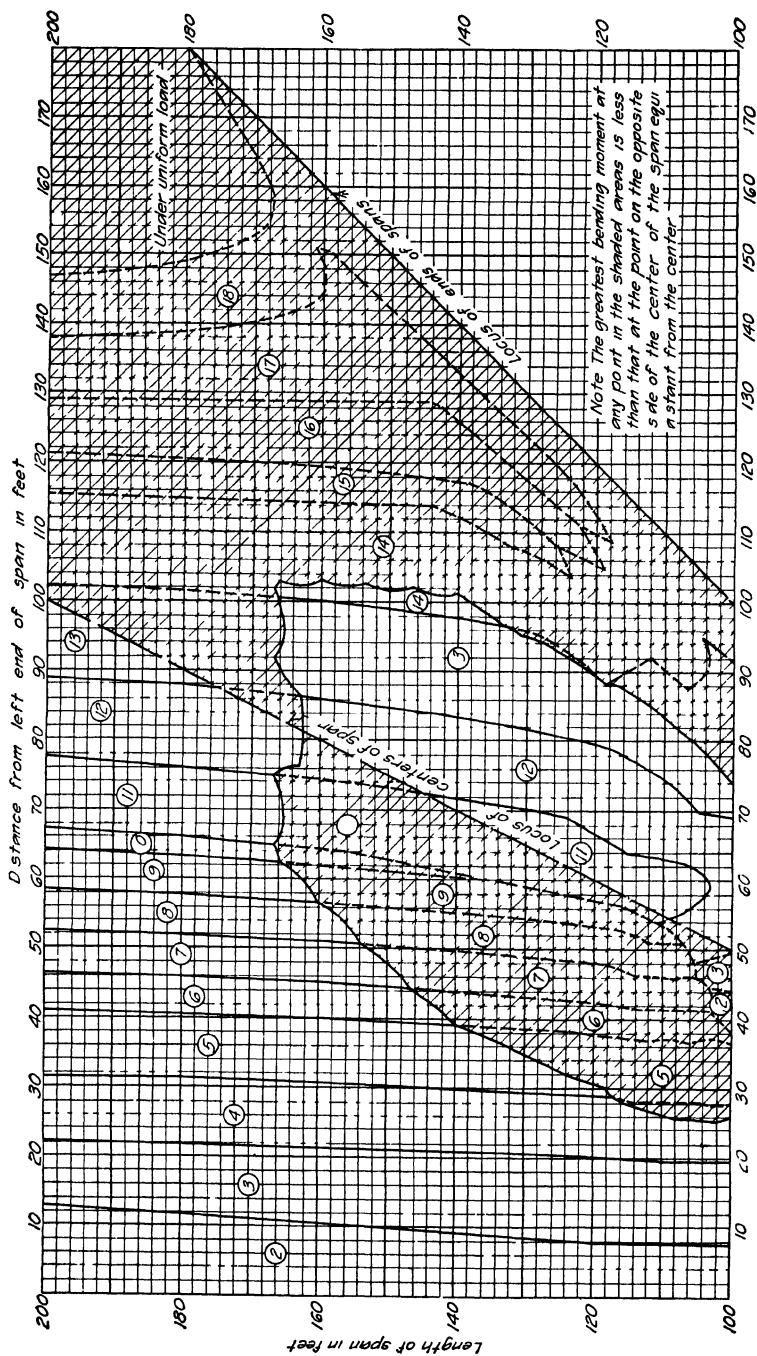
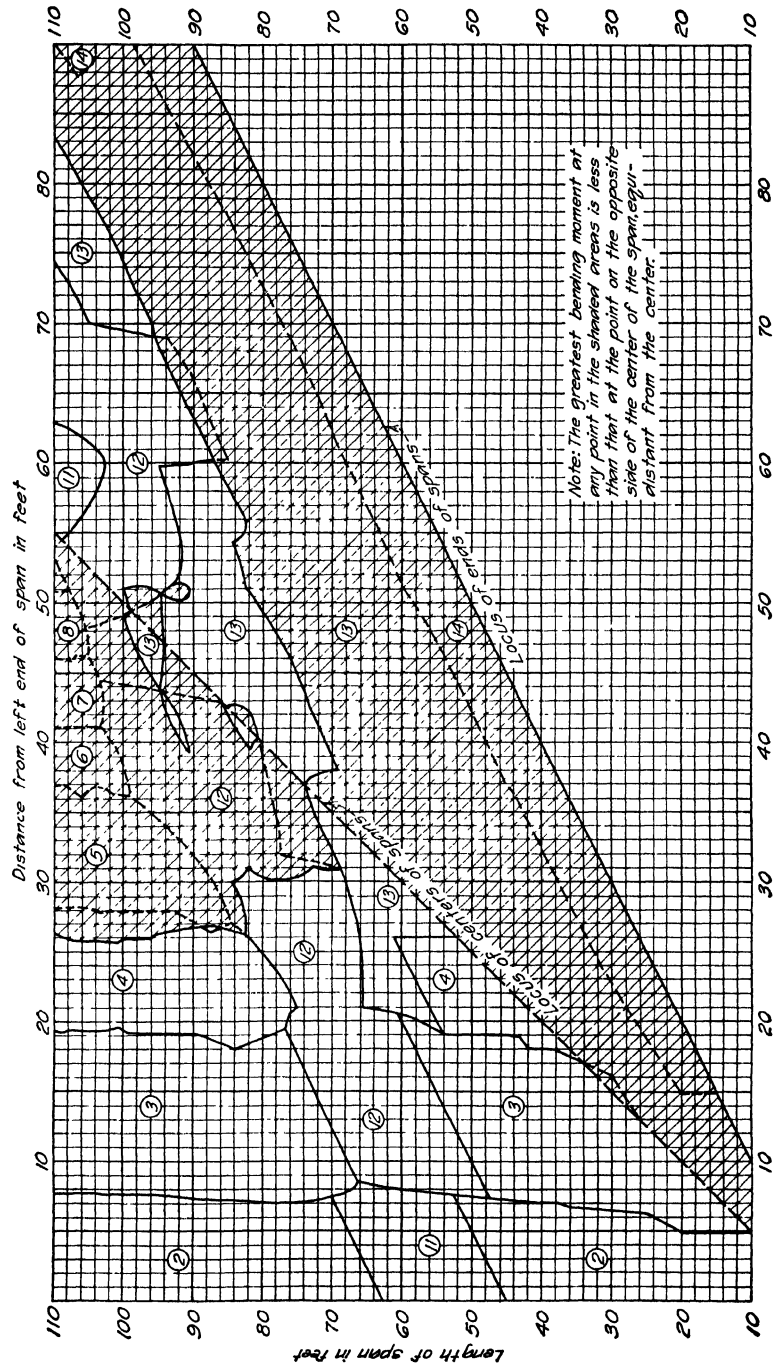



FIG. 142.

conditions, the loads must be moved further to the left until some wheel is reached which will satisfy the criterion. If in any case substitution in the criterion results in two minus signs, a movement of the loads to the right is necessary before a wheel can be found which will answer the criterion.

In some cases it will be found that several consecutive wheels will answer the criterion for maximum moment at a given point. To make certain that all possible wheels are discovered under which a maximum moment may occur, it is best to carry the work far enough to include all wheels which do answer and also locate a wheel on each side of this group for which the criterion is not satisfied. When several wheels satisfy the criterion for maximum moment, the maximum value is determined by calculating the moment under the several wheels. On comparing the calculated values, the maximum is definitely determined.

The determination of the wheel which answers the criterion for moment is greatly facilitated by the use of the diagrams given in Figs. 140, 141 and 142.[1] The use of these diagrams will be illustrated by the solution of the following problem: Determine the wheel which gives maximum moment at the 20-ft. point in a beam 100 ft. long. Refer to Fig. 142 and find the 100-ft. span indicated on the vertical scale. On the horizontal scale locate the 20-ft. point. Through the points thus located project horizontal and vertical lines to an intersection. Note that these lines intersect in an area marked by the number four. This indicates that wheel 4 gives maximum moment at the 20-ft. point in a 100-ft. beam.

While these diagrams were calculated for the load system of Table 1, p. 105, it will be found that they apply also to the load system of Table 2. From these diagrams it is possible to determine by inspection the wheel which gives maximum moment at any point in spans 10 ft. to 700 ft.

A study of the diagrams of Figs. 140 to 142, inclusive, shows that for spans from about 70 to 165 ft., the moment at points between the quarter-point and the center of the beam is greater for points to the right of the beam center than for the corresponding points to the left of the center. Also, for certain spans between 45 and 85 ft. in length, moments at points near the ends of the beams are greatest when the wheels of the second locomotive reach the point in question.

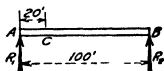



FIG. 143.

Illustrative Problem.—Calculate the maximum moment at point C of the beam of Fig. 143 for Cooper's E-50 loading.

Referring to Fig. 142 we find that wheel 4 should be placed at point C to obtain maximum moment. If wheel 4 is at point C, wheel 16 will be 5 ft. from the right end of the beam. From either Table 1 or Table 2

[1] These diagrams were devised by Prof. W. M. Scheurman, Professor of Civil Engineering, Vanderbilt University, and were published in part 3, vol. 12, *Proceedings* of the American Railway Engineering Association (*Bull.* 128).

$$M_B = \sum_1^{16} M + \left(\sum_1^{16} W\right)5$$

$$= 15{,}051.25 + (322.5)(5) = 16{,}663.75$$

$$R_1 = \frac{M_B}{100} = 166.6375$$

$$M_c = (R_1)(20) - \sum_1^4 M$$

$$= (166.6375)(20) - 600 = 2{,}732.75$$

or moment at c = 2,732,750 ft.-lb.

Illustrative Problem.—Calculate the maximum moment at point C of the beam shown in Fig. 144 due to Cooper's E-50 loading.

Using Table 1.—Referring to Fig. 140 we find that the greatest bending moment occurs under the uniform load and also that the uniform load should begin 142 ft. from the left end of the span or 158 ft. from the right end of the span. From Table 1

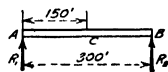



Fig. 144.

$$M_B = 20{,}455 + (158)(355) + \frac{wl^2}{2}$$

$$= 20{,}455 + (158)(355) + \frac{2.5(158)^2}{2} = 107{,}750$$

$$R_1 = \frac{M_B}{300} = 359.17$$

$$M_c = (R_1)(150) - \left[20{,}455 + (8)(355) + \frac{wl^2}{2}\right]$$

$$= (359.17)(150) - \left[20{,}455 + (8)(355) + \frac{2.5(8)^2}{2}\right]$$

$$= 30{,}500.5$$

or

Moment at C = 30,500,500 ft.-lb.

Using Table 2.—In Table 2 the uniform train load has been divided into concentrated loads of 25,000 lb. Therefore in order to determine the position of loads for maximum moment the criterion of Eq. (33) must be used. Try wheel 19 at c. Note that wheel 34 is at the right end of the span.

Wheel 19 to right of C,

$$G\frac{a}{l} - W = 730\tfrac{1}{2} - 355 = +$$

Wheel 19 to left of C,

$$G\frac{a}{l} - W = 755\tfrac{1}{2} - 380 = -$$

Other positions were tried but none were found which would satisfy the criterion.

$$M_B = \sum_1^{34} M = 105{,}480$$

$$R_1 = \frac{M_B}{300} = 351.60$$

$$M_c = (R_1)(150) - 22{,}230 = 30{,}510$$

or

Moment at C = 30,510,000 ft.-lb.

The difference between the values of moment obtained by the two tables is 9,500 ft.-lb. or 0.03 per cent. This is well within the limits of accuracy necessary in work of this character.

68d. Greatest Possible Moment in a Beam.—On comparing the maximum moments calculated at several points on a beam by the methods of the preceding article, it will be found that these maximum values for points near the center of the beam are larger than those for points near the ends of the beam. It will also be found that there is some point for which the moment is greater than for any other point on the beam. This moment is known as the *greatest possible moment in the beam.* It is also called the *absolute maximum moment.* The position of this moment center and the amount of the bending moment will now be determined.

Since the maximum moment due to any set of loads occurs under one of the loads, let us assume that point C of Fig. 145, at a distance z from the left end of the beam, is the point of greatest possible moment, and that wheel P is located at the moment center. Assume that the total load on the beam is G, and that the center of gravity of these loads is located at a distance x from the right end of the beam. Also assume W to represent the load in front of the moment center (P is included in W), and let b represent the distance from P to the center of gravity of loads W. Let c be the distance from G to P, a distance which can be calculated as soon as the load G is known.

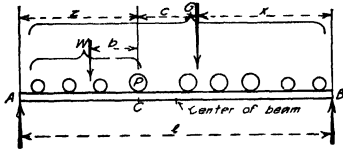



Fig. 145

For the conditions shown in Fig. 145, the moment at C is,

$$M_c = G\frac{x}{l}z - Wb \tag{35}$$

Since x and z are the only variables in Eq. (35) it can readily be seen that a maximum value for M is given when the product xz is a maximum. From Fig. 145 we note that $x + z = l - c$. But $l - c$, and hence also $x + z$ is a constant for any given set of loads. Therefore, it can be shown by the Calculus that the product xz is a maximum when $x = z$. Hence, for a maximum value of M in Eq. (35), *the wheel under which the maximum moment occurs and the center of gravity of all of the loads must be located at equal distances from the ends of the beam.* From Fig. 145 it can be seen that this requires the center of the beam to be placed midway between the wheel under which the moment occurs and the center of gravity of the loads. Placing $x = z$ in Eq. (35), we have

$$M_{\text{Max.}} = G\frac{x^2}{l} - Wb \tag{36}$$

Equation (36) is general in form and may be applied to any set of loads on a simple beam.

In determining the maximum possible moment in a beam for a given set of loads, it is usually necessary to calculate the maximum value for several wheels. On comparing these maximum values, the amount and position of the absolute maximum moment is readily determined. A few of the more important general cases will now be worked out in detail.

Two Equal Loads.—When the applied loads consist of two equal loads spaced at a given distance apart, formulas are readily derived for the value of the greatest possible moment. As stated above, the wheel under which the moment is to be determined and the center of gravity of the two loads are to be placed equidistant from the beam center. Let us calculate the moment under the left-hand wheel. Since the loads are equal, the center of gravity is midway between the two loads. To conform to the above rule for position of loads for greatest possible moment, the loads are to be placed as shown in Fig. 146. For this position of the loads, the moment under the left-hand wheel is

$$M = R_1\left(\frac{l}{2} - \frac{d}{4}\right) = \frac{2P}{l}\left(\frac{l}{2} - \frac{d}{4}\right)^2 \tag{37}$$

It can readily be shown that the absolute maximum moment under the right-hand wheel is the same as given by Eq. (37).

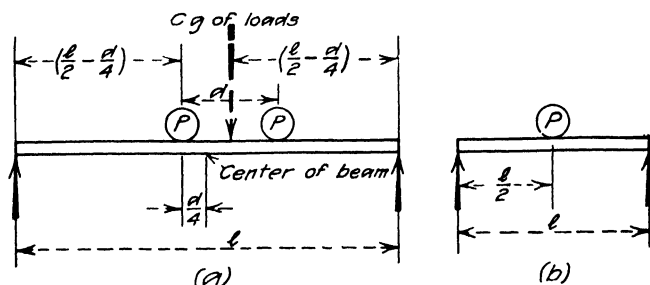



FIG. 146

In the case of beams whose length is somewhat less than twice the distance between wheel loads, it will be found that a single load placed at the beam center, as shown in Fig. 146*b*, will give a greater moment than calculated by Eq. (37). To determine the span length for which the loading conditions of Figs. 146*a* and *b* give equal moments, equate the moment given by Eq. (37) to the moment due to a load *P* at the span center, and solve for *l*, the limiting span length. The moment at the span center due to a single load *P* placed as shown in Fig. 146*b* is $\frac{1}{4}Pl$ Solving the resulting equation, we have

$$l = \frac{d}{2 - \sqrt{2}} = 1.7065d \tag{38}$$

Therefore, when the span is shorter than 1.7065 times the distance between the loads, a single load P placed at the beam center gives a greater moment than the two loads placed as for absolute maximum moment. The value of the maximum moment is $M = \frac{1}{4}Pl$. When the span is equal to 1.7065 times the distance between wheels, either Eq. (37) or the formula $M = \frac{1}{4}Pl$ will give the maximum moment. When the span is greater than 1.7065 times the distance between loads, Eq. (37) must be used.

Illustrative Problem.—Calculate the absolute maximum moment in a simple beam 20 ft long due to two equal concentrated loads of 20,000 lb each, spaced 8 ft apart.

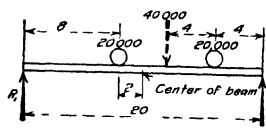



FIG. 147

Since the span of the beam is greater than the limiting span of (1.7065)(8) = 13.65 ft, as given by Eq. (38), the absolute maximum moment may be determined by means of Eq. (37). As stated above, the absolute maximum moments under the two wheels are equal. Consider the case of the left-hand wheel. Since the loads are equal the center of gravity is located midway between the two loads. Figure 147 shows the loads in position for absolute maximum moment. From Eq. (37) with P = 20,000 lb; d = 8 ft.; and l = 20 ft, we have

$$M = \frac{(2)(20,000)}{20}(10 - 2)^2 = 128,000 \text{ ft.-lb}$$

The absolute maximum moment may also be calculated directly from Fig. 147. For the conditions shown, the moment under the left-hand wheel is (R_1) (8), and $R_1 = (40,000)(\frac{8}{20})$ = 16,000 lb. Hence, as before, the absolute maximum moment is M = (16,000)(8) = 128,000 ft-lb. Note from Fig. 147 that the absolute maximum moment occurs at a point 2 ft to the left of the beam center. It will be found that for absolute maximum moment under the right-hand wheel, this wheel is to be placed 2 ft to the right of the beam center, and the moment under the wheel in this position is, as before, 128,000 ft-lb

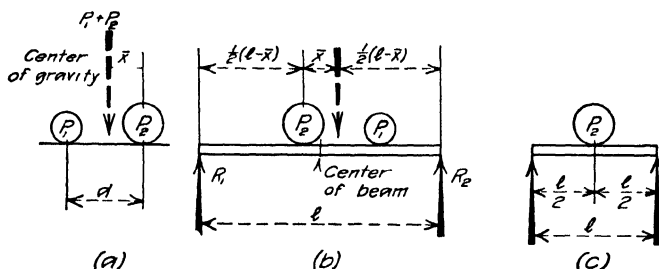



FIG. 148.

Two Unequal Loads.—The case of two unequal loads, representing a road roller or tractor, may be treated in a similar manner. It will be found that the greater moment will occur under the larger load. Therefore only this case will be considered.

The distance from the center of gravity of the group to the heavier wheel, P_2 of Fig. 148a, as determined from moments about P_2 is

$$\bar{x} = \frac{P_1}{P_1 + P_2} d \tag{39}$$

Placing the heavier wheel to the left, with P_2 and the center of gravity equal distances from the beam center, which is the position required for absolute maximum moment, we have the loading conditions shown in Fig. 148b. Taking moments about P_2,

$$M_c = \frac{P_1 + P_2}{4l}(l - \bar{x})^2$$

Substituting the above value of $\bar{x}$,

$$M_c = \frac{(P_1 + P_2)}{4l}\left(l - \frac{P_1}{P_1 + P_2} d\right)^2 \tag{40}$$

Equation (40) gives the amount of the absolute maximum moment, and Fig. 148b shows the position of the critical wheel. Note that when $P_1 = P_2$, or for equal loads, Eq. (40) may be reduced to the form of Eq. (37).

The limiting span for equal moments for a single load equal to the heavier load P_2 placed at the span center and for the two loads of Fig. 148a is found by equating $\frac{1}{4}P_2l$ and M_c of Eq. (40). This limiting span is

$$l = \frac{\dfrac{P_1}{P_1 + P_2}}{1 - \sqrt{\dfrac{P_2}{P_1 + P_2}}} d \tag{41}$$

Illustrative Problem.—Calculate the absolute maximum moment in a 20-ft. beam due to the loads shown in Fig. 149(a).

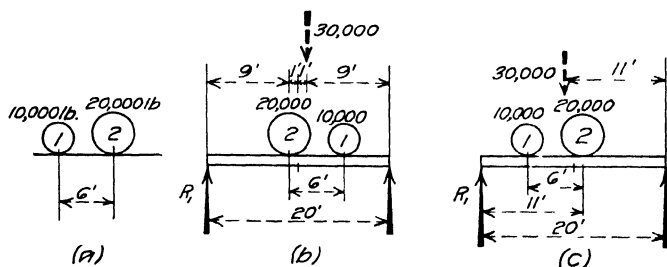



Fig. 149.

As stated above, the greatest moment occurs under the heavier wheel. From Eq. (39), the center of gravity is located at a distance $x = \left(\frac{10}{10 + 20}\right)(6) = 2$ ft. in front of wheel 2. Figures 149(b) and (c) show two possible positions of the loads for absolute maximum moment. From Eq. (40), we have

$$M = \frac{10{,}000 + 20{,}000}{(4)(20)}\left[20 - \frac{(10{,}000)(6)}{10{,}000 + 20{,}000}\right]^2$$

from which

$$M = (375)(18)^2 = 121{,}500 \text{ ft.-lb.}$$

The same result may be obtained directly from Figs. 149(*b*) and (*c*). Thus from Fig. (*b*), $M = (R_1)(9)$, and $R_1 = (30{,}000)(^9\!/_{20})$. Therefore,

$$M = (30{,}000)\left(\frac{9^2}{20}\right) = 121{,}500 \text{ ft.-lb.}$$

Again, from Fig. (*c*), $M = (R_1)(11) - (10{,}000)(6)$, and $R_1 = (30{,}000)(^{11}\!/_{20})$. Therefore,

$$M = (30{,}000)\left(\frac{11^2}{20}\right) - 60{,}000 = 181{,}500 - 60{,}000 = 121{,}500 \text{ ft.-lb.}$$

Note that the load position of Fig. (*b*) leads to simpler moment equations than the one of Fig. (*c*).

Engine Loading.—The absolute maximum moment under engine loading will be found to occur at a point near the center of the beam. To determine this moment and the wheel under which it occurs, a cut and try process must be used. The absolute maximum moment must be determined for several wheels and the values compared.

Experience has shown that the wheel under which the absolute maximum moment occurs is usually the one which answers for maximum moment at the center of the beam. By locating the wheel or wheels which satisfy the criterion of Eq. (33), stated for maximum moment at the beam center, the work may be greatly shortened by applying the conditions for an absolute maximum moment only to these wheels.

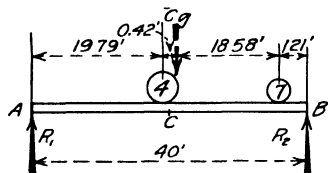



Fig. 150.

On p. 144 is given a table of absolute maximum moments due to E-50 loading for beams from 10 to 250 ft. span length.

Illustrative Problem.—Calculate the absolute maximum moment in a 40-ft. beam for Cooper's E-50 loading.

First, we shall find the wheel which satisfies the criterion of Eq. (33) for maximum moment at the beam center, point C, Fig. 150. Try wheel 4 at c.

Wheel 4 just to right of C,

$$G\frac{a}{l} - W$$

$$= (145)\left(\frac{20}{40}\right) - 62.5 = +$$

Wheel 4 just to left of C,

$$G\frac{a}{l} - W$$

$$= (145)\left(\frac{20}{40}\right) - 87.5 = -$$

Wheel 4 was found to be the only wheel which satisfied the criterion.

The distance from the center of gravity of the loads to wheel 7 equals $\sum_1^7 M$ divided by $\sum_1^7 W = \frac{2{,}693.75}{145} = 18.58$ ft. Wheel 4 is 19 ft from wheel 7. For absolute maxi-

mum moment wheel 4 should be $\frac{19.00 - 18.58}{2} = 0.21$ ft. to the left of C, or 19.79 ft. from the left end of the girder. Wheel 7 is then 1 21 ft. from the right end of the girder.

$$M_B = \sum_1^7 M + \left(\sum_1^7 W\right)1.21$$

$$= 2{,}693.75 + (145)(1.21) = 2{,}869.20$$

$$R_1 = \frac{M_B}{40} = \frac{2{,}869.20}{40} = 71.73$$

$$M_{\text{Max}} = (R_1)(19.79) - \sum_1^4 M$$

$$= (71.73)(19.79) - 600 = 819.54$$

or

$$M_{\text{Max}} = 819{,}540 \text{ ft.-lb.}$$

The absolute maximum moment may also be determined by direct substitution in Eq. (36). With $G = 145$; $x = 19.79$; $l = 40$; and $W_b = 600$, we have

$$M_{\text{Max}} = \frac{(145)(19.79)^2}{40} - 600 = 819.54$$

68e. Position of Loads for Maximum Moment at Any Point in a Girder with Floorbeams.—Two cases will be considered: (1) moment center at a floorbeam, and (2) moment center between floorbeams. The latter case is important in certain forms of trusses (see Art. 2c, p. 229).

Fig. 151.

Moment Center at a Floorbeam.—Assume the loads on a girder with floorbeams to be distributed as shown in Fig. 151. In Art. 65c it was shown that the moment at C due to any given loading is not affected by the presence of floorbeams. It is therefore evident that the criterion for position of loads for maximum moment at a floorbeam is the same as for a corresponding point on a similar girder without floorbeams. Hence the criterion of Eq. (33) applies also to the conditions shown in Fig. 151.

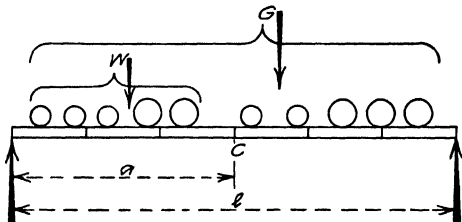



Fig. 152.

Illustrative Problem.—Find the position of loads for maximum moment at point C of the girder of Fig. 152.

Referring to Fig. 142 we find that wheel 3 should be used.

When substituting values of a and l in Eq. (33), the panel length instead of the foot will be used as the unit of length. In a great many problems the work can be simplified by this means. Substituting in Eq. (33) with wheel 3 just to the right of C,

$$G\frac{a}{l} - W = (265)\left(\frac{1}{5}\right) - 37.5 = +$$

Wheel 3 just to left of C,

$$G\frac{a}{l} - W = (265)\left(\frac{1}{5}\right) - 62.5 = -$$

By methods similar to those given in preceding problems we find that the moment at C = 1,788,500 ft.-lb.

Moment Center at a Point between Floorbeams.—Assume that the given moment center is located at a distance a from the left support and at a distance c from the left end of the panel containing the given moment center. Let G represent the total load on the span; let W_2 represent the load in the panel containing the moment center; and let W_1 represent the remaining loads to the left of the moment center. Figure 153 shows

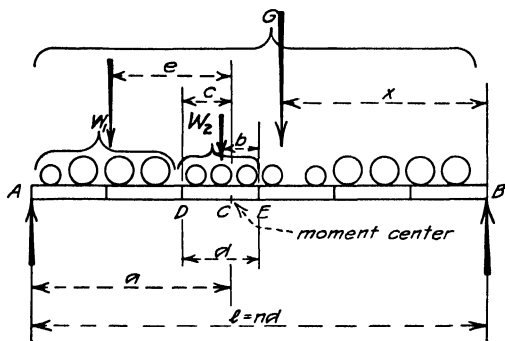



FIG. 153.

the several load groups. On this figure are shown the distances from the centers of gravity of the several load groups to convenient reference points.

For the conditions shown in Fig. 153, the moment at C is,

$$M_c = G\frac{x}{l}a - W_1e - W_2\frac{b}{d}c \tag{42}$$

In this equation, the term $W_2\frac{b}{d}c$ represents the negative moment due to the loads W_2 in panel DE. It can be seen to be equal to the panel reaction at D for the loads W_2 multiplied by the distance from the moment center C to point D.

Now assume that the loads move a small distance Δ to the left. If M_c' represent the new value of the moment at C, we have

$$M_c' = G\frac{(x + \Delta)}{l}a - W_1(e + \Delta) - W_2\left(\frac{b + \Delta}{d}\right)c \tag{43}$$

The change in moment due to the forward motion of the loads, which will be denoted by m, may be determined by subtracting Eq. (42) from Eq. (43), from which

$$m = \left[G\frac{a}{l} - \left(W_1 + W_2\frac{c}{d}\right)\right]\Delta \tag{44}$$

By an analysis similar to the one given in Art. **68b** for the derivation of Eq. (33), it can be shown that for maximum moment, the loads must be so located that

$$G\frac{a}{l} - \left(W_1 + W_2\frac{c}{d}\right) = 0 \tag{45}$$

Equation (45) is the criterion for position of loads for maximum moment at a moment center located between panel points in a girder with floor-beams. Note that $W_2\frac{c}{d}$ is the proportional part of the load carried by that part of the panel which is to the left of the moment center. Therefore, as before, the maximum moment at a moment center between panel points occurs when the average load to the left of the moment center is equal to the average load on the whole span.

The criterion of Eq. (45) may be made to pass through a zero value, or to change from a positive to a negative value, only when some load crosses points D or E. This statement is readily proved by an analysis similar to the one given in Art. **68b**.

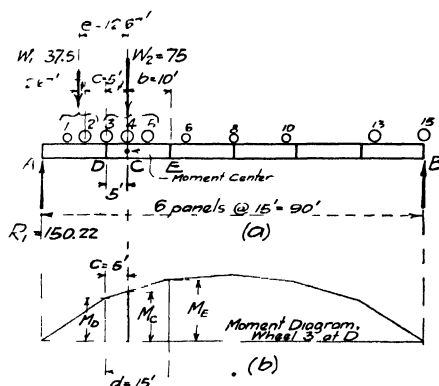



FIG 154.

Illustrative Problem.—Find the position of loads for maximum moment at point C of the girder of Fig. 154.

A position of loads such that wheel 3 was located at D was the only one which would satisfy the criterion of Eq. (45). Fig. 154a shows the position of the loads.

Wheel 3 to right of D,

$$G\frac{a}{l} - \left(W_1 + W_2\frac{c}{d}\right) = (290)\left(\frac{20}{90}\right) - \left[37.5 + (75)\frac{1}{3}\right] = +$$

Wheel 3 to left of D,

$$G\frac{a}{l} - \left(W_1 + W_2\frac{c}{d}\right) = (306.25)\left(\frac{20}{90}\right) - \left[62.5 + (50)\frac{1}{3}\right] = -$$

The moment at C may be determined by direct substitution in Eq. (42). For the load position shown in Fig. 154a the several terms of Eq. (42) have the following values: $W_1 = 37.5$; $W_2 = 75$; $a = 20$ ft.; $b = 10$ ft.; $c = 5$ ft.; $d = 15$ ft.: $e = 12.67$

ft.; and $\frac{Gx}{l} = R_1 = 150.22$. This value of R_1 is calculated by the methods given in the preceding articles. Then from Eq. (42)

$$M_c = (150.22)(20) - (37.5)(12.67) - \frac{(75)(10)(5)}{(15)} = 2{,}279.4 \text{ thousand ft.-lb.}$$

Figure 154b shows the moment diagram drawn for the loads shown in position on Fig. 154a. The moment at C may be determined from the moment diagram of Fig. 154b in the following manner: As shown in Art. 65c the moment diagram consists of straight lines between panel points. It is therefore evident from Fig. 154b that

$$M_C = M_D + (M_E - M_D)\frac{c}{d}$$

For the load position shown, $M_D = 1{,}965.8$ and $M_E = 2{,}906.7$. These moments are calculated by the methods given in the preceding articles. Then,

$$M_C = 1{,}965.8 + (2{,}906.7 - 1{,}965.8)\tfrac{1}{3} = 2{,}279.4 \text{ thousand ft.-lb.}$$

68f. Position of Loads for Maximum Shear in a Beam without Floorbeams. *Shear at Any Point.*—The shear at point C,

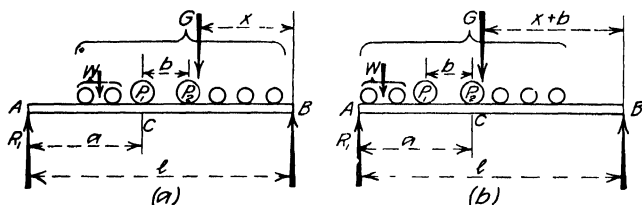



Fig. 155.

Fig. 155(a), assuming that load P_1 is placed a very small distance to the right of C, is

$$V_c = R_1 - W = G\frac{x}{l} - W \tag{46}$$

If the loads be moved a small distance to the left, P_1 crosses point C and must then be included in the load W. The shear at C then becomes

$$V_c' = G\frac{x}{l} - (W + P_1) \tag{47}$$

It is assumed here that the forward motion has been too small to have any effect on the value of x. Hence the shear has been reduced by the amount of the load which crossed point C. Any further movement of the loads will cause an increase in the value of x and therefore in the value of $R_1 = G\frac{x}{l}$. This increase in R_1 will continue until load P_2 moves to a position just to the right of point C, as shown in Fig. 155(b). Assuming that the load group G remains unchanged during the forward movement

of the loads, and that the distance between wheels P_1 and P_2 is b, the shear at C for the conditions shown in Fig. 155(b) is

$$V_c'' = G\frac{(x+b)}{l} - (W + P_1) \tag{48}$$

The difference in the shear at point C for the loading conditions shown in Figs. 155(a) and 155(b) may be found by subtracting Eq. (46) from Eq. (48). If v denotes this difference, we have

$$v = G\frac{b}{l} - P_1 \tag{49}$$

From the definition for shear, and from the statement of Eqs. (46) to (49), it can be seen that $G\frac{b}{l}$ is the change in the left reaction due to the forward motion of the loads, and that P_1 is the load which crosses the point at which the shear is desired. Hence Eq (49) is a statement of the relative changes in left reaction and in load to the left of the section due to a forward motion of the load group G. Therefore, in order that the loading conditions shown in Fig. 155(b) may give a greater shear than the loads shown in Fig. 155(a), the change in R_1 for the forward movement of the loads must exceed P_1, the load which crossed point C. This condition is indicated by a positive sign in Eq. (49). A negative sign in Eq. (49) indicates that the loading condition of Fig. 155(a) gives a greater shear than the load position of Fig. 155(b).

To determine which wheel of a set of concentrated loads will give maximum shear at any point, assume some one of the wheels to be placed at the given point, and by means of Eq. (49) determine whether the following wheel gives a greater or smaller shear. If it gives a greater shear (positive sign in Eq. (49)), move the loads forward until this wheel reaches the point at which the shear is desired. Then repeat the operation, until finally a negative sign is given by Eq. (49). The last wheel which gave a positive sign is the wheel which will give maximum shear. Had a minus sign occurred on the first substitution in Eq. (49), evidently the loads should be moved to the right until a positive sign resulted.

The above discussion assumes that the load group G does not change during the forward movement of the loads through a distance equal to the spacing of the two loads under comparison. However, the total load usually changes, new wheels coming on the span. To take this into account, two values of G must be substituted in Eq. (49); one for P_1 at point C of Fig. 155(a), and the other for wheel P_2 at C of Fig. 155(b). A general rule which will provide for all cases may be stated as follows. The total load G is to be taken as the load which in moving forward causes an increase in the left reaction.

Equation (49) offers a means of determining which of two consecutive wheels gives the greater shear at a given point, but it does not give

any idea of the wheel with which to start. However, it can be seen from Eq. (46) that for maximum positive shear, R_1 must be as great as possible and W must be as small as possible. For the usual engine loadings this result may be obtained by placing the light pilot wheel in front of the point at which the shear is desired and placing the first of the heavy drive wheels just to the right of the point at which the shear is desired.

The table on p. 144 gives values of the shear at the one-fourth and center points of beams varying in span from 10 to 250 ft.

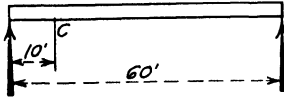



Fig. 156.

Illustrative Problem.—Find the position of loads for maximum positive shear at point C of the beam of Fig. 156 for Cooper's E-50 loading. Determine the amount of the maximum shear.

Consider the relative shears with wheel 1 and wheel 2 at C. The value of G changes from 177.5 to 190 as wheel 10 comes on the span.

$$G\frac{b}{l} - P_1 = (177.5)\left(\frac{8}{60}\right) - 12.5 = +$$
$$(190)(8/60) - 12.5 = +$$

Consider the relative shears with wheel 2 and wheel 3 at C. The value of G changes from 190 to 177.5 as wheel 1 passes off the span.

$$G\frac{b}{l} - P_1 = (190)\left(\frac{5}{60}\right) - 25 = -$$
$$= (177.5)\left(\frac{5}{60}\right) - 25 = -$$

Therefore wheel 2 should be placed at point C for maximum positive shear

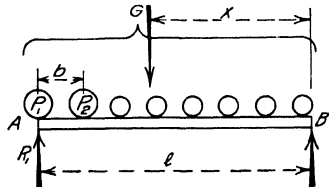
The maximum shear is equal to the left reaction R_1 minus the load on wheel 1.

$$R_1 = \frac{[5,790 + (2)(190)]1,000}{60} = 102,800$$
$$\text{Shear} = 102,800 - 12,500 = 90,300 \text{ lb.}$$

Shear at End of Beam.—The conditions for shear at the end of a beam are as shown in Fig. 157. The maximum positive shear in the beam of Fig. 157 occurs at point A, and its value is

$$V_A = R_1 = G\frac{x}{l} \qquad (50)$$

Fig. 157.

In order that V_A shall have a maximum value, the product Gx must be a maximum. To make G, the total load, a maximum, the heavy drive wheels should be included in the total load on the beam, and to give x, the distance from the right end of the beam to the center of gravity of G, a maximum value, evidently the heavy drive wheels should be placed as close to the left end of the beam as possible. This leads to cut and try methods in the determination of the maximum end shear.

It will be found by trial, that, for all cases except those to be mentioned later, the maximum end shear occurs at the left end of the beam when the first driver is placed at the end of the beam, as shown in Fig. 158. The shear at A is positive and equal to the left reaction. Exceptions to the above rule are found to occur for the conditions shown in Fig. 159. In this case the shear at the right support will be found to be greater than for the conditions shown in Fig. 158. This will be found to occur in beams of spans between 23 and 27⅓ ft.

The shear at point B of Fig. 159 is negative. But since the area required to resist shearing stresses is determined by the amount of the shear, the character (positive or negative) not entering, it is necessary to determine the actual numerical maximum. However, the difference in shear for point A of Fig. 158 and point B of Fig. 159 is so small that it is generally not considered necessary to enter into these refinements in calcula-

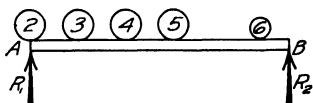
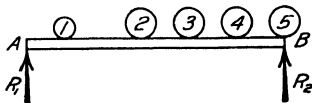



FIG. 158. FIG. 159.

tion in a practical problem. It is mentioned here only as a matter of interest. Similar conditions also hold for shear at points a short distance from the ends of a beam. Since these shears do not in general determine the sectional area required, this case was not mentioned in the discussion relative to Fig. 155.

A direct comparison method similar to the one used in deriving Eq. (49) for shear at any point may also be used for the determination of the wheel which answers for maximum end shear. Assume wheel P_1 of Fig. 157 to be placed just to the right of point A. Then, as in Eq. (50)

$$V_A = R_1 = G\frac{x}{l}$$

As soon as the loads are moved a short distance to the left, P_1 passes off the span and V_A is reduced by the amount of load P_1. We then have

$$V_A' = R_1 - P_1 = G\frac{x}{l} - P_1$$

Now assume that the loads move forward a distance b until P_2 reaches a point just to the right of A. The shear at A is then

$$V_A'' = R_1' - P_1 = G\frac{(x + b)}{l} - P_1$$

The change in end shear, which will be denoted by v, is

$$v = G\frac{b}{l} - P_1$$

Note that this expression is exactly the same as Eq. (49). As before, G is the load which in moving forward causes an increase in the left reaction, and P_1 is the load which passes off the span at the beginning of the forward motion. Since P_1 passes off the span at the beginning of the forward motion, it is not to be included in the value of G.

As before, a positive sign in the above equation indicates that P_2 gives a greater end shear at A than is given by P_1. It will be found on comparing shears by means of the above equation, that the first driver gives maximum shear at the end of the beam. This statement is subject to the same conditions as given for Figs. 158 and 159.

The table on p. 144, gives values of the end shear for beams varying in span from 10 to 250 ft.

Illustrative Problem.—Find the position of loads for maximum positive shear at point A of the beam of Fig. 160 for Cooper's E-50 loading.

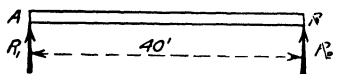



FIG. 160.

Consider the relative shears with wheel 1 and wheel 2 at A. The value of G changes from 132.5 to 148.75 as wheel 8 comes on the span, wheel 1 being off in both cases.

$$G\frac{b}{l} - P_1 = (132.5)\left(\frac{8}{40}\right) - 12.5 = +$$

$$(148.75)\left(\frac{8}{40}\right) - 12.5 = +$$

Consider the relative shears with wheel 2 and wheel 3 at A. The value of G remains constant at 140 for wheel 2 passes off the span as soon as the forward motion begins.

$$G\frac{b}{l} - P_1 = (140)\left(\frac{5}{40}\right) - 25 = -$$

Therefore wheel 2 should be placed at A for maximum positive shear. The shear at A is equal to the left reaction R_1,

$$R_1 = \frac{M_B}{40} = \frac{\sum_2^9 M}{40} = \frac{3{,}770}{40} = 94.25$$

or the shear is 94,250 lb.

68g. Position of Loads for Maximum Shear in any Panel of a Girder with Floorbeams.—The position of loads for maximum shear in any panel of a girder with floorbeams will be determined for the conditions shown in Fig. 161. Let CD be the panel in which the shear is to be determined. Assume that the total load is G, and that it covers the entire span. Let the load in panel CD be denoted by W_2 and let the load to the left of panel CD be denoted by W_1. For the conditions shown,

$$V_{CD} = R_1 - W_1 - \text{Panel load at } C \text{ due to } W_2$$

or

$$V_{CD} = G\frac{x}{l} - \left(W_1 + W_2\frac{c}{d}\right) \tag{51}$$

Assume that the loads move a small distance Δ to the left, no load passing points A, B, C, or D. For the new position of the loads, the shear in CD is

$$V_{CD}' = G\frac{(x + \Delta)}{l} - \left[W_1 + W_2\frac{(c + \Delta)}{d}\right] \tag{52}$$

The change in shear due to the forward motion of the loads may be determined by subtracting Eq. (51) from Eq. (52). If v denotes this difference in shear, we have

$$v = \left(\frac{G}{l} - \frac{W_2}{d}\right)\Delta \tag{53}$$

A positive value from Eq. (53) for a movement of the loads to the left indicates an increasing value of the shear in panel CD, while a negative value indicates a decreasing shear. For a maximum value of the shear—that is, a value which is neither increasing or decreasing—Eq. (53) gives a zero value. The relations between loads and distances are then given by the equation

$$\frac{G}{l} = \frac{W_2}{d}$$

That is, for maximum shear in panel CD, the average load per unit of length for the whole span must be equal to the average load per unit of length in the panel in question. A somewhat more convenient form for this criterion is

$$G\frac{d}{l} - W_2 = 0 \tag{54}$$

In most cases the panels are of equal length. Let n be the number of equal panels of length d composing the span—that is, $l = nd$. Substituting this value of l in Eq. (54), we have,

$$\frac{G}{n} - W_2 = 0 \tag{55}$$

Hence, for equal panels, the maximum shear in any panel occurs when the average panel load for the whole span is equal to the load in the panel in question.

The loading conditions shown in Fig. 161 represent a general case. From Art. **66**a, p. 87, it can be seen that loads W will cause negative shear in panel CD. If positive shear is in question, evidently no load should be placed to the left of panel CD. The term W in the preceding analysis then becomes zero. It can readily be seen that this will have no

effect on the final form of the criteria of Eqs. (54) and (55). However, it is possible in trusses with very short panels or for unusual engine loadings that some of the loads may extend to the left of the panel in question. For this reason the criterion was derived for the general case. Ordinarily there will be no load to the left of the panel in question.

The criteria of Eqs. (54) and (55) may be satisfied exactly by a uniform load. Referring to Fig. 120 of Art. **66***b*, the distance x, in order to conform to the criterion of Eq. (55), and to Eq. (54), must be such that $x:d::a + x:l$. Solving this proportion for x, we have, $x = \frac{ad}{l - d}$. This is the same as the value of x given in Art. **66***a*. Hence the criterion of Eq. (55) and the analysis of Art. **66***b* lead to the same load position for a uniform load.

In the case of concentrated load systems it is not possible to satisfy exactly the criterion of Eq. (55). However, as in the case of the moment

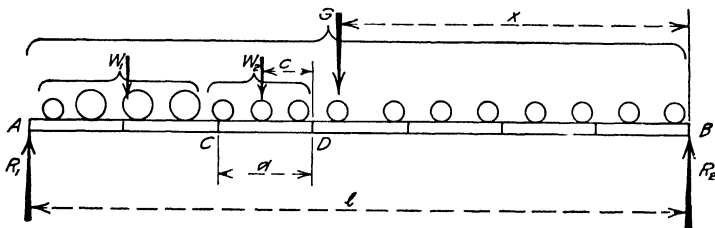



Fig. 161.

criterion, the sign of the criterion for shear may be made to pass through a zero value. By an analysis similar to that given in Art. **68***b* in the derivation of Eq. (33), it can be shown that the criterion may be made to pass through a zero value when some load passes point D of Fig. 161. Thus when the load is considered as to the right of D, substitution in Eq. (55) will give a positive value, and when the load is considered to the left of D, a negative value results. An application of the criterion is given in the problem at the end of this article.

In general the criterion is answered by the wheels near the head of the engine loading. For panels near the left end of the beam, the loads should cover practically the whole panel. For panels near the center of the beam, the load in the panel in question must be reduced, since the average panel load on the whole beam must be equal to the load in the panel in question. This can be done by using wheels near the head of the engine.

Load positions for negative shear in any panel may also be determined by means of the criterion of Eq. (55) This can be done by considering the loads as headed and moving to the right instead of to the left, as in the preceding analysis. Considering forces to the right of any panel, we have negative shear conditions. This same result may also be

obtained by means of the reciprocal relation existing between positive and negative shears at corresponding distances from the ends of the beam to which attention was called in Art. **66b**—that is, positive shears may be calculated for all panels in the girder. Then the negative shear in the third panel from the left end of the girder will be equal to positive shear in the third panel from the right end of the girder.

The maximum end reaction for a girder is calculated by the same methods as used in Art. **68f** for shear at the end of a beam without floor-beams. As before, the first heavy drive wheel placed at the left end of the girder will give the maximum end shear, which will be equal to the left reaction.

Illustrative Problem.—Find the position of loads for maximum positive shear in panel CD of the girder of Fig. 162 for Cooper's E-50 loading. Calculate the resulting maximum shear. Use the criterion of Eq. (55).

A C D B R_1 5 panels @ 14' = 70' R_2

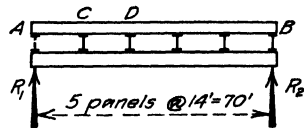



FIG. 162.

$$\text{Wheel 2 at right of } D \quad \frac{G}{n} - W_2 = \frac{177.5}{5} - 12.5 = +$$

$$\text{Wheel 3 at right of } D \quad \frac{G}{n} - W_2 = \frac{177.5}{5} - 37.5 = -$$

Wheels 1 and 3 were also tried, but they did not satisfy the criterion.

Therefore for maximum positive shear in panel CD, wheel 2 should be placed at D. The shear in panel CD = left reaction, R_1, minus joint load at C due to loads in panel CD. With wheel 2 at D, wheel 9 is 2 ft. to the left of B. Hence $R_1 = [4{,}370 + (177.5)(2)]\frac{1}{70} = 67.5$. With wheel 2 at D, wheel 1 is the only load in panel CD. Hence joint load at C due to wheel 1 $= \frac{100}{14} = 7.1$. Therefore, shear in panel $CD = 67.5 - 7.1 = 60.4$, that is, 60,400 lb.

68h. Equivalent Uniform Loads.[1] —On comparing the methods for calculating moments due to concentrated and uniform load systems given in the preceding articles, it will be noted that the former method is more complicated and requires considerably more time than the latter method. When the same system of concentrated loads is to be used in the design of a great many structures it is possible to save considerable time by substituting for the concentrated load system a uniform load system which will give the same result. Such a uniform load system is known as an Equivalent Uniform Load.

To determine the uniform load which will give the same moment as a concentrated load system, let M = moment at any point due to the concentrated load system. From Eq. (6), p. 67, the moment at any point in a beam due to uniform loading is

$$M = \tfrac{1}{2}wx(l - x)$$

[1] Diagrams, in addition to those shown in this article, may be found in "Locomotive Loadings for Railway Bridges," *Trans. Am. Soc. Civil Eng.*, Vol. LXXXVI, p. 606.

Equivalent Uniform Load in Pounds per Foot (One Rail)
4500
4400
4300
4200
4100
4000
3900
3800
3700
3600
3500
3400
3300
3200
3100
3000
2900
2800
2700
2600
2500
0.1 Point
0.2 Point
0.3 Point
0.4 Point
Center
0.1
0.2
0.3
0.4
0.5
50
100
150
200
250
300
350
400
450
500
Length of Span in Feet

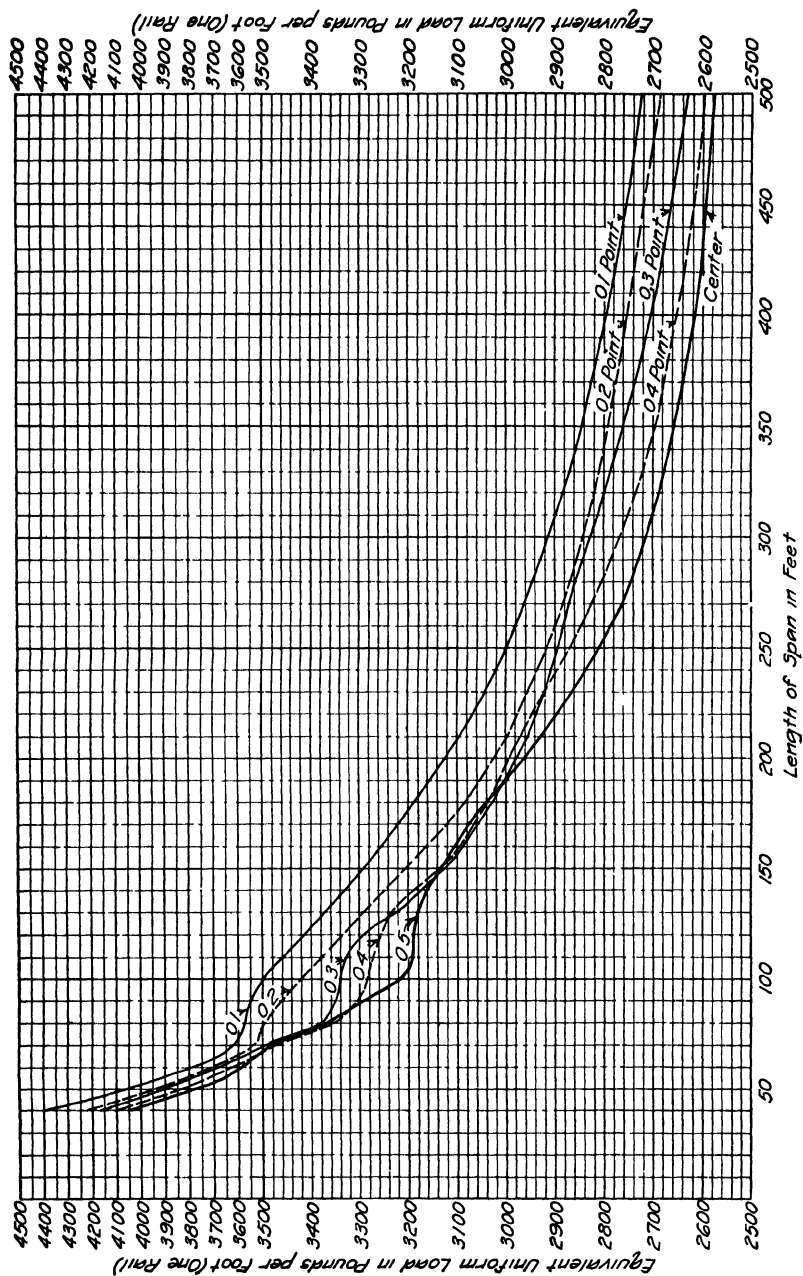



FIG. 163.

where x = distance from end of beam to moment center; l = span of beam, and w = uniform load per foot. Equating the above equation to the calculated value of moment due to concentrated loads, the equivalent uniform load is found to be

$$w = \frac{2M}{x(l - x)} \tag{1}$$

By means of Eq. (1) the equivalent uniform load for moment at the $\frac{1}{10}$th, $\frac{2}{10}$ths, $\frac{3}{10}$ths, $\frac{4}{10}$ths and center points of beams of spans from 40 to 500 ft. have been calculated. These values are plotted to form the diagram of Fig. 163.[1] The values of M substituted in Eq. (1) were calculated for Cooper's E-50 loading by the methods given in Art. **68b**.

Illustrative Problem.—Calculate the moment at a point 40 ft. from the left end of a beam 120 ft. long. Use the equivalent uniform load given by Fig. 163.

The moment center is located at the $\frac{40}{120} = \frac{1}{3}$ point of the beam. From Fig. 163, the equivalent uniform load for the $\frac{1}{3}$ point of a 120-ft. beam is 3,280 lb. per ft. Hence

$$M = \frac{w}{2}x(l - x) = \left(\frac{1}{2}\right)(3{,}280)(40)(120 - 40)$$
$$= 5{,}248{,}000 \text{ ft.-lb.}$$

The exact moment calculated from Table I, p. 105, is 5,255,000 ft.-lb. (See p. 228, calculation of moment for stress in member bc.)

The equivalent uniform loads given by Eq. (1) and Fig. 163 may also be used for the determination of shears in the panels of a truss. However, it is necessary to determine by trial the proper method of selecting the equivalent uniform load for any given structure. It will be found that reasonably accurate results are obtained by using the conventional method of shear calculation (p. 93) and an equivalent uniform load determined as for moment at the right end of the panel in question. Somewhat more accurate results may be obtained by using the exact method of shear calculation (p. 90) and an equivalent uniform load determined for the loaded length of the span, as given by Eq. (16), p. 87.

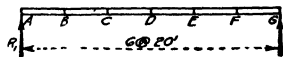



Fig. 164.

Illustrative Problem.—Calculate the shear in panel BC of Fig. 164 using equivalent uniform loads as given by Fig. 163.

Using the conventional method of shear calculation, the uniform load is determined as for point C, the $\frac{1}{3}$ point of a 120-ft. beam. From Fig. 163, w = 3,280 lb. per ft. Panel load = (20)(3,280) = 65,600 lb. Shear in panel $BC = R_1$ for loads at C to F =

$$\frac{65{,}600}{6}(1 + 2 + 3 + 4) = 109{,}200 \text{ lb.}$$

Using the exact method of shear calculation, the loaded length of span must first be determined. From Eq. (16), p. 87, the uniform load extends

$$\frac{ad}{l - d} = \frac{(80)(20)}{(120 - 20)} = 16 \text{ ft.}$$

into panel BC. Hence the total loaded length is 80 + 16 = 96 ft. The equivalent

[1] From "Modern Framed Structures," part I, p. 251.

uniform load is to be determined for the 16-ft. point in a 96-ft. beam. From Fig. 163, w = 3,460 lb. per ft. Substituting this value of w in Eq. (22), p. 92, the resulting shear is

$$V = \frac{1}{2} w \frac{a^2}{(l - d)} = \frac{(\frac{1}{2})(3{,}460)(80)^2}{(120 - 20)} = 110{,}700 \text{ lb.}$$

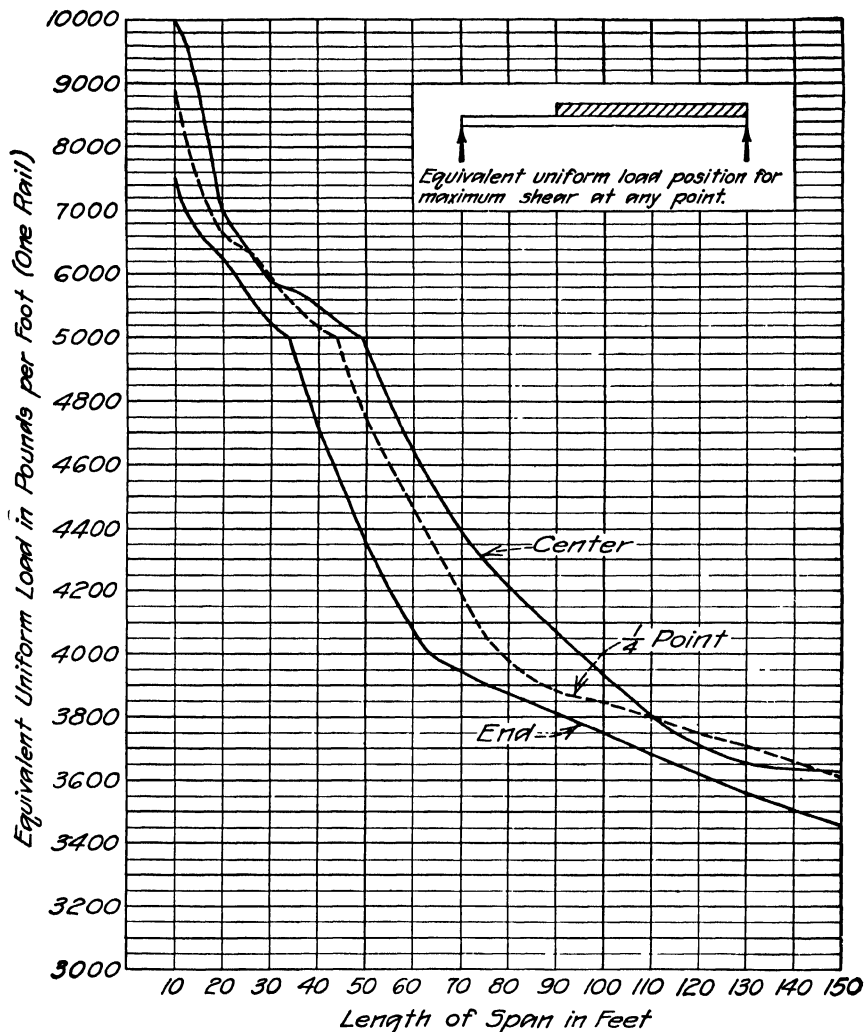



Fig. 165.

The exact shear due to E-50 loading is found to be 111,055 lb. (See p. 233, calculation for shear in panel BC.)

Figure 165 shows equivalent uniform loads for shears in a beam without floorbeams. These curves were obtained from the equation

$$w = \frac{2Vl}{(l - x)^2}$$

This equation was derived by equating V, the shear due to any concentrated load system and Eq. (19), p. 89. In the above equation, w = equivalent uniform load for shear, l = span, and x = distance from left end of beam to point at which the shear is required. In Fig. 165, values of V due to E-50 loading were calculated for the end, ¼ point, and center point of spans varying from 10 to 100 ft.

Illustrative Problem.—Calculate the shear at the 10-ft. point in a 60-ft. beam (see Fig. 156, p. 128), using the equivalent uniform load given by Fig. 165.

From Fig. 165 by interpolation, the equivalent uniform load for the ⅙ point is 4,320 lb. per ft. From Eq. (19), p 89,

$$V = \frac{(4{,}320)(50)^2}{(2)(60)} = 90{,}000 \text{ lb.}$$

The exact value of the shear due to E-50 loading, as calculated on p. 128 is 90,300 lb.

In all of the above problems it will be noted that the results calculated by equivalent uniform load methods differ only by a very small percentage from those calculated by the more exact wheel load method. Although the equivalent uniform loads of Fig. 163 were derived for simple beam conditions, it is often found possible to use them in the analysis of more complicated structures such as swing bridges, arches and double intersection structures.

Other equivalent uniform load systems which have been used to a considerable extent consist of a uniform load and certain excess loads which are intended to represent the effect of the heavy drive wheels of a locomotive. One system makes use of two excess loads spaced 50 ft. apart. Another system uses a single excess load. In both systems the concentrated loads are assumed as rolling on top of the uniform load, and are so placed as to give maximum moment or shear. The uniform load is generally taken equal to the assumed train loading and the concentrated loads are determined by trial so that the resulting moments and shears are equal to those given by the concentrated load system which they replace.

69. Moments and Shears Due to Combined Loading.—The loads carried by any structure consist of the fixed load due to its weight and the moving loads due to the service conditions for which the structure is designed. Hence the moments and shears used in designing the parts of the structure must be calculated for a combination of the dead and live load. These values of moments and shears are called *combined moments* and *combined shears*. To determine these combined moments and shears, values for dead and live loads may be calculated separately by the methods given in the preceding articles.

Moments calculated for dead and live load are always positive. Hence the combined moment is also positive and is equal to the sum of the dead and live load moments. For example, let Fig. 166 show a beam supporting uniform loads due to dead and live load. Ordinate fe represents the calculated dead load moment at any point and ordinate fg represents the live load moment at the same point. The sum of these ordinates, as

represented by *ge* is then the combined moment due to both loads. Similar methods may be used when the live load consists of a concentrated load system.

The shear in a beam due to live load may be either positive or negative, depending upon the position of the live load. To determine combined shear, the dead load shear must be combined with positive live load shear or with negative live load shear. Since the live load positive and negative shears are due to different loading conditions it is evident that separate combinations must be made for positive and negative live load shears. Dead load shear must be included in all combinations for it is present on the structure at all times.

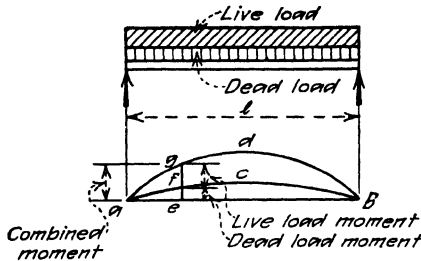



Fig. 166.

As an example of combined shears, consider a simple beam supporting a dead and live load both of which are uniform per foot. Figure 167*b* shows the dead load shear diagram; Fig. 167*c* shows the maximum positive live load shear diagram; and Fig. 167*d* shows the maximum negative live load shear diagram. These diagrams are constructed by the methods given in the preceding articles. The combined shear diagram for dead load and positive live load shears is shown by the line *CHE* of Fig.

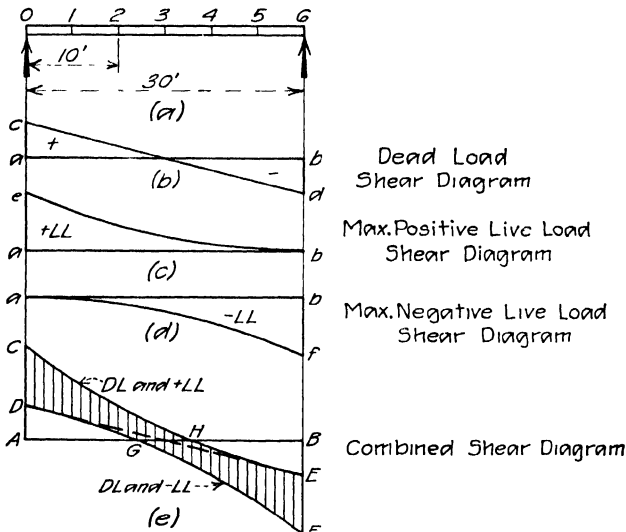



Fig. 167.

167*e*. Combined dead load and negative live load shears are shown by the line *DGF*. Note that the combined shear from *A* to *G* of Fig. 167*e* is always positive; from *H* to *B*, the combined shear is always negative; and from *G* to *H*, the combined shear may be either positive or negative.

Illustrative Problem.—Calculate the combined moments and combined shears at points 5 ft. apart on a simple beam of 30-ft. span due to a uniform dead load of 1,000 lb. per ft. and a moving uniform live load of 3,000 lb. per ft.

Combined Moments.—Dead and live load moments may be calculated by means of Eq. (6), Art. **65***a*. The necessary calculations are given in the table below. Figure 167*a* shows the beam under consideration.

TABLE 1.—COMBINED MOMENTS

Section	x	$(l - x)$	$x(l - x)$	Dead load moment	Live load moment	Combined moment
0	0	30	0	0	0	0
1	5	25	125	62,500	187,500	250,000
2	10	20	200	100,000	300,000	400,000
3	15	15	225	112,500	337,500	450,000

Symmetrical about center of beam
Moments given in foot pounds

Combined Shears.—Dead load shears may be calculated from Eq. (8), Art. **65***a*; positive and negative live load shears may be calculated from Eqs. (19) and (20) of Art. **65***b*. The calculated values are given in the following table.

TABLE 2.—COMBINED SHEARS

Section	Dead load shear	Live load shear		Combined shear	
		Max. pos.	Max. neg.	*DL* and +*LL*	*DL* and −*LL*
0	+15,000	+45,000	0	+60,000	+15,000
1	+10,000	+31,250	− 1,250	+41,250	+ 8,750
2	+ 5,000	+20,000	− 5,000	+25,000	0
3	0	+11,250	−11,250	+11,250	−11,250
4	− 5,000	+ 5,000	−20,000	0	−25,000
5	−10,000	+ 1,250	−31,250	− 8,750	−41,250
6	−15,000	0	−45,000	−15,000	−60,000

Shears are given in pounds

Moment and Shear Diagrams.—Moment and shear diagrams for the values given in the above tables are given by Fig. 166 for moment and Fig. 167*e* for shear.

70. Maximum and Minimum Moments and Shears.—The combinations of fixed and moving load moments and shears to be used depends upon the design methods adopted in proportioning the parts of the structure in question. In some cases only the maximum value of the combined moment or shear is required. In other cases maximum and

minimum values of combined moment or shear are required, for the parts of the structure are to be proportioned with reference to the *range* of moment or shear to which that part of the structure is subjected. Hence, when the dead load moments or shears are of the same kind (both positive or both negative), the maximum combined value is the sum of the dead and live load values. When the live load moment or shear may be either positive or negative, depending upon the position of the live load, the maximum combined value is the dead load value plus the live load value of the same sign, and the minimum combined value is the dead load value and the live load value of opposite sign.

Illustrative Problem.—Determine maximum and minimum values of combined moment and combined shear, using the values of moments and shears calculated in the problem given in the preceding article.

Maximum and Minimum Moment.—Moments due to dead and live load are both positive. Hence: Maximum Moment = dead load moment + live load moment; Minimum Moment = dead load moment. The desired results are given in Table 1, p. 139. Maximum moments are given in the column of combined moments and minimum moments given in the column of dead load moments.

Maximum and Minimum Shear.—Live load shears may be either positive or negative. For points to the left of the beam center, Maximum Shear = dead load and positive live load shears; Minimum Shear = dead load and negative live load shear. The desired values are given in the last two columns of Table 2, p. 139. To the right of the center, the combinations are the reverse of those given above.

71. Panel Concentrations. *Floorbeam Loads and Pier Reactions.*—In computing moments, shears and stresses in certain types of beams or trusses with floorbeams, it is often necessary to determine the loads at the panel points for some given position of the concentrated load system. These loads are known as *panel concentrations*, and also as *floorbeam loads*.

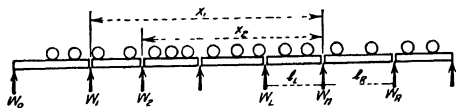



FIG. 168.

Figure 168 shows a system of concentrated loads carried by a series of stringers. Let the loads W_0, W_1, W_n, etc., represent the reactions at the junction points between stringers due to the given concentrated load system. When the series of stringers shown in Fig. 168 is supported by a beam or truss, W_0, W_1, W_n, etc., act as a load system on the beam or truss and become the panel concentrations mentioned above.

A general expression for value of any one of the panel concentrations of Fig. 168, as for example W_n, may be determined subject to the condition that the moment at any point due to the concentrated load system is equal to the moment at the same point due to the panel concentrations. Let W_n = any panel concentration; W_L and W_R = the panel concentrations on the left and right of W_n; M_L, M_n, and M_R = the summation of moments due to the concentrated load system about the position of

W_L, W_n, and W_R; x_1, x_2, etc. = distance from loads W_1, W_2, etc. to W_n; and l_L and l_R = length of panels adjacent to load W_n. Then

$$M_n = W_0x_0 + W_1x_1 + \text{etc.} = \sum_0^L Wx$$

where $\sum_0^L$ denotes that the summation of moments for panel concentrations covers loads W_L to W_0.

Also

$$M_L = \sum_0^L W(x - l_L) = M_n - l_L\sum_0^L W$$

and

$$M_R = M_n + \sum_0^n Wl_R = M_n + l_R\sum_0^L W + W_nl_R$$

On solving the last of these equations for W_n, and substituting for $\sum_0^L W$ its value as given by the second equation, we readily derive

$$W_n = \frac{M_L - M_n}{l_L} + \frac{M_R - M_n}{l_R} \tag{1}$$

which is a general equation for value of any panel concentration W_n of Fig. 168.

When the panels are all of equal length, the usual case in a truss or beam bridge, $l_L = l_R = d$, a panel length. Eq. (1) then becomes

$$W_n = \frac{M_L - 2M_n + M_R}{d} \tag{2}$$

Illustrative Problem.—A beam 100 ft. long is divided into 5 equal panels of 20 ft each, as shown in Fig. 169. Determine the panel concentrations W_1 to W_6 when wheel 10 of the E-50 train load is placed at a point 60 ft. from the left end of the beam, as shown in Fig. 169.

Fig. 169.

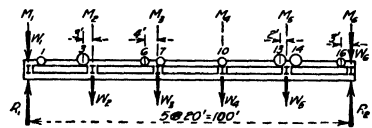
From the moment diagram of p. 103, $M_1 = 0$; $M_2 = 475$; $M_3 = 2{,}565$; $M_4 = 5{,}790$; $M_5 = 10{,}115$; and $M_6 = 16{,}018.75$.

From Eq. (2), the panel concentrations are found to be

$$W_2 = \frac{M_1 - 2M_2 + M_3}{20} = \frac{-950 + 2{,}565}{20} = 80.75$$

$$W_3 = \frac{M_2 - 2M_3 + M_4}{20} = \frac{475 - 5{,}130 + 5{,}790}{20} = 56.75$$

$$W_4 = \frac{M_3 - 2M_4 + M_5}{20} = \frac{2{,}565 - 11{,}580 + 10{,}115}{20} = 55.0$$

$$W_5 = \frac{M_4 - 2M_5 + M_6}{20} = \frac{5{,}790 - 20{,}230 + 16{,}018.75}{20} = 78.9375$$

In some cases the panel concentrations W_1 and W_6 are also desired. To determine W_1 note that M_L of Eq. (2) is zero. Then

$$W_1 = \frac{-2M_1 + M_2}{20} = \frac{0 + 475}{20} = 23.75$$

To determine W_6 assume that a panel containing no load is added at the right of load W_6. The moment at the right end of this span is found to be $M = M_6 + (322.5)(20) = 22{,}468.75$ (322.5 = total load on beam). Then

$$W_6 = \frac{M_5 - 2M_6 + M}{20} = \frac{10{,}115 - 32{,}037.5 + 22{,}468.75}{20} = 27.3125$$

As a check on the calculations, the sum of the panel concentrations must equal the total load on the structure.

The position of loads for maximum value of W_n of Eq. (1) may be determined by the methods used in preceding articles. Let G = total load on the series of stringers; G_L and G_R = loads on spans to left and right of W_n; and Δ = a small forward movement of the load system. The change in the several moments due to a forward movement of the loads can be shown to be as follows: Change in $M_R = G\Delta$; change in $M_n = (G - G_R)\Delta$; and change in $M_L = (G - G_L - G_R)$. Using these values, it can readily be shown from Eq. (1) that

$$\text{Change in } W_n = \frac{G_R}{l_R} - \frac{G_L}{l_L}$$

that is, the panel concentration is a maximum when the average load in the adjoining panels is equal. Note that this requirement is the same as for load position for maximum moment at any point in a beam, as given in Art. **68***b*, p. 109. Hence the proper load position for maximum W_n may be determined by means of Eq. (33), p. 108, applied as for a point distance l_L from the left end of a beam of length $(l_L + l_R)$, where l_L and l_R = lengths of panels adjoining W_n (see Fig. 168).

In a bridge consisting of a series of simple beams or trusses resting on piers, the load on any pier is the *panel concentration* for the spans resting on the pier in question. This load is known as a *pier reaction*. It may be calculated by means of Eqs. (1) or (2).

Illustrative Problem.—Determine the maximum value of the pier reaction at R_3 for the structure shown in Fig. 170. Assume E-50 train loading.

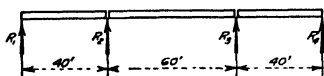



FIG. 170.

Maximum pier reaction occurs when loads are placed in position for maximum moment at the 60-ft. point in a 100-ft. beam. From Eq. (33), p. 108, or Fig. 142, p. 115, wheel 12 is to be placed at R_3. Let M with subscripts denote the moments at the several points. Then $M_1 = 0$; $M_2 = 137.5$; $M_3 = 8{,}385$; and $M_4 = 20{,}455$. From Eq. (1), with $l_L = 60$ and $l_R = 40$, we have

$$R_3 = \frac{M_2 - M_3}{60} + \frac{M_4 - M_3}{40} = \frac{137.5 - 8,385}{60} + \frac{20,455 - 8,385}{40}$$
$$= 163,300 \text{ lb.}$$

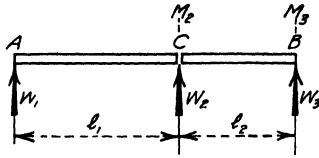



FIG. 171.

When the structure under consideration contains but two panels, Eqs. (1) and (2) reduce to somewhat simpler forms. Let W_2 = panel concentration for two consecutive spans, Fig. 171, of length l_1 and l_2, and let M_2 and M_3 be the moment of the concentrated loads to the left of W_2 and W_3.

For unequal spans, Eq. (1) becomes

$$W_2 = \frac{M_3 - M_2}{l_2} - \frac{M_2}{l_1}$$

This may be written in the form

$$W_2 = \left(M_3\frac{l_1}{l_1 + l_2} - M_2\right)\frac{l_1 + l_2}{l_1 l_2}$$

From Fig. 171, it can readily be seen that the expression in parenthesis is equal to the moment at C in a beam of span $(l_1 + l_2)$ due to the given concentrated load system. Let M_c denote this moment.

Then

$$W_2 = M_c\frac{l_1 + l_2}{l_1 l_2} \tag{3}$$

When the two adjoining spans are equal, $l_1 = l_2 = d$, and Eq. (3) becomes

$$W_2 = \frac{2M_c}{d} \tag{4}$$

Values of W_2 as given by Eqs. (3) or (4) are called the *floorbeam reactions* for the spans in question. These floorbeam reactions will have their maximum values when M_c is a maximum. The proper load position is given by Eq. (33), p. 108, or Fig. 142, p. 115.

On p. 142 is given a table of floorbeam reactions for equal spans varying in length from 10 to 250 ft.

Illustrative Problem.—Determine the floorbeam reaction for a beam with 25-ft. panels.

The value of M_c to be used in Eq. (4) is the maximum moment at the center of a 50-ft. beam. From Fig. 142, p. 115, wheel 4 at the beam center gives maximum moment. Wheel 8 is at the right end of the 50-ft. beam

$$M_c = (1,000)[(\tfrac{1}{2})(3,563.75) - 600] = 1,181,875 \text{ ft lb.}$$

From Eq. (4)

$$W_2 = \frac{2M_c}{25} = \frac{1,181,875}{12.5} = 94,600 \text{ lb.}$$

Table 3.—Maximum Moments, Shears, and Floor-beam Reactions for Cooper's E-50 Loading

Values given for one rail

Moments given in thousands of foot-pounds, shears and floor-beam reactions given in thousands of pounds. Floor-beam reactions are calculated for two panels each equal to the tabulated span. L and R denote that critical wheel is to the left or right of the beam center for absolute maximum moment.

Span in feet	Absolute maximum moment			Maximum shear			Floor-beam reaction
	Under wheel	Distance from beam center to critical wheel	Moment	End	¼ Point	Center	
10	2	1.25*L*	70.4	37.5	25.0	12.5	50.0
11	2	1.25*L*	82.1	40.9	26.1	13.6	54.5
12	3	0	100.0	43.8	27.1	14.6	58.4
13	3	0	118.8	46.2	27.9	15.4	61.6
14	3	0	137.5	48.2	29.5	16.2	65.2
15	3	0	156.3	50.0	31.3	16.6	68.3
16	3	0	175.0	53.1	32.9	17.1	71.1
17	3	0	193.8	55.9	34.3	17.3	73.5
18	3	0	212.5	58.3	35.4	17.4	75.9
19	3	1.25*L*	233.2	60.5	36.5	17.5	78.6
20	3	1.25*L*	257.9	62.5	37.5	17.5	81.9
21	3	1.25*L*	282.5	64.3	39.2	18.1	84.9
22	3	1.25*L*	307.1	65.9	40.9	18.8	87.6
23	3	1.25*L*	331.8	67.4	42.4	19.3	90.2
24	3	1.25*L*	356.5	69.3	43.8	19.8	92.4
25	3	1.25*L*	381.3	71.0	45.0	20.2	94.6
26	3	1.25*L*	406.0	72.6	46.1	20.6	97.1
27	3	1.25*L*	430.8	74.0	47.2	21.1	100.1
28	3	0.39*L*	456.9	75.5	48.2	21.4	102.8
29	3	0.39*L*	485.0	76.9	49.1	21.8	105.4
30	3	0.39*L*	513.0	78.8	50.0	22.1	107.9
31	3	0.39*L*	541.1	80.5	50.9	22.7	110.6
32	3	0.39*L*	569.3	82.1	51.8	23.4	113.7
33	3	0.39*L*	597.4	83.7	52.5	24.0	116.7
34	3	0.39*L*	625.8	85.1	53.5	24.6	119.4
35	4	0.94*R*	653.8	86.5	54.4	25.1	122.0
36	4	0.94*R*	685.8	88.2	55.1	25.8	124.4
37	4	0.94*R*	717.9	89.8	56.0	26.2	126.9
38	4	0.94*R*	750.0	91.4	56.7	26.6	129.7
39	4	0.21*L*	783.3	92.9	57.5	27.1	132.3
40	4	0.21*L*	819.5	94.3	58.5	27.5	135.0
42	4	0.21*L*	892.0	97.6	60.2	28.3	140.2
44	4	0.21*L*	964.5	100.7	61.9	29.0	145.6
46	4	0.21*L*	1,037.3	103.5	63.4	29.6	150.9
48	4	0.21*L*	1,109.5	106.3	65.1	30.2	156.0
50	4	1.45*L*	1,188.6	109.0	66.8	31.1	161.0
52	4	1.45*L*	1,269.0	111.8	68.5	31.9	166.6
54	13	0.13*L*	1,351.8	114.5	70.1	32.6	172.5
56	13	0.13*L*	1,440.5	117.2	71.8	33.3	178.5
58	13	0.13*L*	1,529.2	119.8	73.5	34.0	185.1
60	13	1.37*L*	1,624.5	122.5	75.2	34.9	191.5
62	13	1.37*L*	1,721.2	125.2	76.6	35.6	197.7
64	13	0.07*L*	1,819.4	128.2	78.0	36.4	203.6
66	13	0.07*L*	1,924.4	131.2	79.5	37.1	209.7
68	13	0.07*L*	2,029.4	134.8	81.0	37.8	215.6
70	13	0.07*L*	2,134.4	138.1	82.4	38.4	221.3
72	13	0.35*L*	2,241.2	141.7	83.8	39.2	226.9
74	13	0.61*L*	2,349.0	145.3	85.0	40.0	232.4
76	13	0.81*R*	2,465.0	148.8	86.5	40.8	238.0
78	13	0.54*R*	2,581.2	152.1	88.2	41.5	243.3
80	13	0.27*R*	2,700.6	155.3	89.6	42.1	248.6
82	12	0.14*L*	2,820.9	158.6	91.2	43.0	253.6
84	12	0.42*L*	2,945.4	161.8	93.0	43.7	258.7
86	13	1.15*R*	3,074.5	165.1	94.3	44.5	263.0
88	13	0.88*R*	3,205.3	168.4	96.5	45.2	268.3
90	13	0.60*R*	3,338.1	171.5	98.4	45.9	273.2
92	13	0.34*R*	3,470.9	174.7	100.4	46.6	278.0
94	13	0.07*R*	3,606.6	178.0	102.1	47.3	282.7
96	13	0.22*L*	3,743.1	181.0	104.1	47.9	287.5
98	13	0.49*L*	3,883.1	184.3	106.2	48.5	292.0
100	12	0.15*R*	4,024.9	187.5	108.2	49.2	296.5
105	12	2.01*R*	4,422.0	195.1	112.7	50.7	307.5
110	11	1.90*R*	4,858.5	202.5	117.4	52.3	318.5
115	11	3.52*R*	5,306.2	209.9	121.9	53.9	329.0
120	11	2.85*R*	5,767.6	217.1	126.4	55.6	340.0
125	11	2.19*R*	6,245.5	224.2	130.9	57.5	351.2
150	12	1.77*R*	8,827.9	259.2	152.2	68.0	406.7
175	12	1.79*L*	11,690.6	292.1	172.9	78.2	464.6
200	13	3.68*L*	14,841.2	326.3	191.8	88.0	523.8
225	14	5.67*L*	18,278.2	359.0	210.8	97.3	583.6
250	16	4.62*L*	21,990.6	391.5	229.6	106.3	644.0

INFLUENCE LINES

72. Influence Lines Defined.—If the load on a structure is always applied at the same point, the determination of stresses in the structure involves only the determination of the stresses in certain members due to a known load at a given point. There are, however, many structures designed to support loads which, instead of being applied at certain points, move along the structure. This is notably true of train loads on railway bridges; truck loads on highway bridges; and crane loads on crane runways. In designing structures supporting moving loads, it is not only necessary to determine the stresses due to a load at a given point, but it is also necessary to determine the position of the load which will produce the greatest stresses. The influence diagram has been developed to assist in the determination, in the case of a moving load, of the position of the load which produces the maximum stresses.

The sketch AC of Fig. 172 represents any simple beam. A load of

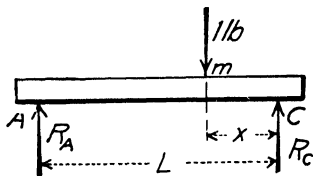
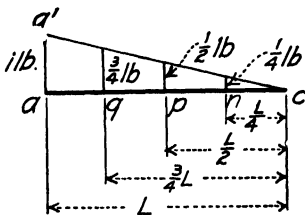



Fig. 172. Fig. 173.

1 lb. applied at m, a distance x from C, will produce a reaction R_A at A equal to $(1)\left(\frac{x}{L}\right)$—that is, the reaction at A is a function of x. By assigning different values to x we can compute the values of R_A for the corresponding positions of the 1-lb. load. Thus, if $x = 0$, $R_A = 0$; if $x = \frac{1}{4}L$, $R_A = \frac{1}{4}$ lb.; if $x = \frac{1}{2}L$, $R_A = \frac{1}{2}$ lb.; if $x = \frac{3}{4}L$, $R_A = \frac{3}{4}$ lb.; and if $x = L$, $R_A = 1$ lb.

In Fig. 173, draw the line ca representing the length of the beam CA. From c lay off the distance cn equal to $\frac{1}{4}L$. At n erect an ordinate whose length represents $\frac{1}{4}$ lb. This ordinate represents the reaction R_A when the 1-lb. load is a distance of $\frac{1}{4}L$ from C. Also on ac lay off cp equal to $\frac{1}{2}L$, and at p erect an ordinate whose length, to the same scale as was used at n, represents $\frac{1}{2}$ lb. This ordinate represents the reaction at R_A when the 1-lb. load is a distance of $\frac{1}{2}L$ from C. Likewise the ordinate at q represents the value of R_A when the 1-lb. load is $\frac{3}{4}L$ from C, and the ordinate at a represents the value of R_A when the 1-lb. load is a distance of L from C. The line $a'c$ passes through the upper extremity of the ordi-

nates at a, q, p, n and c. That is, the line $a'c$ is so constructed that its ordinate at any point represents, to scale, the reaction at A due to a load of 1-lb. applied at *a point on the beam corresponding to the position of the ordinate of the diagram.*

The line $a'c$ of Fig. 173 is the influence line, and the figure aca' is the influence diagram for the reaction at A. This diagram shows graphically the relative magnitudes of the reactions produced by a 1-lb. load in different positions. It shows the influence that the position of the load has upon the effect of the load.

The diagram of Fig. 173 shows the influence of the position of a 1-lb. load upon the reaction at A. Influence diagrams can also be drawn representing the influence of the position of a 1-lb. load upon any function of the load—such as the reaction at the end, the shear at the end, the moment at the center, etc. We have, therefore, finally as the definition of an influence line:

An influence line is a line whose ordinate at any point represents to scale the value of the function of a force for which the influence line is drawn when a 1-lb. load is applied at the point on the structure corresponding to the position of the ordinate on the diagram.

It should be noted that with an influence line the *ordinates of all points* represent the *same* function. The position of the load is the variable. Before drawing and before using an influence diagram it is important to consider carefully the function for which the diagram is drawn. It is not sufficient to realize that a certain influence line is the influence line for moment. We must clearly conceive that it is an influence line for moment *at a particular section*, and that the *whole line* is for the moment at that section and that section only.

73. Influence Lines for Simple Cases. *End Reactions of Simple Beams.*—The method of drawing the influence diagram for the reaction at A of the beam of Fig. 172 is described in the preceding paragraph. We found that the reaction at A due to a 1-lb. load at a distance x from C is given by the equation $R_A = (1)\left(\frac{x}{L}\right)$—that is, the reaction varies directly as the ordinates of a straight line so that the influence diagram for the reaction at A is a straight line whose ordinate represents zero at one end and 1-lb. at the other end.

Shear at End of Simple Beam.—For a simple beam the shear at the end is the same as the end reaction. The influence lines for the two functions are therefore identical.

Moment at Any Section of a Simple Beam.—In Fig. 174, AC represents any simple beam. It is required to draw an influence line for the moment at B.

With the load to the right of B, $M_B = (R_A)(a)$. But $R_A = (1)\left(\frac{x}{L}\right)$

Therefore, for the portion of the beam to the right of B, $M_B = \frac{xa}{L}$. That is, the influence line for the portion of the beam BC is a straight line. Its ordinate at C is zero and at B is $\frac{ab}{L}$.

With the load to the left of B, $M_B = (R_c)(b)$. But $R_c = (1)\left(\frac{L - x}{L}\right)$. Therefore, for the portion of the beam to the left of B, $M_B = (b)\left(\frac{L - x}{L}\right)$. That is, the influence line for the portion of the beam to the left of B is also a straight line. Its ordinate at A, where $x = L$, is zero, and at B its ordinate is $\frac{ab}{L}$. The influence diagram is therefore as shown in Fig. 174.

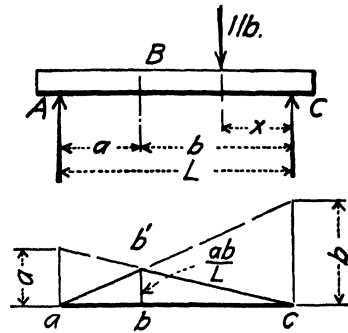



Fig. 174.

Shear at the Supported End of a Cantilever.—The member AB of Fig. 175 represents a cantilever. It is required to draw an influence line for the shear at A. In this case the shear at A is 1 lb. no matter where the load is applied so long as it is to the right of A. The influence

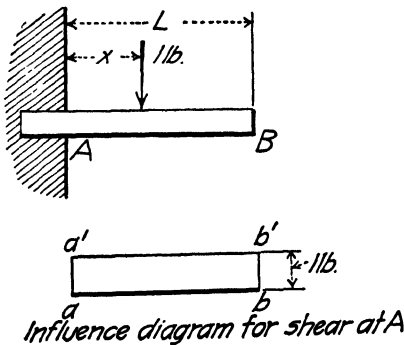
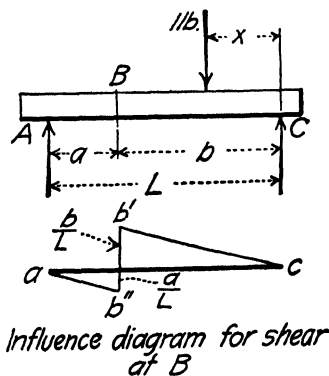



Fig. 175. Fig. 176.

line for the shear at A is therefore a horizontal line at a distance representing 1 lb. above the base.

Shear at an Intermediate Section of a Simple Beam.—The sketch AC of Fig. 176 represents a simple beam. It is required to draw the influence line for the shear at B.

When the load is to the right of B, the shear at B equals the reaction at A. When the load is to the left of B, the shear at B equals the reaction at A minus 1 lb. Therefore we have with the load on the portion of the beam BC, shear at B equals $\left(\frac{x}{L}\right)(1)$. The influence line is therefore a straight line whose ordinate at c is zero and whose ordinate at b is $\frac{b}{L}$. It is represented in the figure by $b'c$. And with the 1-lb. load on the portion of the beam AB, the shear at B is $\frac{x}{L} - 1$ or $\frac{x - L}{L}$. The influence line is therefore a straight line whose ordinate at A is zero and whose ordinate at B is $-\frac{a}{L}$. It is represented in the figure by ab''. The entire influence line is therefore $ab''b'c$.

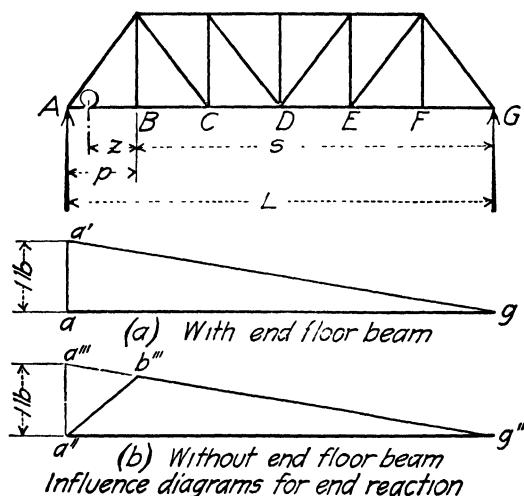



Fig. 177.

End Reaction of a Through Truss.—Figure 177 represents a truss of a through truss bridge. It is required to draw the influence line for the reaction at A.

There are two types of construction in common use. With one type of construction an end floorbeam carries the load from the stringers in the end panel into the truss at the end panel point. With the other type of construction, there is no floorbeam at the end panel point. Instead, the end stringers rest directly on the masonry so that the portion of the load in the panel AB that goes to A is not delivered to the truss, but goes directly into the masonry.

We will first consider the case in which there is an end floorbeam. In this case the end reaction is the same as for a beam having the same length and the same loading as the truss. The magnitude of the reaction

varies directly as the distance from G and the influence line is the straight line $a'g$, Fig. 177a, having an ordinate equal to zero at one end, and equal to 1 lb. at the other.

If the truss has no end floorbeam, the influence line will be as shown at b of Fig. 177.

For the portion of the beam to the right of B, the influence line is identical with the corresponding portion of the influence line for a truss having end floorbeams.

If the load is to the left of B, the end reaction of the truss equals the end reaction of a beam of the same length as the truss, less the stringer reaction at A. That is, $R_A = (1)\left(\frac{s+z}{L}\right) - (1)\left(\frac{z}{p}\right)$, which shows that the influence line from A to B is a straight line.

When z equals zero, $R_A = \frac{s}{L}$, the same as for a truss having end floorbeams. When $z = p$, $R_A = 0$. We have, therefore, the following construction for the influence line for the end reaction of a truss without end floorbeams:

From the base line $a''g''$ lay off the distance $a''a'''$ representing a reaction of 1 lb. Connect a''' and g'' with a straight line. Locate b'''

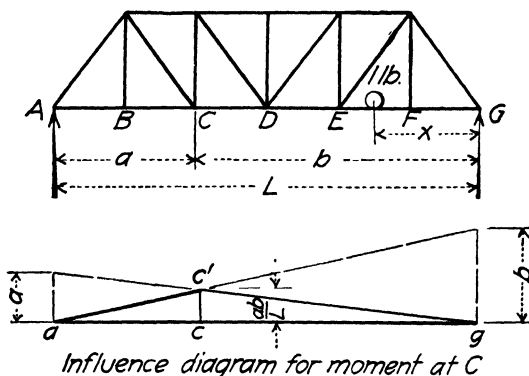



FIG. 178.

directly below B. Then $a''b'''g''$ is the influence line for the reaction of the truss.

Moment at a Panel Point of a Truss.—Figure 178 represents a truss of a through truss bridge. It is required to draw the influence line for the moment at the panel point C. The moment at a panel point of a truss is exactly the same as the moment at the corresponding point of a beam having the same length as the truss. Therefore $ac'g$ constructed exactly the same as the influence line for moment in a simple beam shown in Fig. 174, is the influence line for the moment at C in the truss of Fig. 178.

Shear in Any Panel of a Through Truss.—Figure 179 represents a truss of a through truss bridge. It is required to draw the influence line for the shear in the panel *BC*.

If the load is to the right of *C*, the shear in the panel *BC* equals the left-hand reaction. If, therefore, we draw *ag* as a base line, lay off the distance *aa′* to represent a shear of 1 lb., and connect *a′* and *g* with a straight line, then the portion of the line *a′g* to the right of *c′* is the influence line for the portion of the truss to the right of *C*.

If the load is to the left of *B*, the shear in the panel *BC* equals the reaction at the right-hand end of the truss. And the portion of *ag′* to the left of *b′* is the influence line for the portion of the truss to the left of *B*.

Influence diagram for shear in panel BC

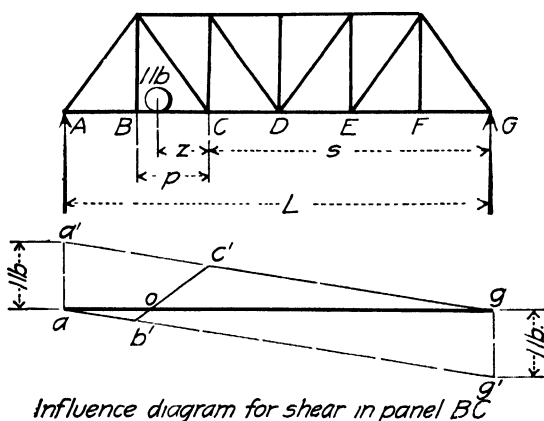



Fig. 179

We have yet to determine the influence line for the portion of the truss from *B* to *C*.

When the 1-lb. load is in the panel *BC*, a part of the load is delivered by the stringers through the floorbeam to *B* and a part to *C*. The shear in the panel *BC* due to a load of 1 lb. in *BC* equals the truss reaction at *A* less the floorbeam reaction at *B*. (This shear also equals the truss reaction at *G* less the floorbeam reaction at *C*.) The truss reaction at *A* equals $(1)\left(\frac{s+z}{L}\right)$. The floorbeam reaction at *B* equals $(1)\left(\frac{z}{p}\right)$. The shear in the panel *BC* therefore equals $\frac{s+z}{L} - \frac{z}{p}$. This is an expression of the first degree and therefore the shear due a 1-lb. load in the panel *BC* varies directly as the distance of the load from *C*. This being true, the influence line for the portion of the truss *BC* is a straight line. As one point on this influence line is *c′*, and another point is *b′*, the influence line for *BC* is *b′c′*. From an inspection of the influence line, it is apparent that a load to the right of *o*, produces a positive shear and a load to the

left of o produces a negative shear; that for a load to produce a maximum positive shear, it should be placed at C, and for it to produce a maximum negative shear, it should be placed at B; that to produce a maximum positive shear due to a uniform load, the load should extend from o to g, and that to produce a maximum negative shear, the load should extend from o to a.

74. Interpretation of Influence Lines.—It was stated in connection with the definition of influence lines that an influence line shows the influence which the position of a load has upon some function of the load. Thus Fig. 173 shows that the reaction at A produced by a load increases as the load moves from C to A. It is apparent from this influence line that a greater reaction at A will be produced by a load near A than by an equal load near B.

Figure 174 shows that as a load moves across the beam from C to A the moment at B increases until the load passes B and then decreases as the load moves away from B, and it shows that the moment at B due to a given load is a maximum when the load is at B.

Figure 175 shows that for a cantilever, the shear at the support is independent of the position of the load.

Figure 176 shows that for a simple beam the shear at an intermediate point increases numerically as the load approaches the point from either direction; and it shows that the shear passes from a maximum value of one sign to a maximum value of the opposite sign as the load passes the point for which the shear is to be determined. This diagram also shows that, in their effect upon the shear at B, loads placed, one to the right of B and one to the left of B, tend to offset each other.

75. Uses of Influence Lines.—Influence lines are used for three purposes:

(1) To determine the magnitudes of functions (shears, moments, stresses, or reactions) due to given loads.

(2) To locate a given load so that it will give a maximum value to a given function.

(3) To determine the portion of a structure to be loaded with either a uniform load or a series of concentrated loads in order that a given function may have a maximum value.

76. Determination of the Magnitude of a Given Function.

76*a*. Due to Concentrated Loads.—In Fig. 173 the ordinate at q represents ¾ lb. This means that a load of 1 lb. applied at a distance of ¾ L from C produces a reaction at A of ¾ lb. As the reaction is directly proportional to the load, 100 lb. ¾ of L from C will produce a reaction of (100)(¾) = 75 lb. From this illustration we can draw the general rule:

The function (moment, shear, stress, or reaction) due to a known load at a specified point is the product of two factors, one factor is the load, and

the other factor is the ordinate[1] of the influence line corresponding to the point at which the load is applied.

Illustrative Problem.—Figure 180 represents a beam carrying loads as shown. It is required to find the moment at B due to these loads.

The ordinate of the influence diagram at B is $\frac{ab}{L}$ (see Fig. 174) or $\frac{(8)(16)}{24}$. The ordinates at n and m are each equal to half the ordinate at b. The moment at B is therefore

$$(10)\left(\frac{1}{2}\right)\left[\frac{(8)(16)}{24}\right] + (20)\left[\frac{(8)(16)}{24}\right] + (40)\left(\frac{1}{2}\right)\left[\frac{(8)(16)}{24}\right] = 240 \text{ ft.-lb.}$$

Influence diagram for M_B

FIG. 180.

Influence diagram for shear at A

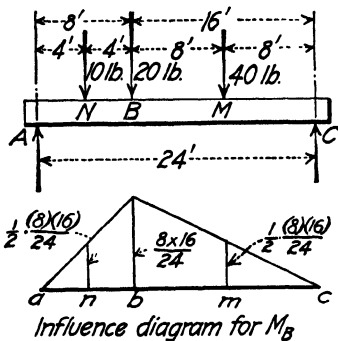
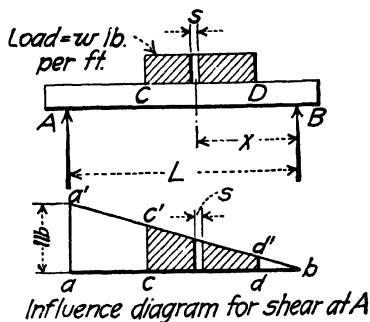



FIG. 181.

76b. Due to a Uniform Load.—In Fig. 181, AB represents a simple beam carrying a uniform load of w lb. per ft. on the portion CD. It is required to find the shear at A due to the uniform load CD.

The influence line is the straight line $a'b$ whose ordinate at a' represents 1 lb. and at b is zero. Consider a short section of the beam of length s that is located at a distance x from B. The magnitude of the load on this s is ws. The ordinate of the influence diagram at a distance x from B is $\frac{x}{L}$ (see footnote 2). Therefore the shear at A due to the load on the portion s is $\frac{wsx}{L}$.

The area of the influence diagram corresponding to the portion of the beam s is $\frac{xs}{L}$ (see footnote 2). This multiplied by the rate of loading, w,

[1] The ordinate of an influence diagram may be considered as representing an abstract number or it may be considered as representing all, or a part, of a unit in terms of which the function is usually expressed. If the ordinate represents an abstract number, then the ordinate is multiplied by the load in pounds (or tons). If the ordinate represents all or a part of a unit in terms of which the function is expressed, then the ordinate is multiplied, not by the load in pounds, but by the number of pounds in the load. Thus, if the load is 500 lb., in the first case we would multiply by 500 lb., in the second case by 500. The writer has preferred to use the latter method.

[2] These statements are approximately true when s is a very small finite quantity and may be considered as true when s is a differential quantity.

gives $\frac{wsx}{L}$, an expression identical with the expression for the shear at A due to the load on the portion s of the beam. We can therefore say that for the small length s of the beam the shear at A equals the area of the influence diagram corresponding to the portion s of the beam multiplied by the rate of loading. If this is true for each small s of the portion CD, it is true for the entire loaded portion.

We have finally, then, that in the case of a uniform load the value of a function is equal to the area of that portion of the influence diagram corresponding to the portion of the structure that is loaded, multiplied by the rate of loading.

Illustrative Problem.—The beam AB of Fig. 182 carries a uniformly distributed load as shown. It is required to determine the shear at A.

The influence diagram is aba'. The area of the diagram corresponding to the portion of the beam loaded is $\frac{3/4 + 1/4}{2}(10) = 5$ ft.-lb. The shear is therefore $(5)(1{,}000) = 5{,}000$ lb.[1]

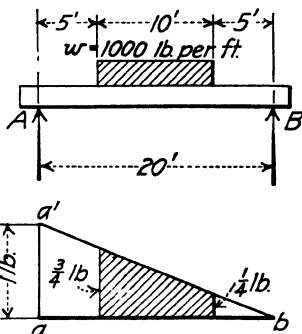



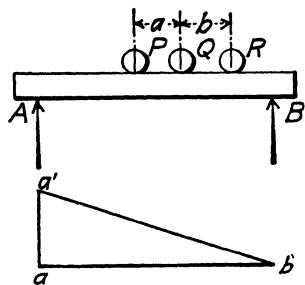
Fig. 182.

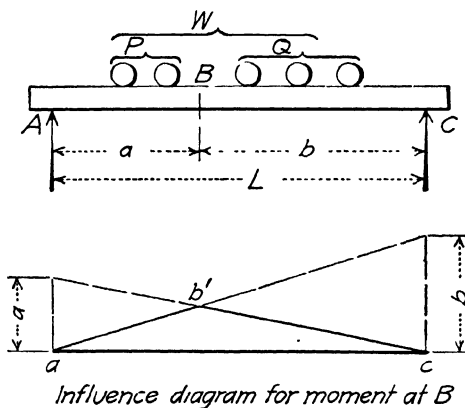
77. Location of a Concentrated Load to Give a Maximum Value to a Function.—Since the magnitude of a function due to a given load is the product of the load and the ordinate of the influence diagram, the position of a load for a maximum value of a function is apparent. Thus from Fig. 173 the maximum reaction at A due to a single load is obtained by placing the load at A where the ordinate of the influence diagram is a maximum; from Fig. 174 the maximum moment at B due to a single load is obtained by placing the load at B; from Fig. 176, the maximum shear at B of one sign is obtained when the load is just to the left of B, and the maximum shear of the other sign is obtained by placing the load just to the right of B.

78. Location of a Uniform Load, or of a Series of Concentrated Loads, to Produce a Maximum Value of a Function.—The most important application of influence lines is their use in determining the location of a uniform load or of a series of concentrated loads which will give the maximum value to a given function.

With certain types of structures and with certain functions, the position of a series of moving loads which gives the function its maximum value is apparent from a glance at the influence diagram. Thus in Fig. 183, $aa'b$ is the influence diagram for the shear at A. Since the

[1] Since the area of the influence diagram represents foot-pounds, the rate of loading is represented, not as the load per foot, but as the number of pounds per foot. Not 1,000 lb. per ft., but 1,000 per ft.

ordinates of the influence diagram increase from b to a', it is apparent that the loads should be placed as far to the left as possible without causing the load P to pass off of the span. For as the loads move to the left, the ordinate of the influence diagram for each load increases and therefore the shear produced by each load increases.

Influence diagram for shear at A
FIG. 183.

With certain other types of structures or with certain other functions, the position of a series of moving loads which gives the function its maximum value is not apparent by inspection, but can be determined from the geometrical construction of the influence diagram. Thus in Fig. 184 $ab'c$ is the influence diagram for the moment at section B of the beam AC. Q represents the sum of all the loads to the right of B, and P represents the sum of all of the loads to the left of B. As the loads move, say to the left, the ordinates of the influence diagram for the loads which help to make up Q increase, whereas, the ordinates of the influence diagram for the loads which help to make up P decrease. Apparently therefore as the loads move to the

Influence diagram for moment at B
FIG. 184.

left, the moment due to Q increases and the moment due to P decreases. We wish to find the position of the loads which will make the sum of these two moments, one of which is increasing, and the other of which is decreasing, a maximum.

If the line $b'c$ is extended, it will cut the vertical through A at a distance a above a. The slope of $b'c$ is therefore $\frac{a}{L}\cdot$ Likewise if the

line ab' is extended it will cut the vertical through C at a distance b above c. The slope of ab' is therefore $\frac{b}{L}$.

Consider that all of the loads have moved a small distance to the left, but that no load has passed B—that is, neither P nor Q have changed in magnitude. Let the movement to the left be represented by s. The ordinate of the influence diagram corresponding to each load composing Q, has been increased by an amount $\frac{sa}{L}$. As the ordinate for each load has been increased by this amount, and as the increase in the ordinate times the load equals the increase in the moment, the moment due to Q increased by the amount $Q\left(\frac{sa}{L}\right)$ when the loads moved a distance s to the left. Likewise, when the loads moved to the left, the ordinate of the influence diagram for each load composing P decreased by the amount $\frac{sb}{L}$, and the moment due to P decreased by the amount $\frac{Psb}{L}$. The resultant change in the moment due to all of the loads, is therefore, $\frac{Qsa}{L} - \frac{Psb}{L}$. If the first term of this expression is greater than the second, the moment increases as the loads move to the left. If the first term is less than the second, the moment decreases as the loads move to the left.

The only quantities in these expressions that can change in value are Q and P. When a load passes B, Q decreases and P increases by the amount of the passing load. Suppose that just before a certain load passes B, $\frac{Qsa}{L}$ is greater than $\frac{Psb}{L}$, and that after the load passes B, $\frac{Qsa}{L}$ is less than $\frac{Psb}{L}$. This means that as the load approaches B, the sum of the moments of all the loads is increasing and that as the load leaves B, the sum of the moments of all of the loads is decreasing. If the moment increases until the load reaches B and decreases after the load leaves B, the moment is a maximum when the load is at B. We can therefore say that if $\frac{Qsa}{L}$ changes from *greater than* to *less than* $\frac{Psb}{L}$ as the load passes B, the moment is a maximum with the load at B. Sometime during this change

$$\frac{Qsa}{L} = \frac{Psb}{L}$$

Eliminating the common terms s and L

$$Qa = Pb$$

Adding Pa to both terms

$$Qa + Pa = Pb + Pa$$

or

$$Wa = PL$$

Dividing both terms by aL

$$\frac{W}{L} = \frac{P}{a}$$

The expression $\frac{W}{L}$ is the total load divided by the total length, or the rate of loading for the whole span. And the expression $\frac{P}{a}$ is the rate of loading for the left hand segment. We have therefore, finally, as the criterion for determining the position of a series of moving loads to produce a maximum moment at a given section:

If, as a load approaches the section from the right, the average load on the left-hand segment is less than the average load on the whole span, and, as the load leaves the section, the average load on the left-hand segment is greater than the average load on the whole span, then the moment is a maximum at the section when the load is at the section.

If the series of concentrated loads is replaced by a uniformly distributed load, the criterion takes the form:

The moment at a given section of a beam is a maximum when the average load on one segment (either segment) equals the average load on the whole span.

GENERAL METHODS OF COMPUTING STRESSES IN TRUSSES

79. Two Methods Used.—The stresses in the members of a truss may be computed either by a "method of sections" or by a "method of joints." It is often convenient to compute the stresses in some of the members of a truss by one method and the stresses in the remaining members by the other method.

In either method the necessary procedure, in order to determine stresses for a given loading, is to separate the given truss into two parts by an imaginary section, either plane or curved; the part of the truss to one side of the section is removed (that is, considered so) together with all external forces, and the members that are cut by the section are replaced by the stresses acting in those members. By so doing, the part of the truss considered will be in equilibrium due to the outer forces acting on that portion of the truss and the stresses in the members cut. If the section is taken completely across the truss, as XX' or YY', Fig. 185a, so that the members cut *do not* all intersect in one point, then the method

used is the *method of sections*. If the section is so taken that the members *do* all intersect in one point, as ZZ', Fig. 185*a*, then the method used is the *method of joints*.

80. Algebraic Treatment.—The algebraic treatment of the *method of sections* will be explained with reference to the truss shown in Fig. 185*a* which is subjected to moving loads transmitted to the lower panel points. Assume that the maximum stresses in members 1, 2 and 3 of the truss are required, these members being cut by the section XX'. Consider the portion of the truss shown in Fig. 185*b*. For a definite loading the forces are all in equilibrium as explained above and, since only three members are cut, any or all of the three equations of equilibrium can be used; namely, $\Sigma H = 0$, $\Sigma V = 0$, and $\Sigma M = 0$ (see Art. **38,** p. 19).

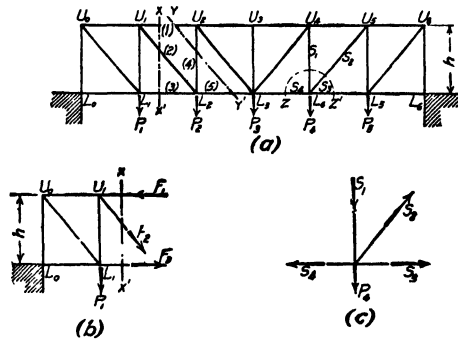



FIG. 185.

First use the equation $\Sigma M = 0$. This equation is true about any point in the plane of the truss but, in order to get the stress in a given member directly, it is necessary to take the center of moments at the intersection of the other two members. For example, the stress in F_3 for a given loading can be found by taking moments about the point U_1. It should be noticed that U_1 is vertically above L_1 and, since the loads are all vertical, the moments at U_1 and L_1 are equal. The *maximum* stress in F_3, then, occurs with the loading which gives maximum moment at the first panel point from the left support (see chapter on "Shears and Moments"). Call this maximum moment M_1. The moment of F_3 (when F_3 is a maximum) about the point U_1 must be equal and opposite to M_1 in order that ΣM may equal zero. Thus

$$(\text{Max. } F_3)(h) = M_1$$

or

$$\text{Max. } F_3 = \frac{M_1}{h}$$

In the same manner, calling M_2 the maximum moment at the second panel point,

$$\text{Max. } F_1 = \frac{M_2}{h}$$

It should be observed (using $\Sigma M = 0$) that the stress in the upper chord acts toward the section, thus denoting compression, while the stress in the lower chord acts away from the section, thus denoting tension; that is, F_1 = compression and F_3 = tension. This is true of all the upper and lower chords throughout the truss.

The maximum stress F_2 remains to be found. This may be accomplished by using the equation $\Sigma V = 0$. The vertical component of the maximum stress in F_2 is equal to the maximum positive shear in the second panel from the left support. Call this component V_2. Then

$$\text{Max. } F_2 = V_2 \frac{U_1L_2}{h}$$

In using the equation $\Sigma V = 0$, observe that the stress acts away from the section, thus denoting tension.

Let the maximum stress be required in members 1, 4, and 5, Fig. 185*a*. Take the section YY'. Using $\Sigma H = 0$, and knowing that the loads are all vertical, the stress in member 1 is seen to be equal and opposite to the stress in member 5. This applies for any loading, hence the loading giving maximum stress in member 1 will also give a maximum stress in member 5 of the same amount; that is, the loading giving the maximum moment at the second panel point from the left support will cause maximum stress in both members 1 and 5. The maximum stress (compression) in member 1 is, as before, $\frac{M_2}{h}$, using $\Sigma M = 0$. This same amount of tension, then, occurs in member 5. The maximum stress in member 4 is directly the maximum positive shear in the third panel from the left support, using the equation $\Sigma V = 0$. Stress in member 4 is compression.

In the method of sections, the section should always be taken so as to cut only three members whose stresses are unknown. If more than three members are cut, there are more unknown quantities than can be found by the principles of statics.

The method of joints is only a name given to the manner of determining stresses from the conditions of equilibrium of concurrent forces, namely, $\Sigma H = 0$ and $\Sigma V = 0$. It should be clearly understood that this method can be applied to a joint only when there are two unknown stresses. In solving a truss by this method, it is evident that a joint must be selected where but two members meet and then proceed from this to other joints.

In the algebraic method of joints, if a maximum stress is desired in a certain member of a truss, all the joints from one end of the truss up to the member considered must be computed for the loading giving maximum stress in that member only. For this reason the algebraic method of joints, although perfectly general, is too laborious to be employed in practice in determining the maximum stresses in all the members of an

ordinary truss. It may be used with great advantage, however, for certain specific members, and should be understood. A graphical method based upon the same principles is well adapted for many types of trusses, particularly roof trusses with non-parallel chords. In roof trusses, the conditions for probable maximum stress in the given members are few, and usually all the stresses may be computed graphically for each loading in much shorter time than it would take to compute the stresses throughout the truss algebraically for any *one* condition of loading.

Illustrative Problem.—Roof truss of Fig. 186*a*; loads as shown. (*a*) Required the stresses in all members algebraically by the method of sections. (*b*) By the method of joints.

(*a*) **Method of Sections**

To find the stresses in member L_0U_1 and L_0L_1 pass a section *a-b* cutting these members. Consider the truss to the left of the section. Figure 186*b* shows the joint at L_0 removed and the known loads applied, together with the unknowns S_1 and S_2, assumed to act as shown. Consider upward forces and forces to the right as positive; downward forces and forces to the left as negative. The two equations, $\Sigma V = 0$ and $\Sigma H = 0$, may be employed to find the two stresses S_1 and S_2.

$\Sigma V = 0.\quad 4\,000 - 1{,}000 - S_1 \sin \theta = 0$

$S_1 = (3{,}000)\left(\frac{22.36}{10}\right) = 6{,}710$ lb. (compression, as assumed, since result is positive).

$\Sigma H = 0.\quad S_2 - S_1 \cos \theta = 0$

$S_2 = (6{,}710)\left(\frac{20}{22.36}\right) = 6{,}000$ lb. (tension, as assumed, since result is positive).

To find the stresses in members U_1U_2, U_1L_2, and L_1L_2, pass a section *c-d* cutting these members and consider the portion of the structure to the left (Fig. 186*c*). The

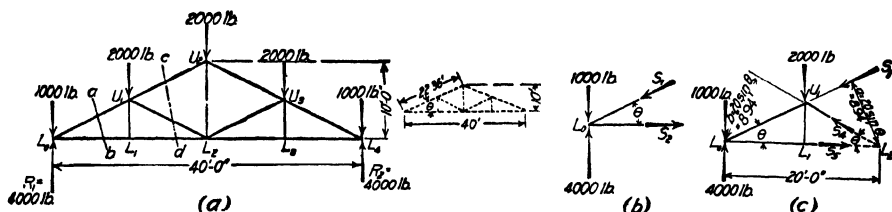



Fig. 186.

three equations of equilibrium may be used to determine the three unknown stresses, but the solution may be simplified by employing only $\Sigma M = 0$ three times. This equation should be applied at the intersection of two members to find the stress in the third. Thus, to determine the stress in U_1U_2, take moments about L_2, the intersection of U_1L_2 and L_1L_2. Then, considering clockwise moments as positive,

$$4{,}000(20) - 1{,}000(20) - 2{,}000(10) - S_3(a) = 0$$
$$S_3 = 4{,}470 \text{ lb. (compression)}$$

The stress in S_4 may be obtained by taking moments about L_0, the intersection of U_1U_2 and L_1L_2.

$$2{,}000(10) - S_4(b) = 0$$
$$S_4 = 2{,}240 \text{ lb. (compression)}$$

The stress in S_5 may be found by taking moments about U_1, the intersection of U_1L_2 and U_1U_2.

$$(4{,}000 - 1{,}000)(10) - S_5(5) = 0$$
$$S_5 = 6{,}000 \text{ lb. (tension)}$$

Other sections should now be taken cutting only three members whose stresses are unknown and the moment equation again applied. Proceeding in this manner the stresses in all the members may be determined.

(a)

(b)

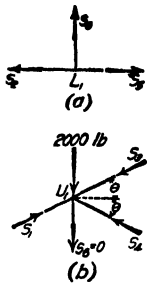



FIG. 187.

(b) **Method of Joints**

The stresses in members L_0U_1 and L_0L_1 are determined as for the method of sections and the solution will not be repeated here.

Passing now to the next joint at which only two unknowns exist, joint L_1 will be selected, shown in Fig. 187a.

$$\Sigma V = 0. \quad S_6 = 0$$
$$\Sigma H = 0. \quad S_5 - S_2 = 0$$
$$\text{or } S_5 = S_2 = 6{,}000 \text{ lb. (tension)}$$

Next pass to joint U_1, which is shown in Fig. 187b. The two unknown forces are S_3 and S_4.

$$\Sigma V = 0. \quad S_1 \sin\theta + S_4 \sin\theta - S_3 \sin\theta - 2{,}000 = 0$$
$$S_4 \sin\theta - S_3 \sin\theta = -1{,}000$$
$$\Sigma H = 0. \quad S_1 \cos\theta - S_4 \cos\theta - S_3 \cos\theta = 0$$
$$S_4 \cos\theta + S_3 \cos\theta = 6{,}000$$

These independent equations involve only the unknowns S_3 and S_4. Solving simultaneously

$$S_4 - S_3 = -2{,}236$$
$$S_4 + S_3 = +6{,}708$$
$$S_3 = 4{,}470 \text{ lb. (compression)}$$
$$S_4 = 2{,}240 \text{ lb. (compression)}$$

The stresses at joint U_1 are now completely determined. In the same way pass to the other joints until all the stresses in the members of the truss are determined.

81. Graphical Treatment.—In the graphical method of sections it is necessary to commence at one end of the structure and pass a section cutting but *two* members. The stresses in these members can be determined by the single condition that the force polygon, drawn from the forces on one portion of the structure, must close. Next a section is taken cutting *three* members, one of which has already been determined, and the two unknowns can be found by the force polygon method as before. By successive sections taken in this manner, all the stresses can be determined by simple force polygons.

The graphical construction resulting from the method of joints is identical with that resulting from the method of sections. The only difference is the sections taken and, consequently, the order in which the lines are drawn. The method of joints is generally preferred in practice on account of its simplicity and this method only will be illustrated here.

Illustrative Problem.—Required the stresses in all members of the roof truss shown in Fig. 188*a* by the graphical method of joints; loads as shown.

It will simplify matters to draw a sketch of the truss to some suitable scale and show on it all the outer forces including reactions. Also, to designate all the forces and members on this sketch by letters so located that each force and each member will lie between two letters and only two, as illustrated in 188*a*.

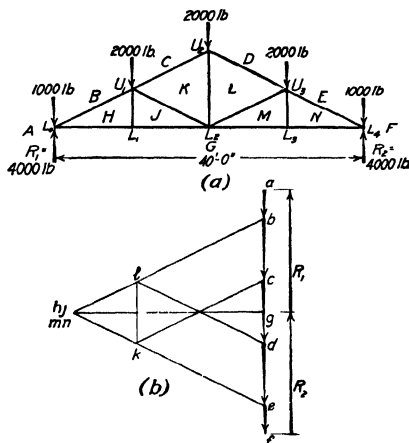
Now any force, as AB, for example, in this figure may be designated in the graphical solution by a line having a length corresponding to the magnitude of the force and with the letter a at one end and the letter b at the other. By going through the graphical construction in this manner one letter only need be placed at each apex of a force polygon and the work is greatly simplified.

The next step is to draw a force polygon for the outer forces to a scale of sufficient size to give the desired accuracy. The force polygon is *abcdefga* in Fig. 188*b* and is a straight line, since all the forces are vertical. The external forces should be plotted in the order obtained by going around the figure in a clockwise direction. $ab = 1{,}000$. $bc = cd = de = 2{,}000$. $ef = 1{,}000$. $fg = R_2 = 4{,}000$. $ga = R_1 = 4{,}000$. The right and left reactions must previously be computed either algebraically or graphically (see chapter on "Reactions").

Fig. 188.

The force polygon should now be drawn for joint L_0. The unknown forces which act at this joint are the stress in BH and the stress in HG. bh and hg are known in direction but not in magnitude, hence, there are but two unknowns and these can be found by the polygon of forces. The figure *abhga*, Fig. 188*b*, is this polygon obtained by drawing from b a line parallel to BH, and from g a line parallel to HG. The lines bh and hg may now be scaled from the force polygon to obtain the magnitude of the stresses in the two members intersecting at L_0. The character of these stresses must also be found. The forces at joint L_0, being in equilibrium, must follow in order around the corresponding force polygon. Reading around joint L_0 in a clockwise direction gives bh acting downward to the left, or toward the joint L_0, thus showing compression, and hg acting toward the right, or away from the joint L_0, showing tension.

The joint L_1 is the next one at which only two unknowns exist. The stress in GH is known from joint L_0, and the stresses in HJ and JG are unknowns. The corresponding force polygon hjg for this joint must close. Since gh and jg have the same line of action, the line in the force polygon representing the magnitude of hj will be a point, thus having no length. The stress in HJ is, therefore, zero. This might have been seen by inspection, as there is no load at L_1 to cause stress in this member. In reading around joint L_1 in a clockwise direction, the line JG is from left to right, and the stress acts away from joint L_1, denoting tension.

Now pass to joint U_1. The stresses in CK and KJ are the unknowns. To obtain them draw ck and jk in the force polygon parallel respectively to the corresponding members in the truss. (The stress being zero in JH, the whole space occupied by J and H may conveniently be called J.) Reading around joint U_1 in a clockwise direction gives both ck and kj acting toward the joint U_1, hence, denoting compression in both these members. The polygon considered is *bckjb*. In a similar manner the stresses in the other bars may be determined.

SECTION 2

ROOF TRUSSES

SIMPLE ROOF TRUSSES

1. Methods of Computing Stresses.—The two general methods of computing stresses in trusses are the "method of sections" and the "method of joints," as explained in the preceding chapter.

2. Algebraic Method of Sections.—To determine the direct stress in the member of a truss, the following procedure should be used:

(1) Pass a section through the unknown member and remove part of the truss to one side of the section.

(2) Replace cut members by forces, assuming the directions of the forces.

(3) Take moments about a point which is common to the lines of action of all unknowns but the one desired.

(4) Determine the magnitude and direction of the unknown force by equating the algebraic sum of the moments to zero.

If the force which is to be determined acts toward the section, the member will be in compression; if it acts away from the section, the member will be in tension.

Illustrative Problem.—The stresses in the Pratt truss shown in Fig. 1 will be determined by the algebraic method, for the loads shown. Before beginning the determination of moments acting on sections of the truss, it will be convenient to determine the right-angle distances of upper chord members from lower panel points and the right-angle distances of web members from the heel joint, L_0.

The first section is taken through L_0U_1 and L_0L_1 and the part of the truss to the right of the section is removed, as shown in Fig. 1*b*. The members are replaced by forces, as indicated by the arrows. In order to determine the stress in L_0U_1, the moments are taken about L_1, so as to eliminate the stress in L_0L_1, from the computations. In order to determine the stress in L_0L_1, the moments are taken about U_1 for a similar reason. The solutions of the equations give

$$L_0U_1 = (3{,}000)\left(\frac{10}{4.47}\right) = 6{,}710 \text{ lb.}$$

$$L_0L_1 = (3{,}000)\left(\frac{10}{5}\right) = 6{,}000 \text{ lb.}$$

Because the sum of the moments about L_1 must equal zero, the force L_0U_1 must be directed toward the section; therefore the member L_0U_1 will be in compression.

Because the sum of the moments about U_1 must equal zero, the force L_0L_1 must be directed away from the section; therefore, the member L_0L_1 will be in tension.

The second section is taken as shown in Fig. 1c, the cut members being replaced by forces. In order to determine the stress in U_1L_1 the moments are taken about L_0; and in order to determine the stress in U_1U_2 the moments are taken about L_1. The directions of the forces are determined as before.

The third section is taken as shown in Fig. 1d and the cut members are again replaced by forces. The stresses and their directions are determined as in the previous cases.

It should be observed that, if a section is passed through three unknowns, any one of them can be determined by taking moments of all the forces acting about the intersection of the other two unknowns.

The stresses in a symmetrical truss loaded symmetrically need be determined only for one-half the truss.

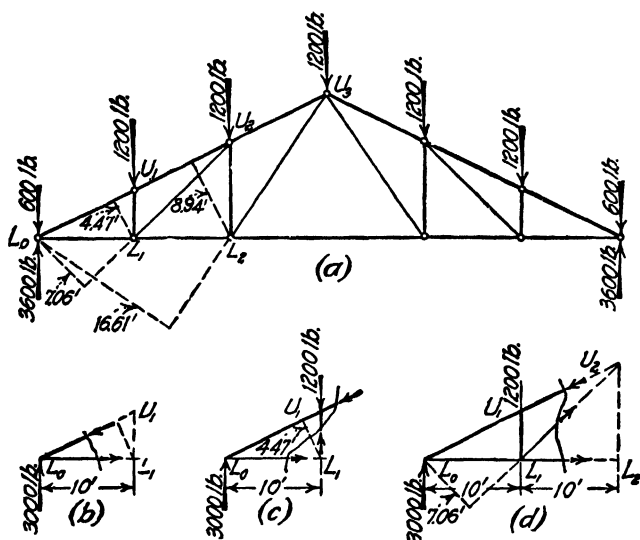



FIG. 1.

3. Methods of Equations and Coefficients.—The method of determining the stresses in symmetrical trusses, symmetrically loaded, by means of equations or coefficients involves the least amount of labor.

Equations for stresses in members can be determined in terms of the panel load and the ratio of span to height of truss, by the algebraic method of sections, the loads being expressed in panel loads and the moment arms in terms of span divided by height. These equations give constant values, or coefficients, for each member of a truss for each particular ratio of span divided by height. The value for any member, when multiplied by the panel load will give a product, which will be the stress in the member.

The equations for stresses and the coefficient of stresses for the standard simple types of symmetrical trusses are given in the chapter on "Roof Trusses—Stress Data."

4. Graphical Method of Joints.—In the graphical method of computing stresses, joints are considered to be cut from the truss in consecutive order and a force polygon is drawn for the forces at each joint. The stresses should be determined by use of the following procedure:

(1) Draw a scaled diagram of the truss showing all the external forces, and letter each space between forces or members with a capital letter.

(2) Consider each joint separately as a "free body" acted upon by concurrent forces in equilibrium.

(3) Draw a force polygon for each joint showing the external and internal forces and letter each intersection of forces with a small letter corresponding to the space between the forces in the space diagram.

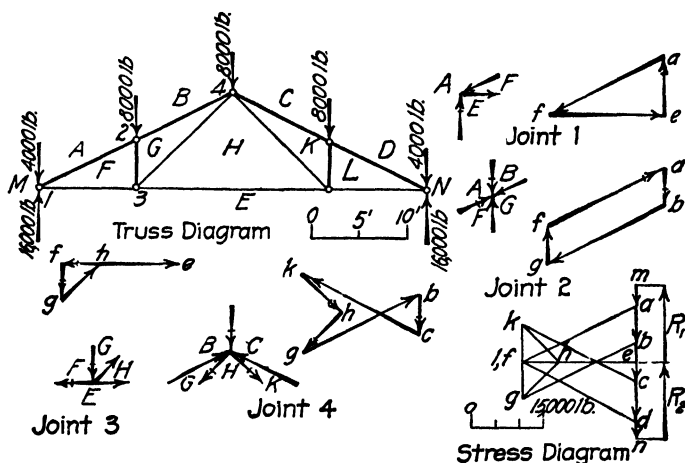



Fig. 2.

Illustrative Problem.—The stresses in the truss of Fig. 2 will be determined by the graphical method for the loads shown.

The heel joint, joint 1, is the first to be solved. The one-half panel load at the joint and the reaction are combined to give the effective reaction. The force polygon for the joint is drawn with the forces parallel to the lines of action shown in the space diagram. Since the sum of the horizontal components and the sum of the vertical components must equal zero for equilibrium, the polygon must close. The order of letters as read around the force polygon indicates the direction of the forces acting at the joint and thereby indicates whether a member is in compression or tension. If the force acts toward the joint, the member which transmits it must be in compression; if it acts away from a joint the member must be in tension.

Joint 2 is the next joint to be solved. The procedure used in the solution of joint 1 is followed. The known forces are marked with a line across the arrow in the space and force diagrams. It should be noted that no more than two unknowns can be determined in the solution of any one joint.

The solutions of joints 3 and 4 follow in order and complete the solutions for the truss.

It is not necessary to draw separate space and force diagrams for each joint, as the *truss diagram* gives the space diagrams for all joints and the force diagrams may be combined into one *stress diagram* as shown in the figure.

Illustrative Problem.—The stresses in the King-rod truss of Fig. 3*a* for the roof and suspended-ceiling loads shown will be determined by the graphical method.

The truss diagram is first drawn to scale and all the external forces (loads and reactions) are indicated on the diagram. To construct the stress diagram, first plot to scale all the loads on the truss rafters, *i.e.*, *ab*, *bc*, *cd*, *de*, and *ef*. R_1 is then laid off from *a* and in opposite direction to *ab*, *bc*, etc., and R_2 is laid off from *f*. The two reactions are found to overlap because the suspended loads on the lower chord have the same line of action as the loads on the rafters at the panel points above. The left-hand heel joint is first considered by plotting the stresses in a clockwise direction around the joint. The stress polygon is obtained by drawing *bm* and *ml*, from *b* and *l*, parallel to *BM* and *ML* respectively. Tracing this joint through by a continuous clockwise

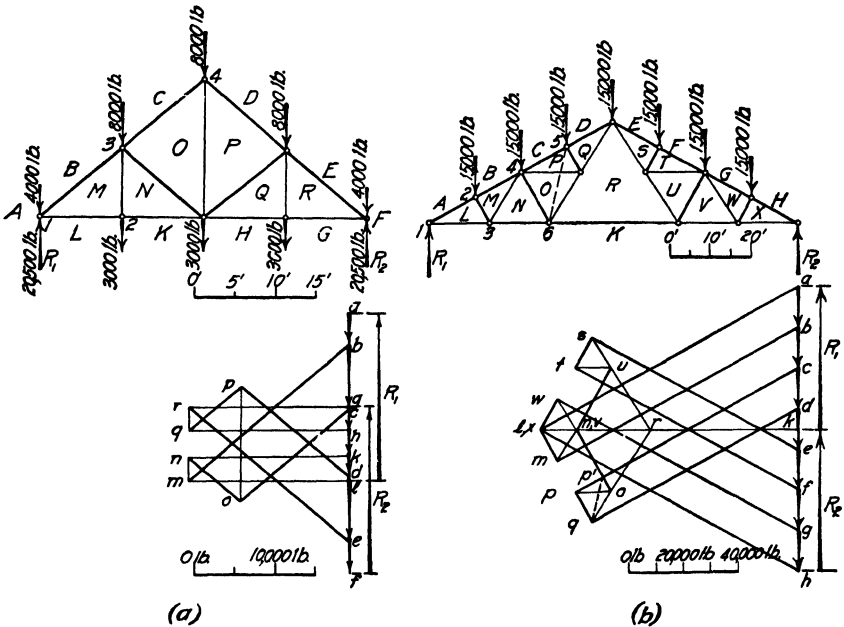



Fig. 3.

reading of the forces, *bm* is found to act toward the joint and *ml* to act away from the joint, which means that these stresses are compression and tension respectively.

The first lower chord joint from the left reaction is next determined. The forces are again traced in a clockwise direction beginning with the known force *kl*. In this force diagram it is found that *mn* and *nk* both act away from the joint and members *MN* and *NK* are, therefore, in tension.

Joints 3 and 4 are solved in the same manner, which completes the determination of stresses, as the stresses on the right-hand side of the truss are equal to those on the left. The stress diagram may be completed as a check on the work.

Illustrative Problem.—The dead load stresses in the Fink truss shown in Fig. 3*b* will be determined by the graphical method. A special feature of this solution is the condition encountered at joint 4 which may at first appear to be an indeterminate condition.

The truss diagram is drawn to scale and the loads and effective reactions are plotted.

The joints are solved in the usual manner in the order indicated on the truss diagram. Bringing the solution from left to right, a condition which cannot at once be solved is met at joint 4. There are three unknowns *cp*, *po*, and *on*. It is seen on inspection that the stress in the members *DQ*, *QR*, and *RK* will remain the same regardless of the web members toward the left. *OP* and *PQ* are, therefore, cut out and replaced by the dotted member *P'Q*. Joints 4, 5, and 6 are determined with this assumed member in place, and joint 6 is then corrected by throwing out the dotted

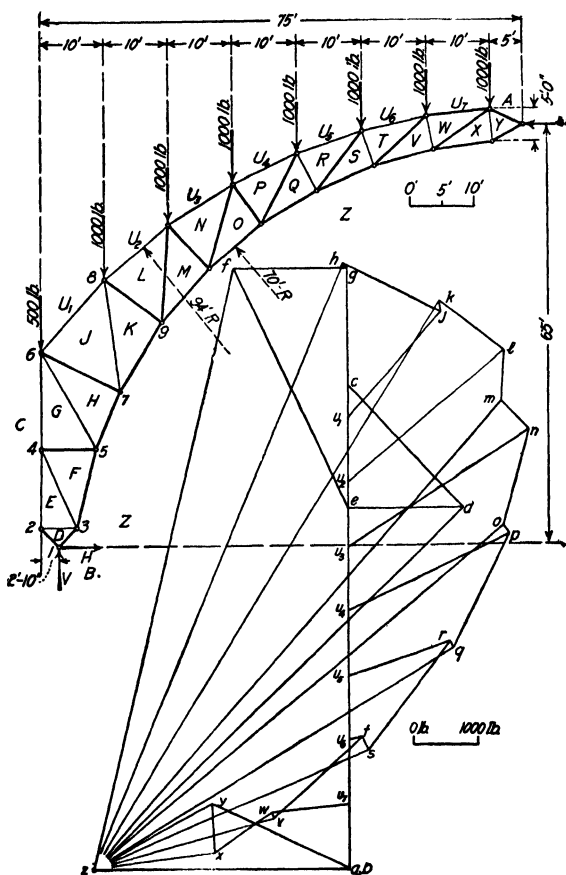



Fig. 4.

member and replacing the members *OP* and *PQ* The stresses in the members *OP* and *PQ* are then determined by the solution of the joint at their intersection.

The solution may be obtained in another manner, by solving algebraically for the stress in *RK* and laying it off to scale on the stress diagram, so that joint 6 can be determined before joint 4.

Illustrative Problem.—The stresses are required in the three-hinged arch truss of Fig. 4.

The reactions may be found graphically but the algebraic solution is more simple. After the components of the reactions are determined the stresses may be found by the usual stress diagram beginning at either reaction and determining stresses at

consecutive joints, as shown in Fig. 4. The solution could, of course, be accomplished by beginning at the crown hinge.

Illustrative Problem.—The stresses are required in a cantilever truss loaded as shown in Fig. 5*a*.

The reactions of the truss are determined graphically in Fig. 5*a*, as explained in the

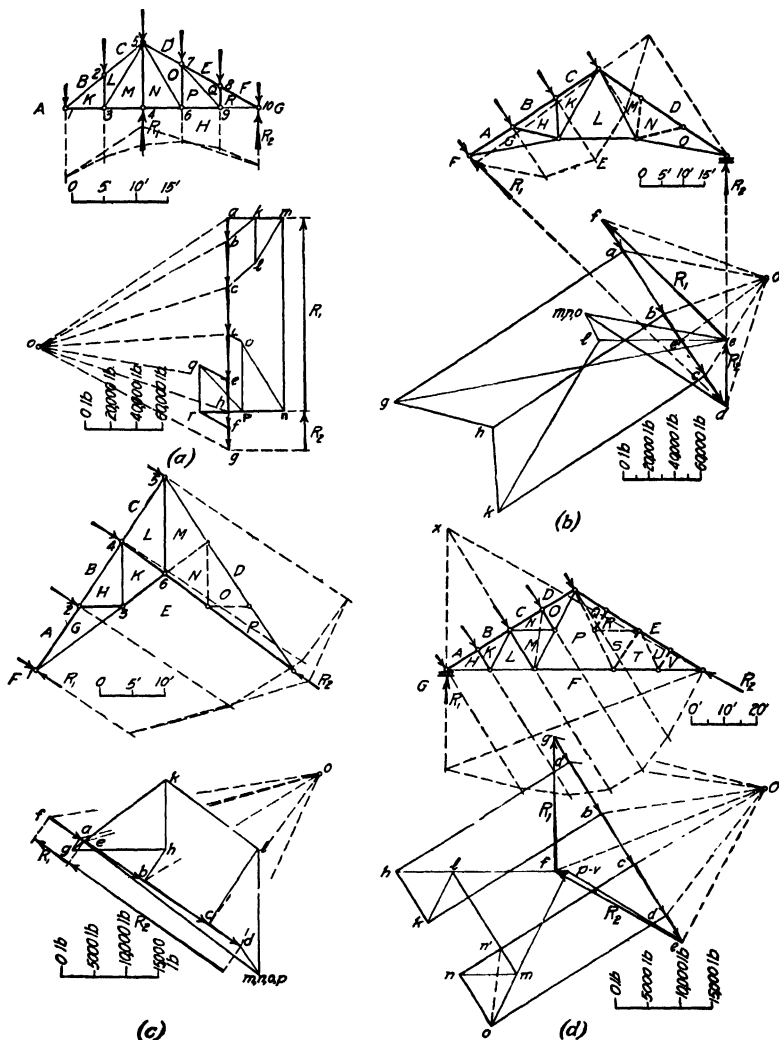



FIG. 5*a* to *d*.

chapter on "Reactions" in Sec. 1. The method of determining the stresses is the same as in the preceding illustrative problems.

5. Wind Load Stresses by the Graphical Method.—In the illustrative problems which follow, stresses will be found in trusses due to wind load under the following conditions: (1) Rollers on the leeward side of truss.

(2) both ends fixed, and (3) rollers on the windward side of truss. The wind load is considered as that component of a horizontal wind force, normal to the plane of the roof, except in the last problem (5e) in which the recommendations of Sub-committee 31, Committee on Steel, American Society of Civil Engineers, are used.

Illustrative Problem.—In Fig. 5*b*, the external force polygon is first drawn with the loads parallel to the wind loads on the truss. The reaction, R_2, can be drawn vertically because it is transmitted through rollers, but the direction of R_1 is not known so the polygon cannot be completed. The reactions will, therefore, be determined by means of the force and equilibrium polygons. R_1 will be assumed as parallel to the wind load and the closing string will give the direction of the ray *Oe′*. Now because R_1 must take the entire horizontal component of the wind load and because R_2 acts vertically, a horizontal line drawn from *e′* to *e* will give the point of intersection of the two reactions. These reactions may be checked by considering the total wind load and the two reactions as three forces acting on the truss. Since the directions and points of application of the resultant of the wind load and the reaction R_2 are known, the two forces may be extended to their point of intersection, *d*; and, since the point of application of R_1 is known, the direction of the force will be from *d* to the point of left reaction. The determination of this direction makes it possible to complete the external force polygon and obtain a check on the first solution for reactions.

The stresses are now determined by drawing a force polygon for each joint. It should be noted that the web members in the leeward side received no stress.

Illustrative Problem.—The wind stresses in the Scissors truss of Fig. 5*c* will be determined by the graphical method under the assumption that the reactions are parallel when both ends of the truss are fixed by an anchorage to solid masonry walls.

The space diagram is drawn with the lines of action of the loads extended so that the equilibrium polygon can be drawn. The reactions are determined by the ray, *Oe*, which is parallel to the closing string of the equilibrium polygon.

The stresses are determined by beginning at the left-hand heel joint and following through in the order indicated. As in the previous problem no stress is found in the web members on the leeward side of the truss. Some stresses are produced in this truss due to wind load which are opposite in direction to those produced by dead loads. Stresses should be carefully determined in roofs of such extreme pitch.

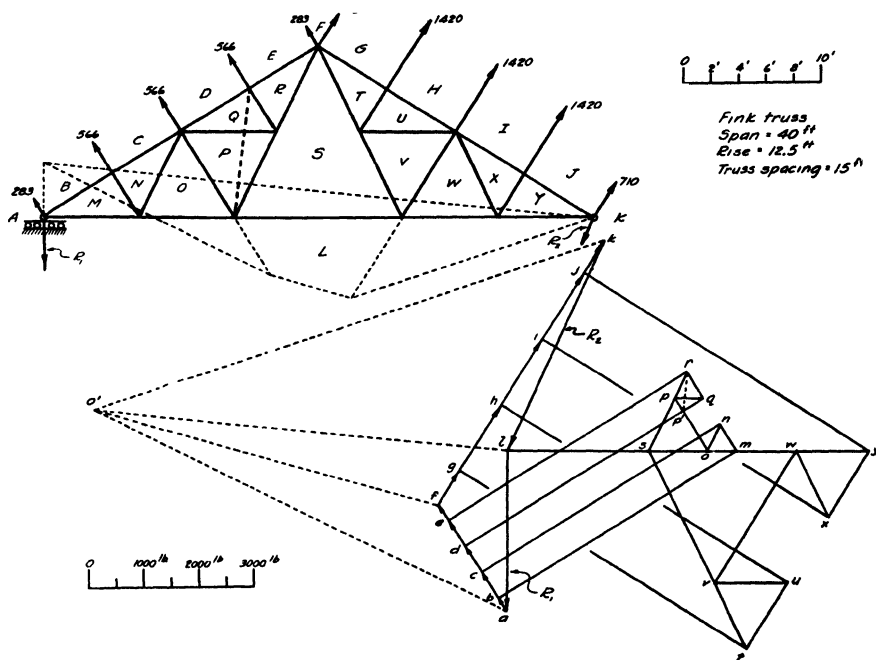
Illustrative Problem.—The wind load stresses are required in the Fink truss of Fig. 5*d*.

The wind load reactions upon the Fink truss of Fig. 5*d* will be determined in a different manner than that used for the determination of the reactions in Fig. 5*b*. The load line is plotted as usual and a pole from which the rays are drawn is selected. The line of action at the left support is known, but the point of application is the only element of the right reaction which is known The equilibrium polygon, is, therefore, begun at the right-hand heel joint so that the intersection of the strings can be made on the line of action of the force. The string parallel to the ray *Oe* is first drawn. The others are drawn in consecutive order from that one parallel to *Od* to the one parallel to *Og*. Since the line of action at the left support is vertical, the point of intersection with the string can be obtained. The closing string between the forces which form the two reactions is then drawn and the ray, *Of*, is drawn parallel to it. The intersection at *f* with the vertical line through *g* gives the left reaction, *fg*. The force *ef*, which is the right reaction, is drawn to the point of intersection of the vertical force through *g* and the ray *Of*.

These reactions may be checked by extending the line of the left reaction and the line of the resultant of the wind loads to a point of intersection shown at *x*, and drawing

the right reaction through the right-hand heel joint and point, x. Since the truss is in equilibrium the two reactions and the resultant of the wind loads must form a system of three concurrent forces. The extended forces drawn to point x give a space diagram from which the force diagram, gef, may be drawn.

The stress diagram is begun at the left-hand heel joint and the joints are taken in consecutive order until the joint at the middle point of the rafter is reached, at which the condition encountered in the Fink truss in Fig. 2b is again met. The difficulty is removed by replacing the members NO and MN by the dotted member shown and carrying the solution through until fp is determined, after which the corrections are made as before. It should be again noted that the web members on the leeward side of the truss take no stress.

FIG. 5e.

Illustrative Problem.—If wind loading is assumed in accordance with the recommendations of Sub-Committee 31, Committee on Steel of the Structural Division of the American Society of Civil Engineers (Sec. 1, Art. **11b**), there may be external pressure or suction, depending upon the pitch of the roof, and internal pressure or suction, depending upon the location and extent of wall and roof openings. With large exposures of glass on the windward side of unpartitioned industrial buildings it would be desirable to assume the possibility of breakage due to flying debris, necessitating the assumption of an internal pressure. Figure 5e provides the reactions and stresses from such assumptions for a Fink truss. The pitch is such as to give a 32-deg. angle of inclination between the roof and horizontal, which results in an external pressure on the windward side of $0.3(32) - 9 = 0.6$ lb. per sq. ft. All leeward slopes, having an angle of inclination of more than zero with the horizontal, carry a suction of 9 lb. per sq. ft. If 10 per cent of the windward surface is assumed to be open, an internal pressure of $4.5 + (^{10}\!/_{30})(12 - 4.5) = 7$ lb. per sq. ft. will exist. The net effect of these pressures provides a suction of 6.4 lb. per sq. ft. on the windward slope and

16 lb. per sq. ft. on the leeward slope. With a truss spacing of 15 ft., the panel loads would be 566 lb. on the windward slope and 1,420 lb. on the leeward slope, both acting as suction. The windward and leeward resultants of the wind loads have been used in determining the reactions by means of force and equilibrium diagrams. It should be observed that the loading shown, in comparison with that of Figure 5*d*, not only produces stresses in all the members but also reverses the type of stress.

ROOF TRUSSES—STRESS DATA

6. Stress Coefficients.—Where the stresses are to be calculated for a great many structures, in which the type of truss and the character of loading are exactly the same, the time spent in stress calculation can be reduced greatly by the use of stress coefficients. A type of structure to which the calculation of stresses by coefficients is readily adapted is the roof truss, for which in general the applied loads consist of equal panel loads placed at the panel points of the truss. Since in general it is possible to arrange the calculations so that the only variable is the amount of the equal applied loads, which for convenience are taken as unit loads, the stresses in all members of the truss can be expressed as a function of the form of the truss and the position of the loads. This factor is known as a stress coefficient. If then, the panel loads are determined, subject to conditions depending upon the size of the truss and the intensity of the applied loads, the stress in any member is obtained by multiplying the actual panel load by the stress coefficient for the member in question.

In the present chapter, tables of stress coefficients have been worked out for some of the standard forms of roof trusses. A general formula is given by which the stress coefficient for any member is expressed in terms of the form of the truss. Special numerical values of these coefficients have been calculated and are tabulated for a few of the pitch ratios generally used in practice. A more complete discussion of the conditions to which the tables apply will be given in the following articles.

The numerical values of the stress coefficients given in the tables at the end of this chapter have been expressed to three significant figures. Therefore, all stresses calculated from these tables are accurate only to three significant figures. For example: Suppose that the panel load for a given truss is 3,520 lb., and suppose that the stress coefficient for the member whose stress is desired is 4.52. Assuming three figure accuracy, the stress in the member is 3,520 × 4 52 = 15,900 lb. It is of course possible to multiply out these quantities, obtaining the result, 3,520 × 4.52 = 15,910.40 lb. But since in calculating the coefficients we retain only three significant figures, the coefficient 4.52 may mean anything from 4.515 to 4.525, and the corresponding products will be 3,520 × 4.515 = 15,892.80, and 3,520 × 4.525 = 15,928.00. However, as the original data is accurate only to three places, it is quite evident that the result of any manipulation of these data can be accurate only to the same number of places. If we decide to retain only three significant figures in the above multiplications, we proceed to discard any figures in the fourth place below a five, and retain any figure in the fourth place above the five by changing the third significant figure to the next higher number. Thus in each case the result is found to be 15,900 lb. It will be noted that in each case the change made is less than 1 per cent of the result. Stresses obtained with this degree of accuracy are close enough for all designing conditions.

If the designer desires more accurate results, he can make the proper substitutions in the general formulas for the stress coefficients, retaining the desired number of significant figures.

7. Arrangement of Tables of Stress Coefficients—Notation Adopted.—The tables of stress coefficients given at the end of this chapter have been made up for some of the standard forms of roof trusses. In each of these tables, a truss diagram shows the form of the truss and the position of the applied loads. Each member of the truss is represented by a number, which is placed on the truss diagram. By locating the member whose stress is desired, its reference number can be determined, and by looking up this reference number in the table, the stress in the member can be determined. Where several members have equal stresses, the same reference number has been used.

Two methods have been used to indicate the kind of stress in the members. One method indicates the character of the stress by the weight of the lines used in the loading diagram at the head of each table. Heavy lines denote compression, light lines denote tension, and dotted lines denote zero stress. The other method indicates the character of the stress by means of the sign used with the numerical value of the stress coefficient. A plus sign is used to indicate tension, and a minus sign is used to indicate compression. There are a few members in the trusses of Tables 27 and 28 for which a reversal of stress occurs. In such cases the sign given with the stress coefficient must be used to obtain the character of the stress.

In deriving the stress coefficients, it was found convenient to express them in terms of the ratio of span length to height of truss at the span center. The resulting ratio, which is denoted by n, is given by the expression $n = \frac{l}{h}$, where l = span length and h = height of truss. It will be noted that this ratio is the reciprocal of the pitch of the truss. In calculating the numerical values of the stress coefficients, substitutions were made in the general formulas for the pitch ratios in general use. If values for other pitch ratios are desired, they can be obtained by interpolation from the values given in the tables, or they can be calculated directly from the general formulas.

8. Stress Coefficients for Vertical Loading.—Tables 1 to 26 give stress coefficients due to vertical loading for several of the types of trusses commonly used for roofs. Two general cases will be considered: (*a*) equal loads applied at all top chord panel points, known also as *roof loads;* and (*b*) equal loads applied at all lower chord points, known also as *ceiling loads.* These cases will be discussed separately.

8*a*. **Roof Loads.**—Tables 1 to 17 give stress coefficients for Fink, Fan, Pratt, and Howe trusses of various numbers of panels due to equal vertical loads applied at the top chord points. Tables 15, 16, and 17 are for Fink trusses for which the lower chord has been cambered for the sake of appearance. This introduces another variable, k, by means of which the rise of the lower chord member is expressed as a fractional

part of the height of the truss. Numerical values of the stress coefficients have been calculated for the usual values of n and for three values of k.

8b. Ceiling Loads.—Where the top and bottom chord panel points lie on the same vertical line, as in the Pratt trusses of Tables 7 to 10 and the Howe trusses of Tables 11 to 14, stress coefficients for panel loads applied at the lower chord points can be obtained from those given for *roof loads* by the application of a simple rule. This rule is as follows: Stress coefficients due to ceiling loads for all members in Pratt and Howe trusses, *except verticals*, are the same as given in Tables 7 to 14 for roof loads. Stress coefficients for stresses in *vertical members* due to ceiling loads can be obtained from the values given in Tables 7 to 14 by adding +1 (algebraic addition) to the stress coefficients for roof loads. By adding +1 algebraically, the sign of the result will indicate the character of stress in the vertical (+ = tension, − = compression) and the numerical value will give the amount of the stress.

As an example of the application of this rule, suppose that the stress coefficients are desired for the vertical members of the Howe truss of Table 12. Note that the stresses in vertical members are independent of the value of n. Applying the above rule to member 6, the stress coefficient for a ceiling load is $0 + 1 = +1$, or a tension of 1, as indicated by the plus sign. Likewise for member 7 we have $+1 + 0.5 = +1.5$, or a tension of 1.5.

Applying the same rule to the Pratt truss of Table 8, the stress coefficient for member 3 due to ceiling loads is $+1 - 1 = 0$, or zero stress. For member 4 we have $-1.50 + 1.00 = -0.50$, or a compression of 0.50. For member 10, we have $0 + 1.0 = 1.0$, or a tension of 1.

The rule given above does not apply to the trusses of Tables 1 to 6 and 15 to 17. Special tables of stress coefficients for ceiling loads are given for these trusses in Tables 18 to 26. Tables 18 to 21 are for unsymmetrical loads such as lines of shafting, heavy pipe lines, or machinery loads. Tables 22 and 23 are for symmetrical loads, such as ceiling or floor loads, and can be made to include the weight of purlins, floor or ceiling joists, floor and ceiling loads, and live loads applied to an attic floor.

If stresses are desired for all lower chord points loaded, the stresses calculated for the partial loads as given by Tables 22 and 23 can be added to obtain the total stresses. It will usually be found that stress calculations can be made by this process in less time than is required by the graphical methods given in the chapter on "Simple Roof Trusses."

Tables 24 to 26 for a cambered Fink truss are similar to Tables 21 to 23 for the straight chord Fink truss.

9. Stress Coefficients for Wind Loading.[1]—It may be pointed out that

[1] Although there is a growing sentiment toward a more precise determination of wind loading conditions, the stress coefficients for wind loads acting normal to the windward side have been retained in Tables 27 to 32 inclusive. If a closer approximation to actual wind loading is felt to be desirable, the reader is referred to the comprehensive report of Sub-committee 31, Committee on Steel, Structural Division of the American Society of Civil Engineers, *Trans.* Vol. 105, 1940, p. 1713. See also Sec. 1-11b, p. 7 of this volume.

TABLE 1.—STRESS COEFFICIENTS—FINK TRUSS

Member	General formula	Value of n				
		3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
1	$-\frac{3}{4}WN$	−2.70	−3.00	−3.35	−4.04	−4.74
2	$-\frac{W}{4N}(3n^2+4)$	−2.15	−2.50	−2.91	−3.67	−4.43
3	$-\frac{Wn}{N}$	−0.832	−0.866	−0.894	−0.929	−0.949
4	$+\frac{1}{4}Wn$	+0.750	+0.868	+1.00	+1.25	+1.50
5	$+\frac{3}{4}Wn$	+2.25	+2.60	+3.00	+3.75	+4.50
6	$+\frac{1}{2}Wn$	+1.50	+1.73	+2.00	+2.50	+3.00

+ = tension — = compression

for trusses of the Fink, Fan, Pratt, and Howe type, wind stresses calculated for a vertical loading represent fairly well the effect of wind loads. The stress coefficients of Tables 1 to 17 can be used for this assumed wind loading.

FIG. 6.

In case a more exact determination of wind stresses is desired, stress coefficients have been worked out for Fink and Howe trusses for wind loads applied normal to the windward roof surface. Since wind loads acting normal to the roof surface cause reactions which have horizontal components, the stress will depend upon the conditions at the points of support. Figure 6 shows the conditions assumed at the supports. Cases I, II, and III are intended to represent conditions in steel trusses, where provision for expansion due to temperature changes must be made at the walls. Three common assumptions are shown in Fig. 6. It will be noted that these assumptions affect the stresses in the lower chord member only.

TABLE 2.—STRESS COEFFICIENTS—COMPOUND FINK TRUSS

Member	General formula	Value of n				
		3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34'$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
1	$-\frac{7}{4}WN$	−6 31	−7 00	−7.83	−9 42	−11.07
2	$-\frac{1}{4}WN(7N^2-8)$	−5 76	−6 50	−7 38	−9 05	−10 75
3	$-\frac{1}{4}WN(7N^2-16)$	−5 20	−6 00	−6 93	−8 68	−10 43
4	$-\frac{1}{4}WN(7N^2-24)$	−4 65	−5 50	−6 48	−8.31	−10 12
5	$-W\frac{n}{N}$	−0 832	−0 866	−0.894	−0.929	−0.949
6	$-2W\frac{n}{N}$	−1 66	−1.73	−1.79	−1.86	−1.90
7	$+\frac{1}{4}Wn$	+0 750	+0.868	+1 00	+1.25	+1 50
8	$+\frac{1}{2}Wn$	+1 50	+1 73	+2.00	+2.50	+3 00
9	$+\frac{3}{4}Wn$	+2.25	+2 60	+3 00	+3.75	+4 50
10	$+\frac{7}{4}Wn$	+5 25	+6.07	+7 00	+8 75	+10 50
11	$+\frac{3}{2}Wn$	+4 50	+5 20	+6 00	+7.50	+9.00
12	$+Wn$	+3.00	+3 46	+4 00	+5 00	+6 00

+ = tension — = compression

and the tabulation of stress coefficients is arranged accordingly. Case IV represents conditions in small steel trusses, and in all spans of wooden trusses, for in these spans expansion due to temperature need not be considered.

TABLE 3.—STRESS COEFFICIENTS—COMPOUND FINK TRUSS

Member	General formula	Value of n: 3, $\theta = 33°-41'$	$2\sqrt{3}$, $\theta = 30°$	4, $\theta = 26°-34'$	5, $\theta = 21°-48'$	6, $\theta = 18°-26'$
1	$-\frac{9}{4}WN$	−8 11	−9 00	−10 06	−12.12	−14.21
2	$-\frac{1}{4}\frac{W}{N}(9n^2+28)$	−7.55	−8 50	− 9 62	−11 75	−13 91
3	$-\frac{1}{20}\frac{W}{N}(37n^2+100)$	−6 00	−6 80	− 7.74	− 9.52	−11.31
4	$-\frac{3}{4}\frac{W}{N}(3n^2+4)$	−6 44	−7.50	− 8 72	−11.00	−13 28
5	$-\frac{1}{4}\frac{W}{N}(9n^2+4)$	−5 88	−7.00	− 8 28	−10.63	−12 98
6	$-W\frac{n}{N}$	−0 832	−0 866	− 0 894	− 0.929	− 0.949
7	$-\frac{3}{20}W\frac{nQ}{N}$	−1 31	−1 38	− 1.45	− 1 56	− 1.66
8	$+\frac{1}{4}Wn$	+0 750	+0.868	+ 1.00	+ 1.25	+ 1.50
9	$+\frac{3}{4}Wn$	+2 25	+2.60	+ 3.00	+ 3.75	+ 4 50
10	$+Wn$	+3 00	+3 46	+ 4.00	+ 5 00	+ 6 00
11	$+\frac{9}{4}Wn$	+6 75	+7.79	+ 9 00	+11.25	+13.50
12	$+2Wn$	+6 00	+6 92	+ 8 00	+10 00	+12 00
13	$+\frac{5}{4}Wn$	+3 75	+4 34	+ 5 00	+ 6 25	+ 7.50

+ = tension — = compression

TABLE 4.—STRESS COEFFICIENTS—FINK TRUSS WITH VERTICALS

Member	General formula	Value of n				
		3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34'$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
1	$-\frac{7}{4}WN$	−6.31	−7.00	−7.83	−9.42	−11.07
2	$-W$	−1.00	−1.00	−1.00	−1.00	−1.00
3	$-2W$	−2.00	−2.00	−2.00	−2.00	−2.00
4	$+\frac{1}{4}WM$	+1.25	+1.32	+1.41	+1.60	+1.80
5	$+\frac{1}{4}Wn$	+0.750	+0.866	+1.00	+1.25	+1.50
6	$+\frac{1}{2}WM$	+2.50	+2.64	+2.82	+3.20	+3.60
7	$+\frac{3}{4}WM$	+3.75	+3.96	+4.23	+4.80	+5.40
8	$+\frac{7}{4}Wn$	+5.25	+6.07	+7.00	+8.75	+10.50
9	$+\frac{3}{2}Wn$	+4.50	+5.20	+6.00	+7.50	+9.00
10	$+Wn$	+3.00	+3.46	+4.00	+5.00	+6.00

+ = tension — = compression

TABLE 5.—STRESS COEFFICIENTS—FAN TRUSS

Member	General formula	Value of n: 3, $\theta = 33° - 41'$	$2\sqrt{3}$, $\theta = 30°$	4, $\theta = 26° - 34'$	5, $\theta = 21° - 48'$	6, $\theta = 18° - 26'$
1	$-\frac{5}{4}WN$	−4.51	−5.00	−5.59	−6.73	−7.91
2	$-\frac{1}{12}\frac{W}{N}(13N^2 - 16)$	−3.54	−4.00	−4.55	−5.59	−6.64
3	$-\frac{1}{12}\frac{W}{N}(15N^2 - 48)$	−3.40	−4.00	−4.70	−5.99	−7.27
4	$-\frac{1}{6}W\frac{n}{N}(n^2 + 36)^{1/2}$	−0.930	−1.00	−1.08	−1.21	−1.34
5	$+\frac{1}{2}Wn$	+1.50	+1.73	+2.00	+2.50	+3.00
6	$+\frac{5}{4}Wn$	+3.75	+4.33	+5.00	+6.25	+7.50
7	$+\frac{3}{4}Wn$	+2.25	+2.60	+3.00	+3.75	+4.50

+ = tension — = compression

TABLE 6.—STRESS COEFFICIENTS—COMPOUND FAN TRUSS

Member	General formula	Value of n: 3, $\theta = 33° - 41'$	$2\sqrt{3}$, $\theta = 30°$	4, $\theta = 26° - 34'$	5, $\theta = 21° - 48'$	6, $\theta = 18° - 26'$
1	$-\frac{11}{4}WN$	−9 92	−11 00	−12 30	−14 81	−17.39
2	$-\frac{1}{12}\frac{W}{N}(31N^2 - 16)$	−8 95	−10 00	−11 25	−13 66	−16 13
3	$-\frac{1}{12}\frac{W}{N}(33N^2 - 48)$	−8 81	−10 00	−11 40	−14 07	−16 76
4	$-\frac{1}{12}\frac{W}{N}(33N^2 - 72)$	−8 25	−9 50	−10 96	−13 70	−16 44
5	$-\frac{1}{12}\frac{W}{N}(31N^2 - 88)$	−7 28	−8 50	−9 91	−12 55	−15 18
6	$-\frac{1}{12}\frac{W}{N}(33N^2 - 120)$	−7 14	−8 50	−10 06	−12 95	−15 93
7	$-\frac{1}{6}W\frac{n}{N}(n^2 + 36)^{\frac{1}{2}}$	−0 930	−1 00	−1 08	−1 21	−1 34
8	$-3\frac{Wn}{N}$	−2 50	−2 60	−2 68	−2 79	−2 85
9	$+\frac{1}{2}Wn$	+1 50	+1 73	+2 00	+2 50	+3 00
10	$+\frac{3}{4}Wn$	+2 25	+2 60	+3 00	+3 75	+4 50
11	$+\frac{5}{4}Wn$	+3 75	+4 33	+5 00	+6 25	+7 50
12	$+\frac{11}{4}Wn$	+8 25	+9 53	+11 00	+13 75	+16 50
13	$+\frac{9}{4}Wn$	+6 75	+7 79	+9 00	+11 25	+13 50
14	$+\frac{3}{2}Wn$	+4 50	+5 20	+6 00	+7 50	+9 00

+ = tension − = compression

TABLE 7.—STRESS COEFFICIENTS—PRATT TRUSS—4 PANELS

Member	General formula	Value of n: 3, $\theta = 33° - 41'$	$2\sqrt{3}$, $\theta = 30°$	4, $\theta = 26° - 34'$	5, $\theta = 21° - 48'$	6, $\theta = 18° - 26'$
1	$-\frac{3}{4}WN$	−2 70	−3 00	−3 35	−4 04	−4 74
2	$-W$	−1 00	−1 00	−1 00	−1 00	−1 00
3	$+\frac{W}{4}(n^2 + 16)^{\frac{1}{2}}$	+1 25	+1 32	+1 41	+1 60	+1 80
4	$+\frac{3}{4}Wn$	+2 25	+2 60	+3 00	+3 75	+4 50
5	$+\frac{1}{2}Wn$	+1 50	+1 73	+2 00	+2 50	+3 00
6	0	0	0	0	0	0

+ = tension — = compression

For loads on lower chord see Art 8b

TABLE 8.—STRESS COEFFICIENTS—PRATT TRUSS—6 PANELS

Member	General formula	Value of n				
		3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34'$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
1	$-\frac{5}{4}WN$	−4.51	−5.00	−5.59	−6.73	−7.91
2	$-WN$	−3.61	−4.00	−4.47	−5.39	−6.32
3	$-W$	−1.00	−1.00	−1.00	−1.00	−1.00
4	$-\frac{3}{2}W$	−1.50	−1.50	−1.50	−1.50	−1.50
5	$+\frac{W}{4}(n^2 + 16)^{\frac{1}{2}}$	+1.25	+1.32	+1.41	+1.60	+1.80
6	$+\frac{W}{4}(n^2 + 36)^{\frac{1}{2}}$	+1.68	+1.73	+1.80	+1.95	+2.12
7	$+\frac{5}{4}Wn$	+3.75	+4.33	+5.00	+6.25	+7.50
8	$+Wn$	+3.00	+3.46	+4.00	+5.00	+6.00
9	$+\frac{3}{4}Wn$	+2.25	+2.60	+3.00	+3.75	+4.50
10	0	0	0	0	0	0

+ = tension — = compression

For loads on lower chord see Art. 8b.

TABLE 9.—STRESS COEFFICIENTS—PRATT TRUSS—8 PANELS

Member	General formula	Value of n				
		3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34'$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
1	$-\frac{7}{4}WN$	−6.31	−7.00	−7.83	−9.42	−11.07
2	$-\frac{3}{2}WN$	−5.41	−6.00	−6.71	−8.08	− 9.49
3	$-\frac{5}{4}WN$	−4.51	−5.00	−5.59	−6.73	− 7.91
4	$-W$	−1.00	−1.00	−1.00	−1.00	− 1.00
5	$-\frac{3}{2}W$	−1.50	−1.50	−1.50	−1.50	− 1.50
6	$-2W$	−2.00	−2.00	−2.00	−2.00	− 2.00
7	$+\frac{1}{4}W(n^2 + 16)^{\frac{1}{2}}$	+1.25	+1.32	+1.41	+1.60	+ 1.80
8	$+\frac{1}{4}W(n^2 + 36)^{\frac{1}{2}}$	+1.68	+1.73	+1.80	+1.95	+ 2.12
9	$+\frac{1}{4}W(n^2 + 64)^{\frac{1}{2}}$	+2.14	+2.18	+2.24	+2.36	+ 2.50
10	$+\frac{7}{4}Wn$	+5.25	+6.06	+7.00	+8.75	+10.50
11	$+\frac{3}{2}Wn$	+4.50	+5.20	+6.00	+7.50	+ 9.00
12	$+\frac{5}{4}Wn$	+3.75	+4.33	+5.00	+6.25	+ 7.50
13	$+Wn$	+3.00	+3.46	+4.00	+5.00	+ 6.00
14	0	0	0	0	0	0

+ = tension — = compression

For loads on lower chord see Art. 8*b*.

TABLE 10.—STRESS COEFFICIENTS—PRATT TRUSS—10 PANELS

Member	General formula	Value of n				
		3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34'$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
1	$-\frac{9}{4}WN$	−8.11	−9.00	−10.06	−12.12	−14.23
2	$-2WN$	−7.21	−8.00	− 8.94	−10.77	−12.65
3	$-\frac{7}{4}WN$	−6.31	−7.00	− 7.83	− 9.42	−11.07
4	$-\frac{3}{2}WN$	−5.41	−6.00	− 6.71	− 8.08	− 9.49
5	$-W$	−1.00	−1.00	− 1.00	− 1.00	− 1.00
6	$-\frac{3}{2}W$	−1.50	−1.50	− 1.50	− 1.50	− 1.50
7	$-2W$	−2.00	−2.00	− 2.00	− 2.00	− 2.00
8	$-\frac{5}{2}W$	−2.50	−2.50	− 2.50	− 2.50	− 2.50
9	$+\frac{W}{4}(n^2 + 16)^{\frac{1}{2}}$	+1.25	+1.32	+ 1.41	+ 1.60	+ 1.80
10	$+\frac{W}{4}(n^2 + 36)^{\frac{1}{2}}$	+1.68	+1.73	+ 1.80	+ 1.95	+ 2.12
11	$+\frac{W}{4}(n^2 + 64)^{\frac{1}{2}}$	+2.14	+2.18	+ 2.24	+ 2.36	+ 2.50
12	$+\frac{W}{4}(n^2 + 100)^{\frac{1}{2}}$	+2.61	+2.65	+ 2.69	+ 2.80	+ 2.92
13	$+\frac{9}{4}Wn$	+6.75	+7.79	+ 9.00	+11.25	+13.50
14	$+2Wn$	+6.00	+6.93	+ 8.00	+10.00	+12.00
15	$+\frac{7}{4}Wn$	+5.25	+6.06	+ 7.00	+ 8.75	+10.50
16	$+\frac{3}{2}Wn$	+4.50	+5.20	+ 6.00	+ 7.50	+ 9.00
17	$+\frac{5}{4}Wn$	+3.75	+4.33	+ 5.00	+ 6.25	+ 7.50
18	0	0	0	0	0	0

+ = tension — = compression

For loads on lower chord see Art. 8b.

TABLE 11 —STRESS COEFFICIENTS—HOWE TRUSS—4 PANELS

Member	General formula	Value of n				
		3 $\theta = 33° - 41$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34$	5 $\theta = 21° - 48$	6 $\theta = 18° - 26'$
1	$-\frac{3}{4}WN$	−2 70	−3 00	−3 35	−4 04	−4 74
2	$-\frac{1}{2}WN$	−1 80	−2 00	−2 24	−2 69	−3 16
3	$-\frac{1}{4}WN$	−0 900	−1 00	−1 12	−1 35	−1 58
4	0	0	0	0	0	0
5	$+W$	+1 0	+1 0	+1 0	+1 0	+1 0
6	$+\frac{3}{4}Wn$	+2 25	+2 60	+3 00	+3 75	+4 50

+ = t nsion — = compression

For loads on lower chord see Art 86

TABLE 12.—STRESS COEFFICIENTS—HOWE TRUSS—6 PANELS

Member	General formula	Value of n				
		3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34'$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
1	$-\frac{5}{4}WN$	−4 51	−5 00	−5 59	−6 73	−7 91
2	$-WN$	−3.61	−4 00	−4 47	−5 39	−6 32
3	$-\frac{3}{4}WN$	−2 70	−3 00	−3 35	−4 04	−4 74
4	$-\frac{1}{4}WN$	−0 900	−1 00	−1 12	−1 35	−1 58
5	$-\frac{1}{4}W(n^2+16)^{\frac{1}{2}}$	−1 25	−1.32	−1 41	−1 60	−1 80
6	0	0	0	0	0	0
7	$+\frac{W}{2}$	+0.500	+0.500	+0.500	+0.500	+0 500
8	$+2W$	+2 00	+2 00	+2 00	+2 00	+2 00
9	$+\frac{5}{4}Wn$	+3 75	+4 33	+5 00	+6 25	+7 50
10	$+Wn$	+3.00	+3 46	+4 00	+5 00	+6 00

+ = tension — = compression

For loads on lower chord see Art 8b.

TABLE 13.—STRESS COEFFICIENTS—HOWE TRUSS—8 PANELS

Member	General formula	Value of n: 3, θ = 33° − 41	$2\sqrt{3}$, θ = 30°	4, θ = 26° − 34	5, θ = 21° − 48	6, θ = 18° − 26′
1	$-\frac{7}{4}WN$	−6.31	−7.00	−7.83	−9.42	−11.07
2	$-\frac{3}{2}WN$	−5.41	−6.00	−6.71	−8.08	−9.49
3	$-\frac{5}{4}WN$	−4.51	−5.00	−5.59	−6.73	−7.91
4	$-WN$	−3.61	−4.00	−4.47	−5.39	−6.32
5	$-\frac{1}{4}WN$	−0.900	−1.00	−1.12	−1.35	−1.58
6	$-\frac{1}{4}W(n^2+16)^{\frac{1}{2}}$	−1.25	−1.32	−1.41	−1.60	−1.80
7	$-\frac{1}{4}W(n^2+36)^{\frac{1}{2}}$	−1.68	−1.73	−1.80	−1.95	−2.12
8	0	0	0	0	0	0
9	$+\frac{1}{2}W$	+0.500	+0.500	+0.500	+0.500	+0.500
10	$+W$	+1.00	+1.00	+1.00	+1.00	+1.00
11	$+3W$	+3.00	+3.00	+3.00	+3.00	+3.00
12	$+\frac{7}{4}Wn$	+5.25	+6.06	+7.00	+8.75	+10.50
13	$+\frac{3}{2}Wn$	+4.50	+5.20	+6.00	+7.50	+9.00
14	$+\frac{5}{4}Wn$	+3.75	+4.33	+5.00	+6.25	+7.50

+ = tension — = compression

For loads on lower chord see Art. 8*b*

TABLE 14.—STRESS COEFFICIENTS—HOWE TRUSS—10 PANELS

Member	General formula	Value of n				
		3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34'$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
1	$-\frac{9}{4}WN$	−8 11	−9 00	− 10 06	−12.12	−14 23
2	$-2WN$	−7 21	−8 00	− 8 94	−10.77	−12 65
3	$-\frac{7}{4}WN$	−6 31	−7 00	− 7 83	− 9.42	−11 07
4	$-\frac{3}{2}WN$	−5 41	−6 00	− 6 71	− 8.08	− 9 49
5	$-\frac{5}{4}WN$	−4 51	−5 00	− 5 59	− 6 73	− 7 91
6	$-\frac{1}{4}WN$	−0 900	−1 00	− 1.12	− 1.35	− 1 58
7	$-\frac{1}{4}W(n^2 + 16)^{\frac{1}{2}}$	−1 25	−1 32	− 1 41	− 1 60	− 1 80
8	$-\frac{1}{4}W(n^2 + 36)^{\frac{1}{2}}$	−1 68	−1 73	− 1 80	− 1.95	− 2.12
9	$-\frac{1}{4}(n^2 + 64)^{\frac{1}{2}}$	−2 14	−2 18	− 2 24	− 2 36	− 2 50
10	0	0	0	0	0	0
11	$+\frac{1}{2}W$	+0 500	+0 500	+ 0 500	+ 0 500	+ 0 500
12	$+W$	+1 00	+1.00	+ 1.00	+ 1 00	+ 1 00
13	$+\frac{3}{2}W$	+1 50	+1 50	+ 1 50	+ 1.50	+ 1 50
14	$+4W$	+4 00	+4 00	+ 4 00	+ 4 00	+ 4 00
15	$+\frac{9}{4}Wn$	+6 75	+7 79	+ 9 00	+11 25	+13 50
16	$+2Wn$	+6 00	+6 93	+ 8 00	+10 00	+12 00
17	$+\frac{7}{4}Wn$	+5 25	+6 06	+ 7 00	+ 8 75	+10 50
18	$+\frac{3}{2}Wn$	+4 50	+5 20	+ 6.00	+ 7 50	+ 9 00

+ = tension — = compression

For loads on lower chord see Art. 8b.

TABLE 15.—STRESS COEFFICIENTS—CAMBERED FINK TRUSS

Member	General formula	k	Value of n: 3, $\theta = 33°-41'$	$2\sqrt{3}$, $\theta = 30°$	4, $\theta = 26°-34'$	5, $\theta = 21°-48'$	6, $\theta = 18°-26'$
1	$-\frac{3}{4}W\frac{(N^2-8k)}{N(1-2k)}$	1/10	−3.17	−3.56	−4.03	−4.90	−5.81
		1/8	−3.32	−3.75	−4.25	−5.20	−6.17
		1/6	−3.64	−4.13	−4.70	−5.78	−6.89
2	$-\frac{1}{4}W\frac{[3n^2-8(1+k)]}{N(1-2k)}$	1/10	−2.62	−3.06	−3.49	−4.54	−5.51
		1/8	−2.77	−3.25	−3.80	−4.83	−5.85
		1/6	−3.09	−3.63	−4.25	−5.41	−6.57
3	$-\frac{Wn}{N}$		−0.832	−0.866	−0.894	−0.929	−0.949
4	$+\frac{1}{4}Wn\frac{D(1+k)}{N(1-2k)(1-k)}$	1/10	+1.08	+1.26	+1.48	+1.87	+2.26
		1/8	+1.20	+1.40	+1.64	+2.08	+2.52
		1/6	+1.43	+1.69	+1.98	+2.52	+3.06
5	$+\frac{3}{4}Wn\frac{D}{N(1-2k)}$	1/10	+2.65	+3.09	+3.62	+4.57	+5.52
		1/8	+2.79	+3.27	+3.83	+4.85	+5.85
		1/6	+3.07	+3.62	+4.24	+5.40	+6.56
6	$+\frac{1}{2}\frac{Wn}{(1-k)}$	1/10	+1.67	+1.93	+2.22	+2.78	+3.33
		1/8	+1.72	+1.98	+2.29	+2.86	+3.34
		1/6	+1.80	+2.08	+2.40	+3.00	+3.60

+ = tension — = compression

TABLE 16.—STRESS COEFFICIENTS—CAMBERED COMPOUND FINK TRUSS

Member	General formula	k	Value of n: 3 ($\theta = 33° - 41'$)	$2\sqrt{3}$ ($\theta = 30°$)	4 ($\theta = 26° - 34'$)	5 ($\theta = 21° - 48'$)	6 ($\theta = 18° - 26'$)
1	$-\frac{7}{4}W\frac{[n^2+4(1-2k)]}{N(1-2k)}$	1/10	−7 39	−8 31	− 9 10	−11 46	−13 56
		1/8	−7.79	−8 75	− 9 93	−12 14	−14 42
		1/6	−8 49	−9 63	−10 96	−13.49	−16 04
2	$-\frac{1}{4}W\frac{[7n^2+20(1-2k)]}{N(1-2k)}$	1/10	−6 84	−7 81	− 8 95	−11 08	−13.25
		1/8	−7 23	−8 25	− 9 48	−11.76	−14.10
		1/6	−7.94	−9 13	−10.51	−13 11	−15.72
3	$-\frac{1}{4}W\frac{[7n^2+12(1-2k)]}{N(1-2k)}$	1/10	−6 29	−7.31	− 8 50	−10 70	−12 94
		1/8	−6 67	−7 75	− 9 03	−11.38	−13 78
		1/6	−7 39	−8 63	−10 06	−12 74	−15.40
4	$-\frac{1}{4}W\frac{[7n^2+4(1-2k)]}{N(1-2k)}$	1/10	−5.74	−6 81	− 8 05	−10 32	−12 63
		1/8	−6.11	−7 25	− 8 58	−11.00	−13 46
		1/6	−6 83	−8 13	− 9 61	−12 37	−15 08
5	$-W\frac{n}{N}$		−0 832	−0 866	− 0 894	− 0 929	− 0.949
6	$-2W\frac{n}{N}$		−1 66	−1 73	− 1.79	− 1.86	− 1 90
7	$+\frac{1}{4}W\frac{nD}{N(1-2k)}$	1/10	+0 884	+1 030	+ 1 20	+ 1 52	+ 1 85
		1/8	+0 933	+1 09	+ 1 29	+ 1 62	+ 1 96
		1/6	+1.02	+1 21	+ 1 41	+ 1 80	+ 2 19
8	$+\frac{1}{2}Wn\frac{D(1+k)}{N(1-2k)(1-k)}$	1/10	+2 16	+2 52	+ 2.95	+ 3.73	+ 4.51
		1/8	+2.40	+2.80	+ 3 29	+ 4 15	+ 5 04
		1/6	+2 87	+3.37	+ 3 96	+ 5.04	+ 6 12
9	$+\frac{1}{4}Wn\frac{D(3+k)}{N(1-2k)(1-k)}$	1/10	+3 04	+3.57	+ 4 15	+ 5 24	+ 6 34
		1/8	+3 32	+3 90	+ 4 56	+ 5 76	+ 7 00
		1/6	+3 88	+4 58	+ 5 37	+ 6 85	+ 8 30
10	$+\frac{7}{4}W\frac{nD}{N(1-2k)}$	1/10	+6.18	+7 22	+ 8 45	+10 68	+12 91
		1/8	+6 54	+7 64	+ 8 95	+11 31	+13 71
		1/6	+7.17	+8 44	+ 9.90	+12 61	+15 71
11	$+\frac{3}{2}W\frac{nD}{N(1-2k)}$	1/10	+5.30	+6.20	+ 7.25	+ 9 15	+11 09
		1/8	+5 61	+6 55	+ 7.68	+ 9 70	+11 76
		1/6	+6 15	+7 23	+ 8 48	+10 81	+13 49
12	$+W\frac{n}{(1-k)}$	1/10	+3 34	+3 85	+ 4.44	+ 5 55	+ 6 66
		1/8	+3 43	+3 96	+ 4 57	+ 5.72	+ 6.86
		1/6	+3 60	+4 16	+ 4 80	+ 6 00	+ 7 20

TABLE 17.—STRESS COEFFICIENTS—CAMBERED FAN TRUSS

Member	General formula	k	Value of n: 3, $\theta = 33° - 41'$	$2\sqrt{3}$, $\theta = 30°$	4, $\theta = 26° - 34'$	5, $\theta = 21° - 48'$	6, $\theta = 18° - 26'$
1	$-\frac{5}{4}W\frac{[n^2+4(1-2k)]}{N(1-2k)}$	1/10	−5 28	−5.94	−6.71	−8.18	− 9.70
		1/8	−5 55	−6.25	−7 09	−8 67	−10 28
		1/6	−6 06	−6 88	−7.83	−9.64	−11.46
2	$-\frac{1}{4}W\frac{[\frac{13}{4}n^2+12(1-2k)]}{N(1-2k)}$	1/10	−4 20	−4.81	−5.52	−6 83	− 8 19
		1/8	−4 44	−5 07	−5.84	−7.26	− 8 70
		1/6	−4 89	−5 63	−6.48	−8 10	− 9.70
3	$-\frac{1}{4}W\frac{[(5n^2+4(1-2k)]}{N(1-2k)}$	1/10	−4 17	−4.94	−5.81	−7.45	− 9 07
		1/8	−4 45	−5 25	−6 20	−7 92	− 9.65
		1/6	−4 96	−5.88	−6 93	−8 89	−10.81
4	$-\frac{1}{6}\frac{Wn}{N}\frac{[(n^2+36(1-2k)^2]^{\frac{1}{2}}}{(1-2k)}$	1/10	−0 981	−1.07	−1.17	−1 34	− 1 52
		1/8	−1 00	−1 09	−1.20	−1 39	− 1.58
		1/6	−1 04	−1 15	−1 26	−1 49	− 1.71
5	$+\frac{1}{4}W\frac{nD}{N}\frac{(2+k)}{(1-2k)(1-k)}$	1/10	+2 06	+2 41	+2 80	+3 56	+ 4 31
		1/8	+2 26	+2 66	+3 10	+3 93	+4.76
		1/6	+2.66	+3 13	+3.67	+4 69	+5.70
6	$+\frac{5}{4}W\frac{nD}{N(1-2k)}$	1/10	+4.42	+5 16	+6 03	+7 62	+ 9.22
		1/8	+4.67	+5 45	+6 39	+8 08	+ 9.80
		1/6	+5.12	+6 03	+7 07	+9.01	+10.92
7	$+\frac{3}{4}W\frac{n}{(1-k)}$	1/10	+2 50	+2 89	+3 34	+4 17	+ 5.00
		1/8	+2 57	+2 97	+3 43	+4 28	+ 5.15
		1/6	+2.70	+3.12	+3 60	+4 50	+ 5.40

+ = tension — = compression

TABLE 18.—STRESS COEFFICIENTS—COMPOUND FINK TRUSS

Member	General formula	Value of n: 3, $\theta = 33° - 41'$	$2\sqrt{3}$, $\theta = 30°$	4, $\theta = 26° - 34'$	5, $\theta = 21° - 48'$	6, $\theta = 18° - 26'$
1	$-\frac{1}{16}P\frac{N}{n^2}(7n^2 - 4)$	−1.479	−1.665	−1 889	−2.305	−2.720
2	$-\frac{1}{16}P\frac{N}{n^2}(3n^2 - 4)$	−0 576	−0 667	−0.769	−0.957	−1.140
3	$-\frac{1}{16}P\frac{N^3}{n^2}$	−0.326	−0.333	−0.349	−0 391	−0.438
4	$-\frac{1}{2}P\frac{N}{n}$	−0.602	−0.576	−0.559	−0.539	−0.527
5	$+\frac{1}{4}P\frac{N^2}{n}$	+1.083	+1.160	+1.250	+1.450	+1.667
6	$+\frac{1}{8}P\frac{N^2}{n}$	+0.542	+0 580	+0.625	+0.725	+0.833
7	$+\frac{1}{16}P\frac{(7n^2 - 4)}{n}$	+1.229	+1.442	+1 688	+2.139	+2.585
8	$+\frac{3}{16}P\frac{N^2}{n}$	+0.813	+0.865	+0.936	+1.088	+1.25
9	$+\frac{1}{16}P\frac{N^2}{n}$	+0.271	+0.288	+0.312	+0.362	+0.417
R_1	$\frac{1}{8}P\frac{(7n^2 - 4)}{n^2}$	0 819	0.833	0.844	0.855	0.861
R_2	$\frac{1}{8}P\frac{N^2}{n^2}$	0.181	0.167	0.156	0.145	0 139
r	$\frac{1}{8}l\frac{N^2}{n^2} = \frac{1}{8}h\frac{N^2}{n}$	0.181l 0.543h	0.167l 0.578h	0.156l 0.625h	0 145l 0.725h	0 139l 0 833h

+ = tension — = compression

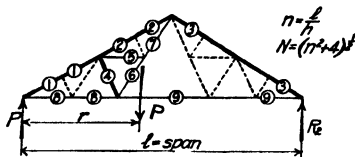
Stress is zero for dotted members.

TABLE 19.—STRESS COEFFICIENTS—COMPOUND FINK TRUSS

Member	General formula	Value of n				
		3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34'$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
1	$-\frac{1}{8}P\frac{N}{n^2}(3n^2 - 4)$	−1.152	−1.335	−1.538	−1.915	−2.228
2	$-\frac{1}{8}P\frac{N^3}{n^2}$	−0 652	−0 667	−0 699	−0 783	−0 877
3	$+\frac{1}{4}P\frac{N^2}{n}$	+1 083	+1.160	+1.250	+1.450	+1.667
4	$+\frac{1}{8}P\frac{(3n^2 - 4)}{n}$	+0 958	+1.152	+1 372	+1.775	+2.167
5	$+\frac{1}{8}P\frac{N^2}{n}$	+0 542	+0.580	+0 625	+0.725	+0.833
R_1	$\frac{1}{4}P\frac{(3n^2 - 4)}{n^2}$	0 639	0 667	0.688	0.710	0.723
R_2	$\frac{1}{4}P\frac{N^2}{n^2}$	0 361	0 333	0 312	0 290	0.277
r	$\frac{1}{4}l\frac{N^2}{n^2} = \frac{1}{4}h\frac{N^2}{n}$	0 361l 1.086h	0.333l 1.156h	0 312l 1.25h	0.290l 1.45h	0.277l 1.667h
x	$\frac{1}{16}\frac{l}{n^2}(5n^2 - 12)$	0 229l	0.250l	0 266l	0.282l	0 292l

+ = tension — = compression

Stress is zero for dotted members.

TABLE 20.—STRESS COEFFICIENTS—COMPOUND FINK TRUSS

Member	General formula	Value of n				
		3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34'$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
1	$-\frac{1}{16}P\frac{N}{n^2}(5n^2 - 4)$	−1.027	−1.165	−1.329	−1 630	−1 930
2	$-\frac{1}{16}P\frac{(9n^4 - 16)}{n^2N}$	−1 37	−1 667	−2 00	−2 60	−3.20
3	$-\frac{1}{16}P\frac{N}{n^2}(3n^2 + 4)$	−0 776	−0 832	−0 910	−1 065	−1.228
4	$-\frac{1}{2}P\frac{(n^2 - 4)}{Nn}$	−0.231	−0 289	−0 336	−0 390	−0 422
5	$+\frac{1}{4}P\frac{(n^2 - 4)}{n}$	+0.417	+0.576	+0 750	+1 050	+1 33
6	$+\frac{1}{8}P\frac{(n^2 - 4)}{n}$	+0 208	+0 288	+0 375	+0 525	+0 667
7	$+\frac{1}{8}P\frac{(3n^2 + 4)}{n}$	+1 291	+1 44	+1.625	+1 975	+2 333
8	$+\frac{1}{16}P\frac{(5n^2 - 4)}{n}$	+0.855	+1.014	+1.188	+1 513	+1 833
9	$+\frac{1}{16}P\frac{(3n^2 + 4)}{n}$	+0.645	+0 720	+0 813	+0 988	+1 167
R_1	$\frac{1}{8}P\frac{(5n^2 - 4)}{n^2}$	0.570	0 583	0 593	0 605	0 611
R_2	$\frac{1}{8}P\frac{(3n^2 + 4)}{n^2}$	0 430	0.417	0.407	0 395	0 389
r	$\frac{1}{8}l\frac{(3n^2 + 4)}{n^2}$	0 430l	0.417l	0.407l	0 395l	0 389l

+ = tension **− = compression**

Stress is zero for dotted members.

TABLE 21.—STRESS COEFFICIENTS—COMPOUND FINK TRUSS

Member	General formula	Value of n				
		3 θ = 33° − 41′	$2\sqrt{3}$ θ = 30°	4 θ = 26° − 34′	5 θ = 21° − 48′	6 θ = 18° − 26′
1	$-\frac{1}{4}PN$	−0 9025	−1.00	−1 117	−1.347	−1 582
2	$+\frac{1}{4}Pn$	+0 75	+0 866	+1 00	+1 25	+1.50
3	$+P$	+1 0	+1 0	+1 0	+1 0	+1.0

+ = tension — = compression
Stress is zero for dotted members.

TABLE 22.—STRESS COEFFICIENTS—COMPOUND FINK TRUSS

Member	General formula	Value of n				
		3 θ = 33° − 41′	$2\sqrt{3}$ θ = 30°	4 θ = 26° − 34′	5 θ = 21° − 48′	6 θ = 18° − 26′
1	$-\frac{1}{2}PN$	−1 805	−2 00	−2 235	−2 695	−3.163
2	$-\frac{1}{4}PN$	−0 903	−1 00	−1 118	−1 347	−1 582
3	$-\frac{1}{2}P\frac{N}{n}$	−0.602	−0.578	−0 558	−0 538	−0.527
4	$+\frac{1}{4}P\frac{N^2}{n}$	+1 083	+1 152	+1.25	+1.45	+1.667
5	$+\frac{1}{8}P\frac{N^2}{n}$	+0 542	+0.576	+0.625	+0.725	+0.833
6	$+\frac{1}{2}Pn$	+1 50	+1 732	+2.00	+2 50	+3.00
7	$+\frac{1}{4}P\frac{N^2}{n}$	+1 083	+1.152	+1.25	+1 45	+1.667
8	$+\frac{1}{8}P\frac{N^2}{n}$	+0 542	+0.576	+0.625	+0 725	+0.833

+ = tension — = compression
Stress is zero for dotted members

TABLE 23.—STRESS COEFFICIENTS—COMPOUND FINK TRUSS

Member	General formula	Value of n				
		3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34'$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
1	$-\frac{1}{2}PN$	−1 805	−2 00	−2 235	−2.695	−3.163
2	$+\frac{1}{4}P\frac{N^2}{n}$	+1.083	+1.152	+1.25	+1.45	+1.667
3	$+\frac{1}{2}Pn$	+1 50	+1.732	+2 00	+2.50	+3 00
4	$+\frac{1}{4}P\frac{N^2}{n}$	+1 083	+1.152	+1 25	+1 45	+1.667

+ = tension — = compression

Stress is zero for dotted members.

TABLE 24.—STRESS COEFFICIENTS—CAMBERED COMPOUND FINK TRUSS

$n = \frac{l}{h}$

$N = (n^2+4)^{\frac{1}{2}}$

$D = [N^2 - 16k(1+k)]^{\frac{1}{2}}$

Member	General formula	k	Value of n: 3, $\theta = 33°-41'$	$2\sqrt{3}$, $\theta = 30°$	4, $\theta = 26°-34'$	5, $\theta = 21°-48'$	6, $\theta = 18°-26'$
1	$-\frac{1}{2}P\frac{[n^2+4(1-2k)]}{N(1-2k)}$	$\frac{1}{10}$	−2.12	−2.38	−2.68	−3.28	−3.88
		$\frac{1}{8}$	−2.22	−2.50	−2.84	−3.46	−4.12
		$\frac{1}{6}$	−2.42	−2.76	−3.14	−3.86	−4.58
2	$-\frac{1}{4}P\frac{[n^2+4(1-2k)]}{N(1-2k)}$	$\frac{1}{10}$	−1.06	−1.19	−1.34	−1.64	−1.94
		$\frac{1}{8}$	−1.11	−1.25	−1.42	−1.73	−2.06
		$\frac{1}{6}$	−1.21	−1.38	−1.57	−1.93	−2.29
3	$-\frac{1}{2}P\frac{[n^2+4(1-2k)]}{nN}$	$\frac{1}{10}$	−0.562	−0.549	−0.537	−0.523	−0.517
		$\frac{1}{8}$	−0.554	−0.542	−0.532	−0.519	−0.515
		$\frac{1}{6}$	−0.538	−0.530	−0.522	−0.514	−0.510
4	$+\frac{1}{4}P\frac{D[n^2+4(1-2k)]}{nN(1-2k)}$	$\frac{1}{10}$	+1.20	+1.31	+1.45	+1.72	+2.01
		$\frac{1}{8}$	+1.24	+1.37	+1.52	+1.81	+2.13
		$\frac{1}{6}$	+1.32	+1.48	+1.65	+1.99	+2.35
5	$+\frac{1}{8}P\frac{D[n^2+4(1-2k)]}{nN(1-2k)(1-k)}$	$\frac{1}{10}$	+0.667	+0.729	+0.806	+0.956	+1.11
		$\frac{1}{8}$	+0.709	+0.782	+0.867	+1.04	+1.22
		$\frac{1}{6}$	+0.793	+0.889	+0.990	+1.19	+1.41
6	$+\frac{1}{2}P\frac{nD}{N(1-2k)}$	$\frac{1}{10}$	+1.77	+2.06	+2.42	+3.04	+3.70
		$\frac{1}{8}$	+1.86	+2.18	+2.56	+3.24	+3.92
		$\frac{1}{6}$	+2.04	+2.40	+2.84	+3.62	+4.38
7	$+\frac{1}{4}P\frac{D[n^2+4(1-2k)]}{nN(1-2k)}$	$\frac{1}{10}$	+1.20	+1.31	+1.45	+1.72	+2.01
		$\frac{1}{8}$	+1.24	+1.37	+1.52	+1.81	+2.13
		$\frac{1}{6}$	+1.32	+1.48	+1.65	+1.99	+2.35
8	$+\frac{1}{8}P\frac{[n^2+4(1-2k)]}{n(1-k)}$	$\frac{1}{10}$	+0.565	+0.610	+0.670	+0.785	+0.910
		$\frac{1}{8}$	+0.570	+0.620	+0.680	+0.800	+0.930
		$\frac{1}{6}$	+0.585	+0.635	+0.700	+0.830	+0.965

+ = tension

TABLE 25.—STRESS COEFFICIENTS—CAMBERED COMPOUND FINK TRUSS

Member	General formula	k	Value of n: 3, $\theta=33°-11'$	$2\sqrt{3}$, $\theta=30°$	4, $\theta=26°-34'$	5, $\theta=21°-48'$	6, $\theta=18°-26'$
1	$-\frac{1}{2}P\frac{[n^2+4(1-2k)]}{N(1-2k)}$	$\frac{1}{10}$	−2 12	−2 38	−2 68	−3 28	−3 88
		$\frac{1}{8}$	−2 22	−2 50	−2 84	−3 46	−4 12
		$\frac{1}{6}$	−2 42	−2 75	−3 13	−3 85	−4 58
2	$+\frac{1}{4}P\frac{D}{nN}\frac{[n^2+4(1-2k)]}{(1-2k)}$	$\frac{1}{10}$	+1 20	+1 31	+1 45	+1 72	+2 01
		$\frac{1}{8}$	+1 24	+1 37	+1 52	+1 81	+2 13
		$\frac{1}{6}$	+1 32	+1 48	+1 65	+1 99	+2 35
3	$+\frac{1}{2}P\frac{nD}{N(1-2k)}$	$\frac{1}{10}$	+1 77	+2 06	+2 42	+3 04	+3 70
		$\frac{1}{8}$	+1 86	+2 18	+2 56	+3 24	+3 92
		$\frac{1}{6}$	+2 04	+2 40	+2 84	+3 62	+4 38
4	$+\frac{1}{4}P\frac{[n^2+4(1-2k)]}{n(1-k)}$	$\frac{1}{10}$	+1 13	+1 22	+1 34	+1 57	+1 82
		$\frac{1}{8}$	+1 14	+1 24	+1 36	+1 60	+1 86
		$\frac{1}{6}$	+1 17	+1 27	+1 40	+1 66	+1 93

+ = tension — = compression

Stress is zero for dotted members

TABLE 26 —STRESS COEFFICIENTS—CAMBERED COMPOUND FINK TRUSS

Member	General formula	k	Value of n				
			3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34'$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
1	$-\frac{1}{4}P\frac{[n^2 + 4(1 - 2k)]}{N(1 - 2k)}$	$\frac{1}{10}$	−1.06	−1.19	−1.34	−1.64	−1.94
		$\frac{1}{8}$	−1.11	−1.25	−1.42	−1.73	−2.06
		$\frac{1}{6}$	−1.21	−1.38	−1.57	−1.93	−2.29
2	$+\frac{1}{4}P\frac{nDk}{N(1 - 2k)(1 - k)}$	$\frac{1}{10}$	+0.0980	+0.114	+0.134	+0.169	+0.206
		$\frac{1}{8}$	+0.133	+0.156	+0.183	+0.232	+0.280
		$\frac{1}{6}$	+0.204	+0.240	+0.284	+0.362	+0.438
3	$+\frac{1}{4}P\frac{nD}{N(1 - 2k)}$	$\frac{1}{10}$	+0.884	+1.03	+1.21	+1.52	+1.85
		$\frac{1}{8}$	+0.932	+1.09	+1.28	+1.62	+1.96
		$\frac{1}{6}$	+1.02	+1.20	+1.42	+1.81	+2.19
4	$+\frac{1}{4}P\frac{n}{(1 - k)}$	$\frac{1}{10}$	+0.675	+0.780	+0.900	+1.13	+1.35
		$\frac{1}{8}$	+0.656	+0.758	+0.875	+1.09	+1.31
		$\frac{1}{6}$	+0.625	+0.722	+0.833	+1.04	+1.25
5	$+P$		+1.0	+1.0	+1.0	+1.0	+1.0

+ = tension — = compression

Stress is zero for dotted members.

TABLE 27.—WIND STRESS COEFFICIENTS—FINK TRUSS

Case	Member	General formula	Value of n				
			3 $\theta = 33° - 41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26° - 34'$	5 $\theta = 21° - 48'$	6 $\theta = 18° - 26'$
I, II, III, and IV	1	$-\frac{1}{2}W\frac{(n^2-2)}{n}$	−1 17	−1.45	−1.75	−2 30	−2 83
	2	$\cdot -\frac{1}{4}W\frac{N^2}{n}$	−0 100	−0.0833	−0.0700	−0 054	−0.0438
	3	$-W$	−1 00	−1.00	−1.00	−1 00	−1.00
	4	$+\frac{1}{4}WN$	+0 900	+1.00	+1.12	+1 35	+1 58
	R_1	$\frac{1}{2}W\frac{(3n^2-4)}{nN}$	1 06	1.15	1 23	1 32	1 37
	R_2	$\frac{1}{2}W\frac{N}{n}$	0 600	0.578	0.559	0 539	0.526
I	5	$+\frac{1}{2}WN$	+1 80	+2 00	+2 24	+2 69	+3 16
	6	$+\frac{1}{4}WN$	+0 900	+1 00	+1.12	+1 35	+1 58
	R_3	$4\frac{W}{N}$	1 11	1.00	0 895	0 742	0 633
	R_4	0	0	0	0	0	0
II	5	$+\frac{1}{2}W\frac{(n^2-4)}{N}$	+0 694	+1.00	+1.34	+1 95	+2 53
	6	$+\frac{1}{4}W\frac{(n^2-12)}{N}$	−0 208	0	+0 224	+0 604	+0 950
	R_3	0	0	0	0	0	0
	R_4	$4\frac{W}{N}$	1 10	1 00	0.895	0.742	0.633
III	5	$+\frac{1}{2}W\frac{n^2}{N}$	+1 25	+1.50	+1.78	+2 32	+2.85
	6	$+\frac{1}{4}W\frac{(n^2-4)}{N}$	+0 347	+0.500	+0 670	+0 975	+1 265
	R_3	$2\frac{W}{N}$	0 555	0 500	0 447	0 371	0 316
	R_4	$2\frac{W}{N}$	0 555	0 500	0.447	0.371	0 316
IV	5	$+\frac{1}{2}WN\frac{(n^2-2)}{n^2}$	+1.41	+1 67	+1.96	+2 46	+2.98
	6	$+\frac{1}{4}WN\frac{(n^2-4)}{n^2}$	+0.502	+0 667	+0 837	+1 13	+1 41
	R_3	$W\frac{(3n^2-4)}{n^2N}$	0.708	0 667	0 616	0 526	0.458
	R_4	$W\frac{N}{n^2}$	0.401	0 333	0.280	0.216	0.175

+ = tension — = compression

Stress is zero for dotted members.

TABLE 28.—WIND STRESS COEFFICIENTS—COMPOUND FINK TRUSS

Case	Member	General formula	Value of n				
			3 $\theta = 33°-41'$	$2\sqrt{3}$ $\theta = 30°$	4 $\theta = 26°-34'$	5 $\theta = 21°-48'$	6 $\theta = 18°-26'$
I, II, III, and IV	1	$-\frac{1}{4}W\frac{(5n^2-8)}{n}$	−3.08	−3.75	−4.50	−5.83	−7.17
	2	$-\frac{1}{2}W\frac{N^2}{n}$	−2.17	−2.31	−2.50	−2.90	−3.33
	3	$-W$	−1 00	−1 00	−1 00	−1.00	−1.00
	4	$-2W$	−2 00	−2 00	−2 00	−2.00	−2.00
	5	$+\frac{1}{4}WN$	+0.902	+1 0	+1 12	+1 35	+1.58
	6	$+\frac{1}{2}WN$	+1 80	+2.0	+2.24	+2.70	+3.16
	7	$+\frac{3}{4}WN$	+2.71	+3.0	+3 35	+4 05	+4 24
	R_1	$W\frac{(3n^2-4)}{Nn}$	2.12	2 31	2.46	2.64	2.74
	R_2	$W\frac{N}{n}$	1.20	1.15	1.12	1.08	1.05

+ = tension — = compression

Stress is zero for dotted members.

TABLE 28 (*Continued*)

Case	Member	General formula	3	$2\sqrt{3}$	4	5	6
I	8	$+\frac{5}{4}WN$	+4.51	+5 00	+5 59	+6 73	+7.90
	9	$+WN$	+3.61	+4 00	+4.47	+5 39	+6.32
	10	$+\frac{1}{2}WN$	+1 80	+2 00	+2 24	+2.69	+3 16
	R_3	$8\frac{W}{N}$	2.22	2 00	1 79	1 49	1.27
	R_4	0	0	0	0	0	0
II	8	$+\frac{1}{4}W\frac{(5n^2-12)}{N}$	+2 28	+3.00	+3 80	+5.26	+6 65
	9	$+W\frac{(n^2-4)}{N}$	−1.39	+2 00	+2.68	+3 92	+5 06
	10	$+\frac{1}{2}W\frac{(n^2-12)}{N}$	−0 415	0	+0.447	+1 21	+1 90
	R_3	0	0	0	0	0	0
	R_4	$8\frac{W}{N}$	2 22	2.00	1 79	1 49	1.27
III	8	$+\frac{1}{4}W\frac{(5n^2+4)}{N}$	+3 40	+4 00	+4.70	+6.02	+7.28
	9	$+W\frac{n^2}{N}$	+2 49	+3 00	+3.58	+4 66	+5.70
	10	$+\frac{1}{2}W\frac{(n^2-4)}{N}$	+0.693	+1.00	+1.34	+1.96	+2 53
	R_3	$4\frac{W}{N}$	1 11	1 00	0 894	0.746	0.633
	R_4	$4\frac{W}{N}$	1 11	1 00	0 894	0.746	0 633
IV	8	$+\frac{1}{4}W\frac{N}{n^2}(5n^2-8)$	+3 71	+4 33	+5.03	+6 31	7 56
	9	$+W\frac{N}{n^2}(n^2-2)$	+2 81	+3.34	+3 92	+4.96	+5.97
	10	$+\frac{1}{2}W\frac{N}{n^2}(n^2-4)$	+1 01	+1.33	+1 68	+2 25	+2 81
	R_3	$2W\frac{(3n^2-4)}{Nn^2}$	1 42	1 33	1 23	1.05	0.915
	R_4	$2W\frac{N}{n^2}$	0 803	0 667	0 558	0 431	0.351

Table 29.—Wind Stress Coefficients—Howe Truss—4 Panels

Case	Member	General formula	Value of n: 3, $\theta=33°-41'$	$2\sqrt{3}$, $\theta=30°$	4, $\theta=26°-34'$	5, $\theta=21°-48'$	6, $\theta=18°-26'$
Case IV	1	$-\frac{1}{2}W\frac{(n^2-2)}{n}$	−1 17	−1.45	−1 75	−2.30	−2.83
	2	$-\frac{1}{4}Wn$	−0 750	−0 867	−1.00	−1.25	−1.50
	3	$-\frac{1}{4}W\frac{N^2}{n}$	−1 08	−1 16	−1.25	−1.45	−1.67
	4	$-\frac{1}{4}W\frac{N^2}{n}$	−1 08	−1.16	−1 25	−1.45	−1.67
	5	$+\frac{1}{2}W\frac{N}{n^2}(n^2-2)$	+1 41	+1 67	+1.96	+2 46	+2.98
	6	$+\frac{1}{4}W\frac{N}{n^2}(n^2-4)$	+0.502	+0 667	+0.837	+1 13	+1.41
	7	$+\frac{1}{2}W\frac{N}{n}$	+0 600	+0 575	+0 559	+0 539	+0.526
	R_1	$\frac{1}{2}W\frac{(3n^2-4)}{n^2}$	1 28	1 33	1 375	1.42	1.445
	R_2	$\frac{1}{2}W\frac{N^2}{n^2}$	0.720	0 665	0 625	0 580	0 555

+ = tension compression

Stress is zero for dotted members.

TABLE 30.—WIND STRESS COEFFICIENTS—HOWE TRUSS—6 PANELS

Case	Member	General formula	Value of n: 3, $\theta = 33° - 41'$	$2\sqrt{3}$, $\theta = 30°$	4, $\theta = 26° - 34'$	5, $\theta = 21° - 48'$	6, $\theta = 18° - 26'$
Case IV	1	$-\frac{1}{8}W\frac{(7n^2-12)}{n}$	−2.12	−2.60	−3.12	−4 07	−5 00
	2	$-\frac{1}{8}W\frac{(5n^2-4)}{n}$	−1.71	−2 02	−2 38	−3 03	−3.67
	3	$-\frac{1}{8}W\frac{(3n^2+4)}{n}$	−1.29	−1 44	−1 63	−1 98	−2.34
	4	$-\frac{3}{8}W\frac{N^2}{n}$	−1.61	−1 74	−1 88	−2 18	−2.50
	5	$-\frac{1}{4}W\frac{N^2}{n}$	−1.08	−1 16	−1 25	−1 45	−1.67
	6	$-\frac{1}{4}W\frac{N}{n}(n^2+16)^{\frac{1}{2}}$	−1.50	−1.53	−1 58	−1 73	−1 90
	7	$+\frac{1}{8}W\frac{N}{n^2}(7n^2-12)$	+2 56	+3 00	+3 49	+4.38	+5.28
	8	$+\frac{1}{8}W\frac{N}{n^2}(5n^2-12)$	+1.66	+2 00	+2 37	+3 04	+3 70
	9	$+\frac{3}{8}W\frac{N}{n^2}(n^2-4)$	+0.752	+1 00	+1 26	+1 69	+2.11
	10	$+\frac{1}{2}W\frac{N}{n}$	+0.600	+0.575	+0 559	+0 539	+0.526
	11	$+W\frac{N}{n}$	+1.20	+1 15	+1.12	+1.08	+1.05
	R_1	$\frac{3}{4}W\frac{(3n^2-4)}{n^2}$	1.92	2 00	2.06	2.13	2 17
	R_2	$\frac{3}{4}W\frac{N^2}{n^2}$	1 08	1.00	0 940	0.867	0.833

+ = tension — = compression

Stress is zero for dotted members.

TABLE 31.—WIND STRESS COEFFICIENTS—HOWE TRUSS—8 PANELS

Case	Member	General formula	Value of n: 3, $\theta = 33°-41'$	$2\sqrt{3}$, $\theta = 30°$	4, $\theta = 26°-34'$	5, $\theta = 21°-48'$	6, $\theta = 18°-26'$
Case IV	1	$-\frac{1}{4}W\frac{(5n^2-8)}{n}$	−3.08	−3.75	−4.50	−5.83	−7.17
	2	$-W\frac{(n^2-1)}{n}$	−2.67	−2.89	−3.75	−4.80	−5.83
	3	$-\frac{3}{4}Wn$	−2.25	−2.60	−3.00	−3.75	−4.50
	4	$-\frac{1}{2}W\frac{(n^2+2)}{n}$	−1.83	−2.02	−2.25	−2.70	−3.17
	5	$-\frac{1}{2}W\frac{N^2}{n}$	−2.11	−2.32	−2.50	−2.90	−3.33
	6	$-\frac{1}{4}W\frac{N^2}{n}$	−1.08	−1.16	−1.25	−1.45	−1.67
	7	$-\frac{1}{4}W\frac{N}{n}(n^2+16)^{\frac{1}{2}}$	−1.50	−1.53	−1.58	−1.73	−1.90
	8	$-\frac{1}{4}W\frac{N}{n}(n^2+36)^{\frac{1}{2}}$	−2.02	−1.97	−2.01	−2.11	−2.24
	9	$+\frac{1}{4}W\frac{N}{n^2}(5n^2-8)$	+3.71	+4.33	+5.03	+6.31	+7.56
	10	$+W\frac{N}{n^2}(n^2-2)$	+2.81	+3.33	+3.91	+4.95	+5.98
	11	$+\frac{1}{4}W\frac{N}{n^2}(3n^2-8)$	+1.91	+2.33	+2.79	+3.60	+4.40
	12	$+\frac{1}{2}W\frac{N}{n^2}(n^2-4)$	+1.00	+1.33	+1.68	+2.26	+2.82
	13	$+\frac{1}{2}W\frac{N}{n}$	+0.600	+0.575	+0.559	+0.539	+0.526
	14	$+W\frac{N}{n}$	+1.20	+1.15	+1.12	+1.08	+1.05
	15	$+\frac{3}{2}W\frac{N}{n}$	+1.80	+1.73	+1.68	+1.62	+1.58
	R_1	$W\frac{(3n^2-4)}{n^2}$	2.56	2.67	2.75	2.84	2.89
	R_2	$W\frac{N^2}{n^2}$	1.44	1.33	1.25	1.16	1.11

− = compression + = tension

Stress is zero for dotted members.

TABLE 32.—WIND STRESS COEFFICIENTS—HOWE TRUSS—10 PANELS

Case	Member	General formula	Value of n: 3, $\theta = 33°-41'$	$2\sqrt{3}$, $\theta = 30°$	4, $\theta = 26°-34'$	5, $\theta = 21°-48'$	6, $\theta = 18°-26'$
Case IV	1	$-\frac{1}{8}W\frac{(13n^2-20)}{n}$	−4.04	−4.91	−5.88	−7.63	−9.34
	2	$-\frac{1}{8}W\frac{(11n^2-12)}{n}$	−3.63	−4.33	−5.13	−6.57	−8.00
	3	$-\frac{1}{8}W\frac{(9n^2-4)}{n}$	−3.21	−3.75	−4.37	−5.52	−6.67
	4	$-\frac{1}{8}W\frac{(7n^2+4)}{n}$	−2.79	−3.18	−3.63	−4.47	−5.33
	5	$-\frac{1}{8}W\frac{(5n^2+12)}{n}$	−2.38	−2.60	−2.88	−3.42	−4.00
	6	$-\frac{5}{8}W\frac{N^2}{n}$	−2.71	−2.89	−3.13	−3.62	−4.17
	7	$-\frac{1}{4}W\frac{N^2}{n}$	−1.08	−1.16	−1.25	−1.15	−1.67
	8	$-\frac{1}{4}W\frac{N}{n}(n^2+16)^{\frac{1}{2}}$	−1.50	−1.53	−1.58	−1.73	−1.90
	9	$-\frac{1}{4}W\frac{N}{n}(n^2+36)^{\frac{1}{2}}$	−2.02	−1.97	−2.01	−2.11	−2.24
	10	$-\frac{1}{4}W\frac{N}{n}(n^2+64)^{\frac{1}{2}}$	−2.56	−2.51	−2.50	−2.54	−2.63
	11	$+\frac{1}{8}W\frac{N}{n^2}(13n^2-20)$	+4.88	+5.67	+6.56	+8.16	+9.80
	12	$+\frac{1}{8}W\frac{N}{n^2}(11n^2-20)$	+3.97	+4.67	+5.41	+6.83	+8.23
	13	$+\frac{1}{8}W\frac{N}{n^2}(9n^2-20)$	+3.07	+3.67	+4.33	+5.48	+6.65
	14	$+\frac{1}{8}W\frac{N}{n^2}(7n^2-20)$	+2.16	+2.67	+3.21	+4.14	+5.08
	15	$+\frac{5}{8}W\frac{N}{n^2}(n^2-4)$	+1.26	+1.67	+2.09	+2.80	+3.50
	16	$+\frac{1}{2}W\frac{N}{n}$	+0.600	+0.575	+0.559	+0.539	+0.526
	17	$+W\frac{N}{n}$	+1.20	+1.15	+1.12	+1.08	+1.05
	18	$+\frac{3}{2}W\frac{N}{n}$	+1.80	+1.73	+1.68	+1.62	+1.58
	19	$+2W\frac{N}{n}$	+2.40	+2.30	+2.24	+2.17	+2.10
	R_1	$\frac{5}{4}W\frac{(3n^2-4)}{n^2}$	3.20	3.34	3.44	3.55	3.61
	R_2	$\frac{5}{4}W\frac{N^2}{n^2}$	1.80	1.66	1.56	1.45	1.39

+ = tension — = compression

Stress is zero for dotted members

TRUSSES WITH KNEE-BRACES

10. General Methods of Stress Determination.—Figure 7 shows a knee-braced bent acted on by wind loads W_1 perpendicular to the side walls, and loads W_2 normal to the roof surface. General methods of

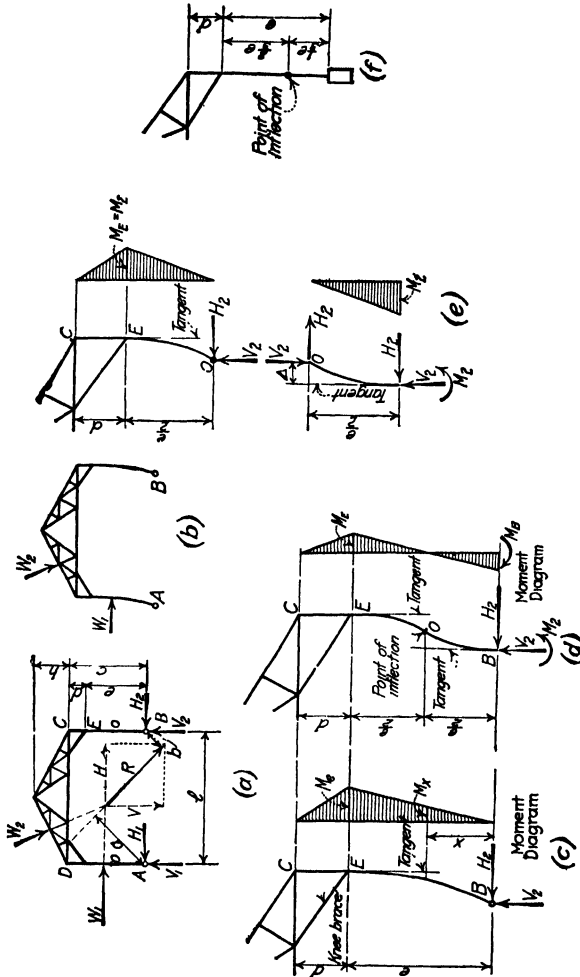



Fig. 7.

stress determination will be developed for the conditions shown in Fig. 7. Assume first that the truss is simply supported at points a and b by hinges, or by some method which will prevent horizontal movement under the action of the applied loads. Let R of Fig. 7a represent the resultant of the loads W_1 and W_2. The reactions at A and B are to be determined for the force R.

For the conditions shown in Fig. 7, it will be noted that there are four unknowns to be determined; a vertical and a horizontal force at A and B. The problem is therefore indeterminate, for, as stated in the chapter on "Principles of Statics" in Sec. 1, only three unknowns can be determined in any system of non-concurrent forces. Some assumption must then be made regarding the relation between certain of these forces before a solution can be made. It will be convenient in this case to consider the relation between the horizontal components of the forces at A and B. The desired relation can be obtained from a principle brought out in the analysis of statically indeterminate structures which states that where there is more than one path over which the stresses due to a given load may pass in order to reach the abutments or points of support, the load will be divided over these paths in proportion to their relative rigidities. It is reasonable to assume in this case that the loads are transmitted from the truss to the columns and thence to the points of support. As the columns are generally made alike, and are therefore of equal rigidity, it is usually assumed that the horizontal components of the applied loads are equally divided between the two points of support. Thus, if H be the horizontal component of R, we have

$$H_1 = H_2 = \frac{H}{2} \tag{1}$$

where H_1 and H_2 represent the horizontal components of the reactions at A and B, Fig. 7*a*. The vertical components of the reactions, shown by V_1 and V_2 in Fig. 7*a*, can be determined by moments. Thus in general terms, we have from moments about B

$$V_1 = \frac{Rb}{l} \tag{2}$$

and from moments about A

$$V_2 = \frac{Ra}{l} \tag{3}$$

The reactions are thus completely determined.

Before proceeding to the determination of the stresses in the truss members, it will be necessary to consider the conditions existing in the columns. As shown in Fig. 7*a*, the horizontal forces are carried to the points of support by means of a vertical member. As the loads act at right angles to the member, it is subjected to bending as well as direct stress. The distortion of the structure as a whole is of the nature shown in Fig. 7*b*. In Fig. 7*c* is shown, to an enlarged scale, one of the distorted columns. Since the column is riveted to the truss at point C, and to the knee-brace at point E, it seems reasonable to assume that E-C remains vertical, and that the distortion of E-B greatly magnified, is as shown in Fig. 7*c*. The column is then a three force piece, as it is subjected to bending moment, shear, and direct stress at all points. If M_x, V_x, S_x represent these quantities at any section a distance x above the base of the column, we have for member B-E of Fig. 7

$$M_x = H_2 x \qquad V_x = H_2 \qquad S_x = V_2 \tag{4}$$

The moment, as given by the first of these expressions, is a maximum at point E, the foot of the knee-brace, varying uniformly to zero at the foot of the column, as shown by the moment diagram of Fig. 7c. Values of the shear and direct stress for member C-E depend on the stress in the knee-brace, which is as yet unknown.

In general the columns are rigidly fastened to the foundations. The distortion of the column is then of the nature shown in Fig. 7d. When the base is fixed, the tangent to the curve at point B can be assumed to be vertical. As the tangent at E is also vertical, the curvature between the two points can be assumed to be a reversed curve, with the point of inflection, or change in curvature, at point O, half-way between E and B. Since a point of inflection is also a point of zero moment, the variation in moment for member B-C is as shown in Fig. 7d. The moment at O is zero, and the moments at points equal distances above and below O are equal in amount, but opposite in kind. It will be noted that the portion O-E of the deformed column of Fig. 7d is similar to the portion B-E of Fig. 7c. Since the moment at O is zero, this point can be regarded as a hinged joint. In the determination of stresses the column can be separated into two parts at point O, as shown in Fig. 7e. The reactions, as given by Eqs. (1), (2) and (3), are to be calculated for a knee-braced bent consisting of that part of the structure above points O of Fig. 7a. The moment at the base of the column can be determined from the conditions shown in Fig. 7e for the lower portion of the column.

The position of the point of inflection has an important bearing on the stresses in the members. It can be seen from Eqs. (1), (2) and (3) and from Fig. 7a, that the values of the reactions depend upon the effective height of the bent. A fixed end bent, considered as hinged at O, midway between the knee-brace and the base, will in general have smaller stresses in its members than one with simply supported ends, considered as hinged at A and B. However, unless the connections at E and C of Fig. 7d are absolutely rigid, and the base of the column is fixed, the point of inflection, O, cannot be assumed as located half-way between the base of the column and the foot of the knee-brace. Any tendency of the tangents to deviate from the vertical will cause the point of inflection to be lowered, the limit being points A and B, or a hinged connection at the base of the columns. Since the base of the column is usually rather wide in the plane of the truss, it can always be considered as partially fixed due to the action of the dead load. In most cases the column is firmly attached to the foundations by means of anchor bolts which are screwed up tight. As long as these bolts remain tight, the base of the column can be considered as fixed. But experience shows that this cannot be relied upon. It seems best, therefore, to assume that the point of inflection is somewhat below the mid-point between the knee-brace and the base of the column. This assumption is on the safe side, as the stresses in the truss members are increased thereby, and the moment to be carried by the columns is also increased.

In the calculations to follow, it will be assumed that the distance from the base of the column to the point of inflection is one-third of the distance from the base of the column to the foot of the knee-brace, as shown in Fig. 7f. There is considerable difference of opinion among designers and writers on this point. The recommendation made above seems to be reasonable and to be founded on conditions which actually exist in the structure; it will therefore be adopted.

11. Stresses in a Truss with Knee-braces.—Methods of stress calculation are best explained by means of a problem. Figure 8 shows the general dimensions of a typical truss of this type.

***11a.* Stresses Due to Vertical Loading.**—Under the action of vertical forces, the horizontal forces H_1 and H_2 of Fig. 7 do not exist.

The resulting stress diagram may then be constructed by the methods explained on p. 165 for Fig. 3*b*.

11*b*. Stresses Due to Wind Loads.—In a framework of the type shown in Fig. 8, it may be assumed that (*a*) the action of the wind causes normal loads to act on the vertical side of the building and on the roof surface of the windward side of the building or (*b*) that the wind causes forces to act in accordance with the recommendations of the Sub-Committee of the American Society of Civil Engineers previously quoted (Sec. 1, Art. **11*b***). For both cases the wind loads which act on the vertical wall are to be determined for the side wall area above the point of inflection. Roof loads are determined for the roof area tributary to each roof panel. Methods for the determination of the applied loads are given in the volume on "Steel and Timber Structures."

It will be assumed that the bases of the columns are partially fixed, and that the point of inflection is located at a point above the base of the column equal to one-third of the distance between the base and the foot of the knee-brace, as shown in Fig. 8.

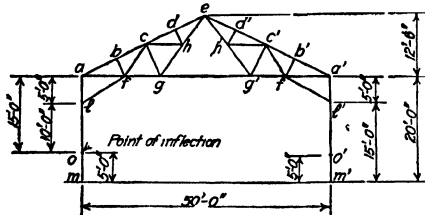



Fig. 8.

Case (*a*), Fig. 9*a*, shows the portion of the bent above the assumed points of inflection, with the applied loads in position.

The reactions at the points of inflection, O and O' of Fig 8, assumed to be points of support for a hinged knee-braced bent, can be calculated by the methods given in Sec. 1. From Fig 9*a*, the total horizontal component of applied loads is $4{,}500 + 6{,}260 \sin 26° 34' = 4{,}500 + 6{,}260 \times 0.447 = 4{,}500 + 2{,}800 = 7{,}300$ lb. The horizontal components of the reactions, as determined from Eq. (1), are

$$H_1 = H_2 = \frac{H}{2} = \frac{7{,}300}{2} = 3{,}650 \text{ lb.}$$

The forces act as shown in Fig. 9*a*. The vertical reactions are determined from moments about the bases of the columns, using Eqs. (2) and (3). Thus for R_2, from moments about O with dimensions and loads as shown on Fig. 9*a*, we have

$$R_2 = \frac{6{,}260 \times 20.71 + 4{,}500 \times 7.5}{50} = 3{,}260 \text{ lb.}$$

and

$$R_1 = \frac{6{,}260 \times 23.99 - 4{,}500 \times 7.5}{50} = 2{,}340 \text{ lb.}$$

These forces are shown in position on Fig. 9*a*. All external forces are thus completely determined.

The next step in the calculations is the determination of the stresses in the members of the truss. In general it will be found that graphical

methods of stress determination are preferable for this purpose. Algebraic methods of stress calculation are somewhat more precise than graphical methods, but in the application of algebraic methods considerable time is consumed in the calculation of lever arms of loads and members. This is avoided by the use of graphical methods, and the results obtained are accurate enough for all practical purposes.

In the application of graphical methods to a knee-braced bent a little difficulty is encountered in the case of the columns. These members are subjected to shear, moment, and direct stress, thus forming three force pieces. The graphical methods given in the chapter on "Simple Roof Trusses" are applicable only to one force pieces—that is, members subjected either to tension or compression. Two methods can be employed

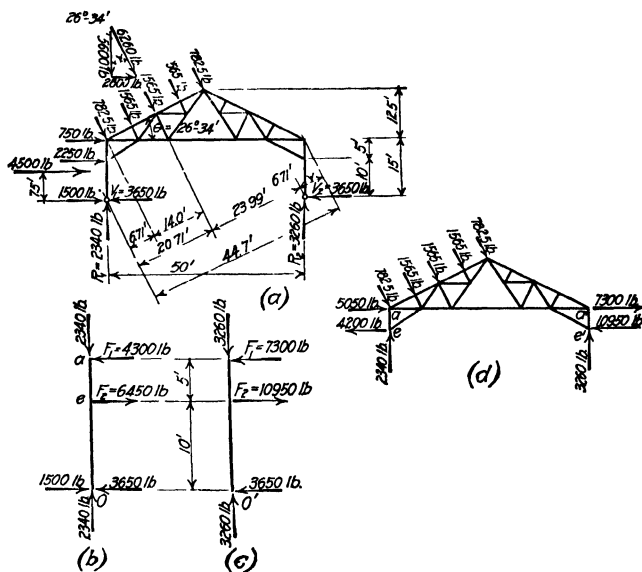



FIG. 9.

for the graphical solution of the case under consideration: (*a*) The columns can be removed and in their place can be substituted a system of forces whose effect on the structure as a whole will be the same as that of the columns, and (*b*) since a moment can be considered as a force times a distance, a temporary frame work can be added to the truss system, arranged so that the moment at the foot of the knee-brace will cause stress in the members of the auxiliary frame work. After the stresses in all members of the truss have been determined, the temporary frame work can be removed and the true stresses in the columns determined. This method is quite similar in principle to the one given in Art. **4**, for the determination of the stresses in certain members of the Fink truss. The methods described above will now be applied to the knee-braced bent of Fig. 9*a*.

The application of the first method outlined above is shown in Figs. 9*b*, *c*, and *d*. Figures 9 *b* and *c* show the columns removed with all forces acting. Forces F_1 and F_2 show the action of the column on the truss. These forces are determined by the methods of statics, subject to the condition that the column is in complete equilibrium. From Fig. 9 *b*, which shows the conditions for the windward column, moments about point *e* give

$$F_1 = (3{,}650 - 1{,}500)^{10}\!/\!_5 = 4{,}300 \text{ lb.}$$

and moments about point *a* give

$$F_2 = (3{,}650 - 1{,}500)^{15}\!/\!_5 = 6{,}450 \text{ lb.}$$

For the leeward column, shown in Fig. 9*c*

$$F_1 = 3{,}650 \times {}^{10}\!/\!_5 = 7{,}300 \text{ lb.}$$

and

$$F_2 = 3{,}650 \times {}^{15}\!/\!_5 = 10{,}950 \text{ lb.}$$

All forces are shown in position in Figs. 9*b* and 9*c*.

Since action and reaction are equal in amount but opposite in direction, forces F_1 and F_2 are to be applied to the truss in directions opposite to those shown in Figs. 9*b*

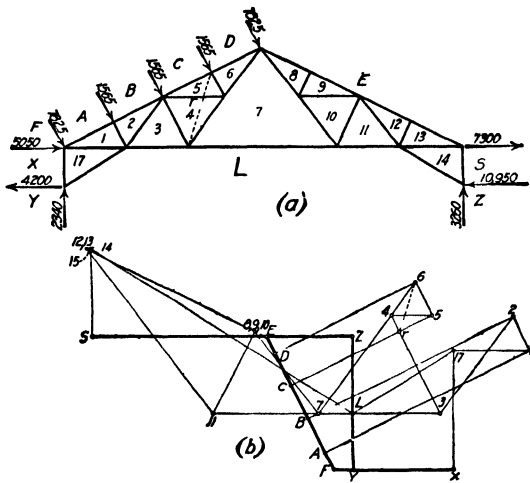



Fig. 10.

and 9*c*. They appear directly on the leeward side, but on the windward side they are to be combined with the loads shown at *a* and *e* of Fig. 9*a*. At *a* the applied load is 4,300 + 750 = 5,050 lb., and at *e* the load is 6,450 − 2,250 = 4,200 lb. These forces are shown in position and direction on Fig. 9*d*. At the foot of the knee-brace, vertical forces equal to the reaction at the foot of the column are applied, as shown in Fig. 9*d*. The resulting forces hold the structure in equilibrium.

Figure 10*b* shows the stress diagram for the forces shown on Fig. 9*d* and repeated on Fig. 10*a*. This stress diagram is constructed by the methods given in the chapter on "Simple Roof Trusses." The stresses in the members, as scaled from the diagram, are recorded in col. 4 and 6 of Table 33*a*. The stresses in the upper portion of the columns are given directly in the stress diagram. In the lower portions of the columns, the stress is equal to the reaction at the point in question, as given in Fig. 9*d*.

The temporary frame work for the second method of stress determination outlined above is shown in Fig. 11*a*. Any convenient arrangement can be used. In this case the top chord member was prolonged to an intersection with a horizontal through the foot of the knee-brace. This point was then connected to the foot of the column by a temporary member. These members are shown by dashed lines in Fig. 11*a*. The loads applied to the windward side of the building are considered as acting at the joints of the auxiliary frame work, as shown in Fig. 11*a*. With the auxiliary frame work in place, it is possible to draw the stress diagrams for all joints. Figure 11*b* shows the complete stress diagram.

The stresses for the columns, as given by the stress diagram of Fig. 11*b*, are not the true stresses for these members, for the addition of the auxiliary frames has affected the stresses in the columns; all other stresses are the true stresses in the members in question. To determine the true stresses in these members, the auxiliary frames must

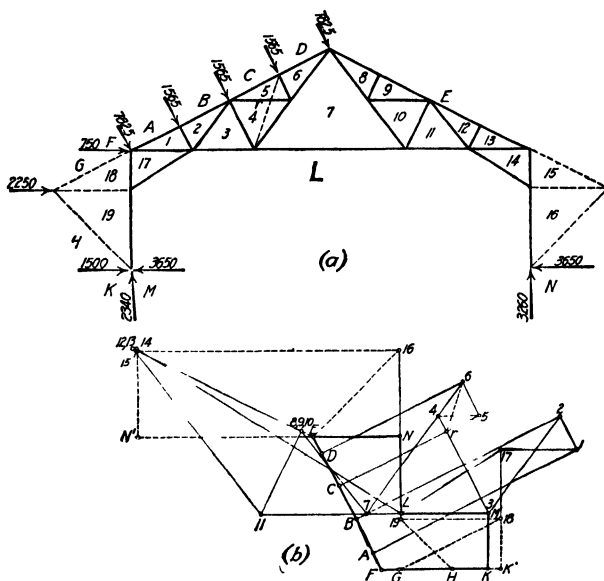
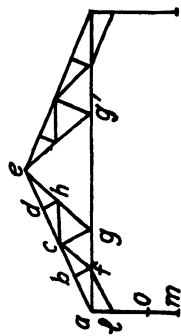



FIG 11.

be removed and the column stresses redetermined, subject to conditions which will be discussed later. Thus for the windward column it can be seen by inspection that as soon as the frame work is removed, the stress in the lower section of the column is a compression which is directly equal to the reaction at the foot of the column, which in this case is 2,340 lb. Consider the upper portion of the column. It is quite evident that the stress in this member must be of such magnitude that it will hold in equilibrium the stress in the lower portion of the column plus the vertical component of the stress in the windward knee-brace. The desired stress can be determined from Fig. 11*b* by locating the forces mentioned and adding them graphically. In Fig. 11*b*, *K–M* represents the reaction at the foot of the column, and *L*–17 represents the stress in the knee-brace. If these forces be projected on a vertical line drawn through point 17, we have as the sum of these forces the component *K'* –17, which represents the amount of the desired stress in the upper portion of the column; the stress as scaled from the stress diagram is 5,000 lb., and the kind of stress is compression. Similar methods are to be used for the leeward column. As before, the stress in the lower portion of the

TABLE 33a —STRESSES IN MEMBERS

Member	Dead load (1)	Snow load (2)	Minimum snow load (½ S. L.) (3)	Wind left to right (4)	Minimum wind left (⅓ W. L.) (5)	Wind right to left (6)	Minimum wind right (⅓ W. R.) (7)	D. L. + S. L. (8)	D. L. + wind (9)	D. L. + ½ S. L. + wind (10)	D. L. + S. L. + ⅓ wind (11)	Maximum stress (12)
ab	−11,660	−16,450	−8,225	− 9,600	−3,200	+ 8,400	+ 2,800	−28,110	−21,260	−29,485	−31,310	−31,310
bc	−11,000	−15,500	−7,750	− 9,600	−3,200	+ 8,400	+ 2,800	−26,500	−20,600	−28,350	−29,700	−29,700
cd	−10,330	−14,550	−7,275	− 6,600	−2,200	+ 550	+ 185	−24,880	−16,930	−24,205	−27,080	−27,080
de	− 9,660	−13,640	−6,820	− 6,600	−2,200	+ 550	+ 185	−23,300	−16,260	−23,080	−25,500	−25,500
bf − *dh*	− 1,335	− 1,880	− 940	− 1,565	− 525	0	0	− 3,215	− 2,900	− 3,840	− 3,730	− 3,840
cg	− 2,670	− 3,760	−1,880	− 4,580	−1,530	+ 3,900	+ 1,300	− 6,430	+ 1,230 − 7,250	− 9,130	− 7,960	− 9,130 + 1,230
lf	0	0	0	+ 4,950	+1,650	−13,000	− 4,340	0	+ 4,950 −13,000	+ 4,950 −13,000	+ 1,650 − 4,340	+ 4,950 −13,000
af	+10,430	+14,700	+7,350	+ 3,220	+1,075	− 220	− 75	+25,130	+13,650	+21,000	+26,205	+26,205
fg	+ 8,940	+12,600	+6,300	+ 3,650	+1,220	− 5,850	− 1,950	+21,540	+12,590	+18,890	+22,760	+22,760

gg	+ 5,940	+ 8,410	+4,205	− 1,450	− 485	− 1,450	− 485	+14,350	+ 4,490	+ 8,695	+13,865	+14,350
fc	+ 1,490	+ 2,100	+1,050	+ 5,050	+ 1,685	− 8,750	− 2,920	+ 3,590	+ 6,540 − 7,260	+ 7,590 − 6,210	+ 5,275	+ 7,590 − 7,260
ch	+ 1,490	+ 2,100	+1,050	+ 1,700	+ 570	0	0	+ 3,590	+ 3,190	+ 4,240	+ 4,160	+ 4,240
gh	+ 2,980	+ 4,200	+2,100	+ 5,120	+ 1,375	− 4,350	− 1,450	+ 7,180	+ 8,100 − 1,370	+10,200	+ 8,555	+10,200 − 1,370
he	+ 4,470	+ 6,300	+3,150	+ 6,850	+ 2,285	− 4,350	− 1,450	+10,770	+11,320	+14,470	+13,055	+14,470
al	− 5,960	− 8,400	−4,200	− 5,000	− 1,670	+ 3,700	+ 1,235	−14,360	−10,960	−15,160	−16,030	−16,030
lo	− 5,960	− 8,400	−4,200	− 2,340	− 780	− 3,260	− 1,090	−14,360	− 8,300	−13,420	−15,450	−15,450
M_l ft.-lb.	0	0	0	21,500	7,170	36,500	12,170	0	36,500	36,500	12,170	36,500
M_m ft.-lb	0	0	0	10,750	3,585	18,250	6,085	0	18,250	18,250	6,085	18,250

+ = tension − = compression. All stresses given in pounds

column is compression, and it is equal to the reaction at the foot of the column. Since the stress in the leeward knee-brace is compression, its vertical component acts downward. Therefore the stress in the upper portion of the column must balance the difference between the stress in the lower portion of the column and the vertical component of the stress in the knee-brace. The desired stress can be determined from Fig. 11*b*. The force *L-N* represents the reaction at the foot of the column, and *L*-14 represents the stress in the leeward knee-brace. If these forces be projected on a vertical line through point 14, the required difference in stress components will be represented by the force *N'*-14. The required stress scales 3,700 lb., and the kind of stress is tension.

Case (b): The same mill bent is shown in Fig. 12 analyzed in accordance with the sub-committee recommendations quoted in Sec. 1, Art. **11*b***.

TABLE 33*b*.—STRESSES IN MEMBERS
(Wind as in Figs. 12*a* and *b*)

Member	Maximum wind	Minimum wind (1/3 W. L.)	D. L. + S. L.	D. L. + 1/3 W. L.	D. L. + W. L.	D. L. + 1/2 S. L. + wind	D. L. + S. L. + 1/3 wind	Maximum stress
ab	+16,350	+5,450	−28,110	−6,210	+4,690	−3,535	−22,660	−28,110
bc	+16,350	+5,450	−26,500	−5,550	+5,350	−2,400	−21,050	−26,500
cd	+13,830	+4,610	−24,800	−5,720	+3,500	−3,775	−20,270	−24,880
de	+13,830	+4,610	−23,300	−4,950	+4,170	−2,650	−18,590	−23,300
bf − dh	+ 1,995	+ 665	− 3,215	− 670	+ 660	− 280	− 3,023	− 3,215
cg	+ 4,580	+1,527	− 6,430	−1,143	+1,910	+ 30	− 4,903	− 6,430
lf	+ 1,830 − 4,120	+ 610 −4,120	0 0	+ 610 −1,373	+1,830 −4,120	+1,830 −4,120	+ 610 − 1,373	+ 1,830 − 4,120
af	−12,720	−4,240	+25,130	+6,190	−2,290	+5,060	+20,890	+25,130 − 2,290
fg	−11,040	−3,680	+21,540	+5,260	−2,100	+4,200	+17,860	+21,540 − 2,100
gg	− 5,900	−1,967	+14,350	+3,973	+ 40	+4,245	+12,383	+14,350
fc	− 4,620	−1,540	+ 3,590	− 50	−3,130	−2,080	+ 2,050	+ 3,590 − 3,130
ch	− 2,270	− 757	+ 3,590	+ 733	− 780	+ 270	+ 2,833	+ 3,590 − 780
gh	− 5,080	−1,693	+ 7,180	+1,287	−2,100	0	+ 5,487	+ 7,180 − 2,100
he	− 6,950	−2,317	+10,770	+2,053	−2,480	+ 670	+ 8,353	+10,770 − 2,480
al	+ 8,100	+2,700	−14,360	−3,260	+2,140	−2,060	−11,660	−14,360
lo	+ 7,270	+2,423	−14,360	−3,537	+1,310	−2,890	−11,937	−14,360
M_l ft.-lb.	+14,450	+4,817	0					+14,450
M_m ft.-lb.	+ 7,225	+2,408	0					+ 7,225

NOTE: See sketch at top of Table 33*a* for lettering of members. D. L. & S. L. stresses used from Table 33*a*.

The wall area likely to be open on the windward side has been assumed at 10 per cent, resulting in an internal pressure of $4.5 + {}^{10}/_{30}(12 - 4.5) =$

Fig. 12a.

Fig. 12b.

7 lb. per sq. ft. The roof angle of 26 deg. 34 min. provides 12 lb. per sq. ft. suction on the windward side and 9 lb. per sq. ft. suction on the leeward side. An external pressure of 20 lb. per sq. ft. exists on the verti-

cal windward wall. These pressures result in the forces shown in Fig. 12, obtained in the same manner as in Figs. 9 and 10.

Maximum stresses are tabulated above, and it may be seen that in every case the recent recommendations result in lower stresses, in comparing with Table 33*a*, owing primarily to the reversal of the type of stress. Considerable care and thought should be exercised, therefore, in using the newer recommendations. Connections and details should receive careful consideration also.

SECTION 3

BRIDGE TRUSSES

The stresses necessary for proportioning the members of a truss depend upon the requirements of the specifications in use. These requirements concern the determination of the working stresses. In some specifications a basic unit stress is specified. The area required for any member is its greatest, or *maximum* stress divided by the specified working stress. In other specifications the working stress is made to depend upon the *range* of stress in the member under consideration. This requires the determination of the *maximum* and *minimum* stresses in each member. As in the case of maximum moments and shears, as defined in Art. **70**, p. 139, the maximum stress in any member is the greatest stress of any kind (tension or compression) and the minimum stress is the least stress of the same kind, or in case of a reversal of stress, the minimum stress is the greatest stress opposite in kind to the maximum stress.

In the work which follows methods will be developed for the determination of the maximum and minimum stresses in the members of bridge trusses in common use. If in any case maximum stresses only are required, the portion of the discussion relative to minimum stress may be omitted, except where a reversal of stress occurs.

TRUSSES WITH HORIZONTAL CHORDS

Trusses of the type to be discussed in this article are shown in Figs. 4*a*, *b* and *c*, p. 4.

1. General Method of Analysis.—Article **80**, p. 157, contains a brief statement of the general methods of stress analysis for trusses with horizontal chords. It is there shown that the stress in any chord member of a truss with horizontal chords and a single web system is equal to the moment at the opposite chord point divided by the height of the truss at that point.

Expressed as a formula, we have

$$S = \frac{M}{h} \tag{1}$$

where M = moment at opposite chord point, and h = height of truss. For the truss of Fig. 1, the stress in member AB is given by moments about point C divided by h, the height of truss. Maximum and minimum stresses are found by combining the dead and live load moments as explained in Art. **70**, p. 139. Thus the maximum stress is the stress due to dead and live load moment and the minimum stress is due to dead load moment only. Stresses in top chord members are compressive and those in bottom chord members are tensile.

In Art. **80**, p. 158, it is shown that the vertical component of stress in any web member of a truss with horizontal chords and a single web system is equal to the shear on a section cutting the member in question and two chord members. The stress in the member is equal to the vertical component of stress multiplied by the secant of the angle which the member in question makes with the vertical.

FIG. 1.

Thus in Fig. 1, the vertical component of stress in member BD is equal to the shear on section x-x. The stress in BD may be expressed by the formula

$$S = V \sec \theta \tag{2}$$

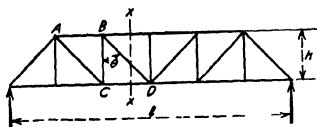
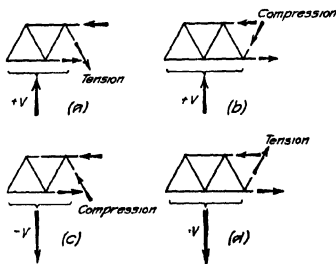
when V = shear on section x-x, and θ = angle between member BD and the vertical.

Maximum and minimum stresses in any member, as BD of Fig. 1, are determined from a combination of dead and live load shears, as explained in Art. **70**, p. 139. Generally the maximum stress in any member is due to the dead and positive live load shears and the minimum stress is due to the dead and negative live load shears.

The character of stress in any member may be determined by referring to Fig. 2.

FIG. 2.

A member sloping downward and to the right will be in tension under positive shear (Fig. 2a) and in compression under negative shear (Fig. 2c). A member sloping upward and to the right will be in compression under positive shear (Fig. 2b) and in tension under negative shear (Fig. 2d).

2. The Warren Truss.—Figure 3 shows a form of truss known as the Warren Truss and also as the Triangular Truss. This form of truss has been used extensively for small span highway and railroad bridges. It may be used either as a through or as a pony, or half-through truss, as

shown in Fig. 3a, or as a deck truss, as shown in Fig. 3b. When used for long spans, the Warren truss is sometimes arranged as shown in Fig. 4. To secure shorter panel lengths, vertical members have been placed at the center of each panel of the chord carrying the floor system. The methods

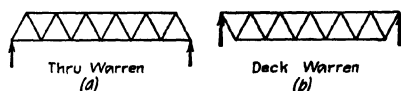



Fig. 3.

of calculation for the truss of Fig. 4 are the same as explained for the Pratt truss in Art. 4.

To illustrate the methods of stress calculation for the Warren truss, assume a truss of the dimensions shown in Fig. 5. Let the dead load

Fig. 4.

be taken as 600 lb. per ft. per truss. Stresses due to live load will be calculated for both uniform and concentrated live load systems. Let the uniform load be taken as 1,500 lb. per ft. per truss, and let the concentrated load system be taken as Cooper's E-50 loading. Required the

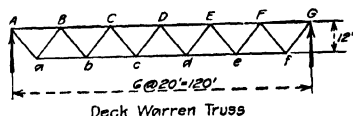



Fig. 5.

maximum and minimum stresses in all members due to the assumed loadings.

2a. Dead Load Stresses.—Two assumptions are generally made regarding the distribution of the dead load. All of the dead load

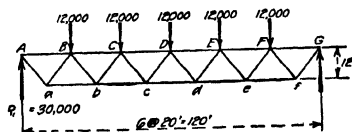



Fig. 6.

may be assumed as concentrated at the panel points of the chord carrying the floor system, or the dead load may be assumed as carried by both chords. Both cases will be given.

Dead Load Carried at Top Chord Panel Points.—Assuming that all of the dead load is carried at the top chord, each panel load is 20 × 600 =

12,000 lb. These loads are placed as shown in Fig. 6. The end reactions are equal and each is equal to half the total load on the truss, or $R_1 = 5/2 \times 12{,}000 = 30{,}000$ lb. Since the loads on the structure are symmetrical about the center of the span, the stresses need be calculated only for one-half of the span.

The chord stresses may be determined from Eq. (1), Art. 1. Substituting in Eq. (1), the chord stresses are found to be as follows (T = tension C = compression).

ab....(30,000) $20/12 = 50{,}000T$
bc....[(30,000) (2) − (12,000) (1)] $20/12 = 80{,}000T$
cd....[(30,000) (3) − (12,000) (1 + 2)] $20/12 = 90{,}000T$
AB....(30,000) (10/12) $= 25{,}000C$
BC....[(30,000) (3/2) − (12,000) (1/2)] $20/12 = 65{,}000C$
CD....[(30,000) (5/2) − (12,000) (1/2 + 3/2)] $20/12 = 85{,}000C$

Since the moment at any lower chord point, as b, Fig. 6, is the average of the moments at points B and C, the stresses in the top chord members may be determined from those given for the bottom chord members. Thus the stress in BC is the average of the stresses in members ab and bc. That is,

$$BC = \tfrac{1}{2}(ab + bc) = \tfrac{1}{2}(50{,}000 + 80{,}000) = 65{,}000$$

Also

$$AB = \tfrac{1}{2}ab = (\tfrac{1}{2})(50{,}000) = 25{,}000$$

and

$$CD = \tfrac{1}{2}(bc + cd) = \tfrac{1}{2}(80{,}000 + 90{,}000) = 85{,}000$$

Stresses in web members may be determined by means of Eq. (2). Since the structure is symmetrical, the value of sec θ is the same for all members. From Fig. 6,

$$\text{Sec } \theta = \frac{\sqrt{10^2 + 12^2}}{12} = 1.30$$

The necessary calculations are as follows:

Member	Shear = V	Stress = V sec θ
Aa	+30,000	(30,000)(1 3) = 39,000T
aB	+30,000	(30,000)(1 3) = 39,000C
Bb	+18,000	(18,000)(1 3) = 23,400T
bC	+18,000	(18,000)(1 3) = 23,400C
Cc	+ 6,000	(6,000)(1 3) = 7,800T
cD	+ 6,000	(6,000)(1 3) = 7,800C

Figure 7 shows the members of the left half of the truss with the dead load stresses indicated on each member.

Dead Load Divided between Top and Bottom Chord Joints.—It will be assumed that two-thirds of the dead load is concentrated at the top chord joints and one-third at the lower chord joints. The top chord panel load is then $(\frac{2}{3})(20)(600) = 8{,}000$ lb., and the lower chord panel load is

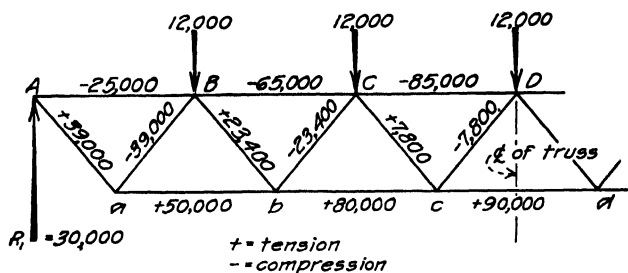



Fig. 7.—Dead load stresses, deck Warren truss. All dead load at top chord joints.

$(\frac{1}{3})(20)(600) = 4{,}000$ lb. These loads are shown in position on Fig. 8. The end reaction is

$$R_1 = \tfrac{1}{2}[(5)(8{,}000) + 6(4{,}000)] = 32{,}000 \text{ lb.}$$

If the true weights of the floor system and truss members are known,

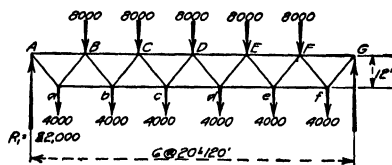



Fig. 8.

the actual panel loads may be computed. However, the distribution of loads assumed in Fig. 8 is sufficiently accurate for small structures.

The stresses in chord and web members are determined by means of Eqs. (1) and (2). On substituting in these equations, the stresses are found to be as follows:

Chord Stresses:

ab .... $[(32{,}000)(1) - (4{,}000)(\frac{1}{2})]\frac{20}{12} = 50{,}000T$
bc .... $[(32{,}000)(2) - (8)(1) - 4(\frac{1}{2} + \frac{3}{2})]\frac{20}{12} = 80{,}000T$
cd .... $[(32{,}000)(3) - 8(1 + 2) - 4(\frac{1}{2} + \frac{3}{2} + \frac{5}{2})]\frac{20}{12} = 90{,}000T$
AB.... $[(32{,}000)\frac{10}{12} = 26{,}700C$
BC.... $[(32{,}000)(\frac{3}{2}) - (8)(\frac{1}{2}) - (4)(1)]\frac{20}{12} = 66{,}700C$
CD.... $[(32{,}000)(\frac{5}{2}) - (8)(\frac{1}{2} + \frac{3}{2}) - (4)(1 + 2)]\frac{20}{12} = 86{,}700C$

Web Stresses:

Member	Shear = V	Stress = V sec θ
Aa	+32,000	(32,000)(1.3) = 41,600T
aB	+28,000	(28,000)(1.3) = 36,400C
Bb	+20,000	(20,000)(1.3) = 26,000T
bC	+16,000	(16,000)(1.3) = 20,800C
Cc	+ 8,000	(8,000)(1.3) = 10,400T
cD	+ 4,000	(4,000)(1.3) = 5,200C

The stresses calculated above are shown on the truss diagram of Fig. 9.

Effect of Dividing the Dead Load.—On examining the stresses for the two conditions of dead load distribution given above, it will be found that

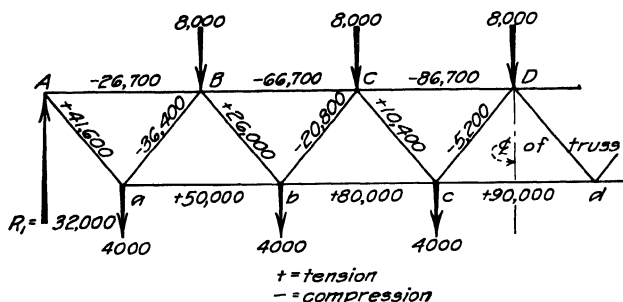



FIG. 9.—Dead load stresses, deck Warren truss dead load divided between top and bottom chord joints.

the stresses in the bottom chord members are the same in both cases. The stresses in the top chord and web members are changed.

To determine the effect of changing the distribution of the dead load on the stresses in the truss members and the reactions, consider the conditions shown in Fig. 10. Let the truss consist of n panels of length d, and let W = total dead panel load = load per foot times the length of a panel. In Fig. 10a all of the dead load is assumed as applied at the top chord points and in Fig. 10b two-thirds of the dead load is assumed as concentrated at the top chord panel points and one-third is concentrated at the lower chord panel points.

Let R_1 and R_1' respectively represent the left reactions for the loadings shown in Figs. 10a and b. It can readily be shown that

$$R_1 = \tfrac{1}{2}(n - 1)W$$

and

$$R_1' = \tfrac{1}{6}(3n - 2)W$$

The change in reaction due to the difference in distribution of dead load may be found by subtracting the first equation from the second, that is, Difference in Reaction $= R_1' - R_1 = \frac{W}{6}$. Hence the reaction has been increased by one-sixth of a panel load. For the truss shown in Fig. 6, the panel load is $W = 12{,}000$ lb. Therefore the change in the reaction will be $(\frac{1}{6})(12{,}000) = 2{,}000$ lb. On comparing values of reactions shown on Figs. 7 and 9, it will be found to check the above statement. Note that the change in reaction is independent of the number of panels in the truss.

General expressions for stresses in the web members in a panel at a distance of m panels from the left end of the truss may be written for the

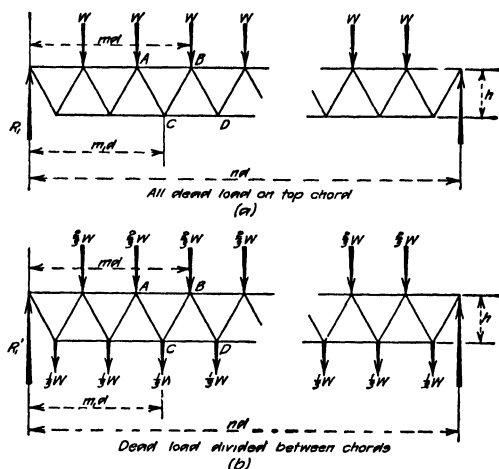



Fig. 10.

conditions shown in Figs. 10a and b. These expressions are as follows:

For Fig. 10a

$$\text{Stress in } AC = [R_1 - (m-1)W] \sec\theta$$
$$= \frac{W}{2}(n + 1 - 2m) \sec\theta \text{ (Tension)}$$
$$\text{Stress in } BC = \frac{W}{2}(n + 1 - 2m) \sec\theta \text{ (Compression)}$$

For Fig. 10b

$$\text{Stress in } AC = [R_1' - (m-1)(\tfrac{2}{3}W + \tfrac{1}{3}W)] \sec\theta$$
$$= \frac{W}{2}\left(n + \frac{4}{3} - 2m\right) \sec\theta \text{ (Tension)}$$
$$\text{Stress in } BC = [R_1' - (m-1)\tfrac{2}{3}W - m\tfrac{1}{3}W] \sec\theta$$
$$= \frac{W}{2}\left(n + \frac{2}{3} - 2m\right) \sec\theta \text{ (Compression)}$$

From these general equations it can readily be seen that the stress in any tension diagonal in Fig. 10b is greater by $\frac{1}{6}W$ sec θ *than the stress in the same member in Fig. 10a.* Also, the stress in any compression diagonal in Fig. 10b is less by $\frac{1}{6}W$ sec θ than the stress in the corresponding member in Fig. 10a. For the conditions shown on Fig. 6, we have $\frac{1}{6}W \sec \theta = (\frac{1}{6})(12{,}000)(1.3) = 2{,}600$ lb. On comparing the stresses for corresponding members in Figs. 7 and 9, it will be found that the above statement is correct.

To determine the effect of the change in distribution of dead load on the stresses in chord members, consider top chord member AB and bottom chord member CD, whose moment centers are respectively at distances of m and m_1 panels from the left support. The general equations for these stresses are readily shown to be as follows:

For Fig. 10a

$$\text{Stress in } AB = \left\{R_1 m_1 - W\left[\frac{1}{2} + \frac{3}{2} + \cdots \left(m_1 - 1\right)\right]\right\}\frac{d}{h}$$
$$= \frac{Wd}{2h}\left[m_1\left(n - m_1\right) - \frac{1}{4}\right] \text{ (Compression)}$$
$$\text{Stress in } CD = \left\{R_1 m - W\left[1 + 2 + \cdots \left(m - 1\right)\right]\right\}\frac{d}{h}$$
$$= \frac{Wd}{2h} m\,(n - m) \text{ (Tension)}$$

For Fig. 10b

$$\text{Stress in } AB = \left\{R_1' m_1 - \frac{2}{3}W\left[\frac{1}{2} + \frac{3}{2} + \cdots \left(m_1 - 1\right)\right] - \frac{W}{3}\left[1 + 2 + \cdots \left(m_1 - \frac{1}{2}\right)\right]\right\}\frac{d}{h}$$
$$= \frac{Wd}{2h}\left[m_1\left(n - m_1\right) - \frac{1}{12}\right] \text{ (Compression)}$$
$$\text{Stress in } CD = \left\{R_1' m - \frac{2}{3}W\left[1 + 2 + \cdots \left(m - 1\right)\right] - \frac{1}{3}W\left[\frac{1}{2} + \frac{3}{2} + \quad \left(m - \frac{1}{2}\right)\right]\right\}\frac{d}{h}$$
$$= \frac{Wd}{2h} m(n - m) \text{ (Tension)}$$

From these general equations it can readily be seen that the stresses in the bottom chord members of Figs. 10a and b are the same. The stresses in top chord members of Fig. 10a and b are seen to differ. By subtracting the value given for Fig. 10a from that for Fig. 10b it will be found that the stress in a top chord member of Fig. 10b exceeds the stress in the corresponding member in Fig. 10a by $\frac{1}{12}W\frac{d}{h}$, where d is a panel

length and h is the height of truss. Also the difference in stress is the same for all top chord members.

On comparing the top chord stress shown in Figs. 7 and 9 it will be found that those in Fig. 9 exceed those in Fig. 7 by a common difference of 1,700 lb. From the discussion given above, this difference is $\frac{1}{12} W \frac{d}{h}$. For $W = 12{,}000$, $d = 20$ and $h = 12$, we have

$$\text{Difference in Stress} = (1/12)(12{,}000)(20/12) = 1{,}700 \text{ lb.}$$

The above discussion suggests a simple means of obtaining the stresses shown on Fig. 9 from those on Fig. 7 by means of certain simple corrections. These corrections are as follows: For top chord members, increase the stresses shown on Fig. 7 by $\frac{1}{12} W \frac{d}{h}$; for tension web members, increase the stress by $\frac{1}{6}W \sec \theta$; for compression web members, decrease the stress by $\frac{1}{6}W \sec \theta$; for lower chord members, no correction is necessary; for reactions, increase the value shown on Fig. 7 by $\frac{1}{6}W$. These corrections are the same regardless of the number of panels in the structure. For through bridges the same corrections may be used. For chord members interchange top and bottom chord corrections, and for web members interchange tension and compression member corrections.

2b. Live Load Stresses—Uniform Live Load.—The uniform live load per foot is 1,500 lb. Hence the live panel load is (1,500) (20) = 30,000 lb. These panel loads are to be applied at top chord points for the deck truss conditions shown in Fig. 6. No part of the live load is assumed as carried by lower chord points.

In determining the chord stresses due to live load, all top chord panel points are loaded with 30,000 lb. each. The loading conditions are similar to those shown in Fig. 6. Left reaction = (5/2)(30,000) = 75,000 lb. The chord stresses are as follows:

ab ....$75{,}000(20/12) = 125{,}000T$
bc ....$[(75{,}000)(2) - (30{,}000)(1)]20/12 = 200{,}000T$
cd ....$[(75{,}000)(3) - (30{,}000)(1 + 2)]20/12 = 225{,}000T$
AB....$(75{,}000)(10/12) = 62{,}500C$
BC....$[(75{,}000)(3/2) - (30{,}000)(1/2)]20/12 = 162{,}500C$
CD....$[(75{,}000)(5/2) - (30{,}000)(1/2 + 3/2)]20/12 = 212{,}500C$

These stresses are shown on Fig. 11.

Since the loading conditions for live load chord stresses are the same as for the dead load distribution shown in Fig. 6, the live and dead load stresses bear the same ratio as the corresponding panel loads. That is:

Live load stress in any member =

$$\text{Dead load stress (Fig. 6)} \times \frac{\text{Live panel load}}{\text{Dead panel load}}$$

Thus for any member, say *BC*,

$$\text{Live load stress} = (65{,}000)(30/12) = 162{,}500$$

Any other stress may be determined in a similar manner.

The live load stresses in web members are determined for partial loading of the truss. Using the conventional method of calculation given on p. 93, Art. **66***b*, the shears in the several panels of the truss shown in Fig. 6 are as follows:

Panel	Shear	
AB	$30/6(1 + 2 + 3 + 4 + 5)$	= 75,000 lb.
BC	$30/6(1 + 2 + 3 + 4)$	= 50,000
CD	$30/6(1 + 2 + 3)$	= 30,000
DE	$30/6(1 + 2)$	= 15,000
EF	$30/6(1)$	= 5,000
FG	$30/6(0)$	= 0

To determine the stresses in the members of the left half of the truss due to positive and negative live load shears, note that the negative shear in any panel of the left half of the truss, as *BC*, is equal to the positive shear in the corresponding panel in the right half of the truss, that is, panel *EF* (see p. 94). The stresses in the members are then as follows:

Member	Panel	Positive Shear Stress	Negative Shear Stress
Aa	*AB*	(75,000)(1.3) = 97,500*T*	0
aB	*AB*	(75,000)(1.3) = 97,500*C*	0
Bb	*BC*	(50,000)(1.3) = 65,000*T*	(5,000)(1.3) = 6,500*C*
bC	*BC*	(50,000)(1.3) = 65,000*C*	(5,000)(1.3) = 6,500*T*
Cc	*CD*	(30,000)(1.3) = 39,000*T*	(15,000)(1.3) = 19,500*C*
cD	*CD*	(30,000)(1.3) = 39,000*C*	(15,000)(1 3) = 19,500*T*

These stresses are shown on the members in Fig. 11.

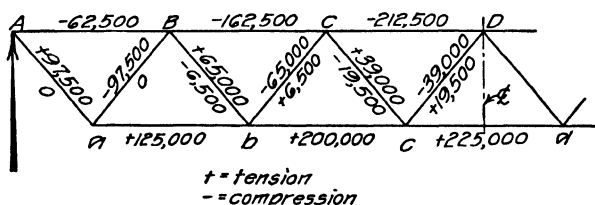



Fig. 11.—Live load stresses, deck Warren truss uniform live load.

2c. Live Load Stresses—E-50 Engine Loading.—Methods for determining the position of the E-50 loading for maximum moment are given in Art. **68***e* p. 123, and Art. **68***g* gives methods for determining the position of load for maximum shear in the panels of the truss. Moment Table I, p. 105, will be used in determining values of moments and shears.

Chord Stresses.—The position of loads for maximum moment may be determined by direct substitution in the criterion of Eq. (33), p. 108, or by means of Fig. 141, p. 114.

Member ab.—Moment center at B, Fig. 6, which is 20 ft. from the left end of the truss. From Fig. 141, p. 114, wheel 3 gives maximum moment. The determination of position of loads by means of the moment criterion will also be given. For the given moment center, noting that $a = 20$ ft. and $l = 120$ ft., the criterion for moment at B is

$$\frac{G}{6} - W = 0$$

Try wheel 3 at B with the train headed to the left. For this position of loads, both engines and 4 ft. of uniform load are on the span. Hence $G = 355 + (4)(2.5) = 365$. The value of W varies from 37.5 (wheel 3 to right of B) to 62.5 (wheel 3 to left of B). Substituting these values in the criterion, we have

For wheel 3 to right of B

$$\frac{G}{6} - W = \frac{365}{6} - 37.5 = +$$

For wheel 3 to left of B

$$\frac{G}{6} - W = \frac{365}{6} - 62.5 = -$$

Since the criterion changed sign as wheel 3 passed point B, wheel 3 gives a maximum moment at B.

Try wheel 4 at B. The uniform load covers 9 ft. of the span and $G = 355 + (9)(2.5) = 377.5$ and $\frac{G}{6} = 62.9$. W varies from 62.5 to 87.5. Substituting in the criterion it will be found that wheel 4 also satisfies the criterion. Wheels 2 and 5 were also tried but it was found that they did not satisfy the criterion.

The moment at B for wheels 3 and 4 at B must now be calculated to determine the true maximum value. These moments are calculated by methods given in Art. **68**c.

For wheel 3 at B, we have

$$M_B = (R_1)(20) - \sum_1^3 W$$

and

$$R_1 = [20{,}455 + (355)(4) + (\tfrac{1}{2})(2.5)(4)^2]\tfrac{1}{120} = 182.5$$

Hence

$$M_B = (182.5)(20) - 287.5 = 3{,}362.5 \text{ (thousand ft.-lb.)}$$

For wheel 4 at B,

$$M_B = (R_1)(20) - \sum_1^4 W$$

and

$$R_1 = [20{,}455 + (355)(9) - (\tfrac{1}{2})(2.5)(9)^2]\tfrac{1}{120} = 197.92$$

Hence

$$M_B = (197.92)(20) - 600 = 3{,}358.4 \text{ (thousand ft.-lb.)}$$

Therefore wheel 3 gives the maximum moment, which checks the information given on Fig. 141.

The stress in ab is then

$$\frac{M}{h} = \frac{3{,}362{,}500}{12} = 280{,}200 \text{ lb., tension}$$

Member bc.—Moment center at C, 40 ft. from left end of the span. From Fig. 141, p. 114, wheel 6 gives maximum moment at this point. However, the diagram shows that the moment at point E (Fig. 6) 40 ft. from the right end of the span is greater than the moment at C. Hence we must calculate the moment at E with wheel 13 at this point and use this moment for points C and E. With wheel 13 at E with the train headed to the left, both engines and 5 ft. of uniform load are on the span. The moment at E is

$$M_E = (R_1)(80) - \sum_1^{13} W$$

and

$$R_1 = [20{,}455 + (355)(5) + (\tfrac{1}{2})(2.5)(5)^2]\tfrac{1}{120} = 185.5$$

Hence

$$M_E = (185.5)(80) - 9{,}585 = 5{,}255 \text{ (thousand ft.-lb.)}$$

The stress in bc is

$$\frac{5{,}255{,}000}{12} = 438{,}000 \text{ lb., tension}$$

When wheel 6 is placed at C, the moment at C is found to be 5,130,000 ft.-lb., a value about 2.4 per cent less than that given above for point E.

Member cd.—Moment center at D, Fig. 6, 60 ft. from the end of the truss. From Fig. 141, p. 114, wheel 11 answers for maximum moment.

$$M_D = (R_1)(60) - \sum_2^{11} W$$

Noting that wheel 1 has passed the left end of the span, we have

$$R_1 = [19{,}092.5 + (342.5)(15) + (\tfrac{1}{2})(2.5)(15)^2]\tfrac{1}{120} = 204.26$$

Hence

$$M_D = (204.26)(60) - 6{,}510 = 5{,}745.6 \text{ (thousand ft.-lb.)}$$

The stress in *cd* is

$$\frac{5{,}745{,}600}{12} = 478{,}800 \text{ lb., tension}$$

Stresses in Top Chord Members.—Moment centers for stress in top chord members are located at the lower chord points, or at points between the top chord panel points. Figure 141, p. 114, will not give the wheels which answer for maximum moment at points located between top chord panel points. We must substitute in the criterion in order to determine the wheels which give maximum moment. The proper criterion is given by Eq. (45), Art. **68***e*, p. 125, for the conditions shown in Fig. 153, p. 124.

Member AB.—Moment center at point a, 10 ft. from the left end of the beam. For the conditions shown in Fig. 153, p. 124, $a = 10$ ft.; $c = 10$ ft.; $d = 20$ ft.; $l = 120$ ft.; and $W_1 = 0$. Eq. (45), p. 125, becomes

$$G\frac{10}{120} - W_2\frac{10}{20} = \frac{G}{12} - \frac{W_2}{2} = 0$$

This may be written

$$\frac{G}{6} - W_2 = 0$$

Note that this criterion is the same as for moment at B. It was found on p. 227 that wheel 3 gave the maximum moment at B. The moment at a is the average of those at A and B, or

$$M_b = \tfrac{1}{2}(0 + 3{,}362{,}500) = 1{,}681{,}250 \text{ ft.-lb.}$$

Hence

$$\text{Stress in } AB = \frac{1{,}681{,}250}{12} = 140{,}100 \text{ lb., compression}$$

Member BC.—Moment center at b. With $a = 30$ ft.; $c = 10$ ft.; $d = 20$ ft.; $l = 120$ ft.; the criterion of Eq. (45), p. 123, becomes

$$\frac{G}{4} - (W_1 + \tfrac{1}{2}W_2) = 0$$

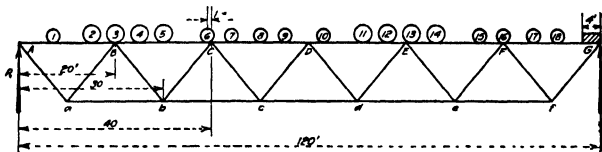



FIG. 12.

To satisfy this criterion, some wheel must be placed at either point B or C of Fig. 6. Try wheel 3 at B. The relative position of the loads is

shown in Fig. 12. For the conditions shown in Fig. 12, $W_1 = 37.5$ to 62.5; $W_2 = 91.25$ to 66.25; and $G = 355 + (4)(2.5) = 365$.

The substitutions in the criterion are as follows:

For wheel 3 to right of B

$$365/4 - [37.5 + 1/2(91.25)] = +$$

For wheel 3 to left of B

$$365/4 - [62.5 + 1/2(66.25)] = -$$

Hence wheel 3 satisfies the criterion. In a similar manner, wheels 2 and 4 were tried at point B, and wheels 6 and 7 were tried at point C, but none of them satisfied the criterion. Since it is possible that the maximum moment at point e on the right side of the truss might be greater than the moment at point b, wheels were also tried at points E and F. It was found that wheel 12 at E and wheel 16 at F also satisfied the criterion.

The moments at b and e for the wheels which satisfied the criterion will now be calculated and compared in order to determine the true maximum moment. For wheel 3 at B, as shown in Fig. 12, the moment at b is the average of the moments at B and C, that is

$$M_b = 1/2(M_B + M_C)$$

Now

$$M_B = (R_1)(20) - M_3$$

$$M_C = (R_1)(40) - \left[M_6 + \left(\sum_1^6 W\right)(1)\right]$$

where

$$R_1 = [20{,}455 + (355)(4) + (1/2)(2.5)(4)^2]1/120$$
$$= 182.45$$

Therefore

$$M_B = (182.45)(20) - 287.5 = 3{,}361.5$$
$$M_C = (182.45)(40) - [2{,}050 + (128.75)(1)] = 5{,}119.25$$

and

$$M_b = 1/2(3{,}361.5 + 5{,}119.25) = 4{,}240.375 \text{ (thousand ft.-lb.)}$$

When wheel 12 is placed at E, $M_e = 4{,}234{,}875$ ft.-lb. and when wheel 16 is placed at F, $M_e = 4{,}222{,}375$ ft.-lb. Hence wheel 3 at B gives the greatest moment.

The maximum stress in BC is therefore found to be

$$\frac{4{,}240{,}375}{12} = 353{,}365 \text{ lb., compression}$$

Member CD.—The maximum stress in CD will be given by the moment at c or d, which ever is the greater. By the methods used for member BC, it will be found that the moment at d is the greater.

The criterion of Eq. (45), when applied for point *d*, takes the form

$$\frac{7}{12}G - (W_1 + \tfrac{1}{2}W_2) = 0$$

Wheels 12 and 13 at *E* are found to satisfy this criterion, and of these, wheel 13 gives the greater moment. The necessary calculations are as follows (load position shown in Fig. 13):

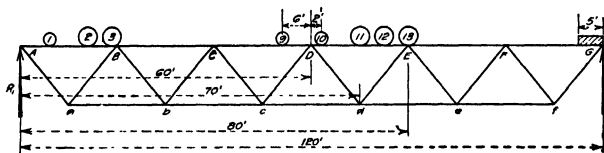



Fig. 13.

Substitution in criterion

$$W_1 = 177.5;\ W_2 = 62.5 \text{ to } 87.5;\ G = 355 + (5)(2.5) = 367.5$$

Wheel 13 to right of *E*

$$(\tfrac{7}{12})(367.5) - [177.5 + (\tfrac{1}{2})(62.5)] = +$$

Wheel 13 to left of *E*

$$(\tfrac{7}{12})(367.5) - [177.5 + (\tfrac{1}{2})(87.5)] = -$$

Therefore wheel 13 at *E* satisfies the criterion.

Calculation of moment at *d*

$$M_d = \tfrac{1}{2}(M_D + M_E)$$

$$R_1 = [20{,}455 + (355)(5) + (\tfrac{1}{2})(2.5)(5)^2]\tfrac{1}{120} = 185{,}510 \text{ lb.}$$

$$M_D = (185{,}510)(60) - (1{,}000)[4{,}370 + (177.5)(6)] = 5{,}695{,}000 \text{ ft.-lb.}$$

$$M_E = (185{,}510)(80) - (1{,}000)(9{,}585) = 5{,}255{,}800 \text{ ft.-lb.}$$

Therefore

$$M_d = (\tfrac{1}{2})(5{,}695{,}000 + 5{,}255{,}800) = 5{,}475{,}400 \text{ ft.-lb}$$

Hence

$$\text{Stress in } CD = (\tfrac{1}{12})(5{,}475{,}400) = 456{,}300 \text{ lb.}$$

When wheel 12 is placed at *E*,

$$M_d = 5{,}465{,}900 \text{ ft.-lb.}$$

The position of loads for maximum moment at *c* was also determined. It was found that wheel 6 at *C* and wheel 10 at *D* both satisfied the criterion. The moment at *c* was found to be the same for both load positions, and $M_c = 5{,}406{,}500$ ft.-lb. Therefore moment at *d* gives the

greatest chord stress, as calculated above. The chord stresses calculated above are shown on Fig. 14.

Stresses in Web Members.—Stresses in web members of a Warren truss due to train loading are determined by the same general methods as used on p. 226 for uniform live load. The process differs only in the methods used in calculating the shears in the several panels.

The position of loads for maximum shear in any panel is determined by means of the criterion of Eq. (55), p. 131. For the truss under con-

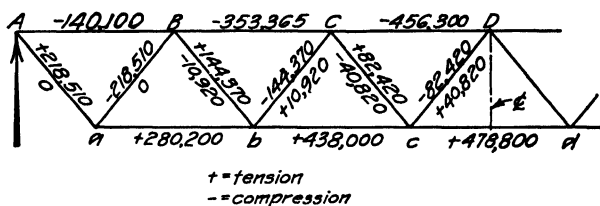



Fig. 14.—Live load stresses, deck Warren truss Coopers E–50 loading.

sideration there are six panels. Hence $n = 6$ in Eq. (55), and the criterion becomes

$$\frac{G}{6} - W_2 = 0$$

where G = total load on truss and W_2 = load in panel for which the shear is desired. The same criterion holds for all panels. Detailed calculations for the shears in the several panels of the truss shown in Fig. 6 will now be given.

Shear in Panel AB.—Try wheel 3 at B with the train headed to the left. The uniform load covers 4 ft. of the right end of the bridge. Hence $G = 355 + (4)(2.5) = 365$, and $\frac{G}{6} = 60.8$. For wheel 3 to the right of B, $G_1 = 37.5$ and for wheel 3 to the left of B, $G_2 = 62.5$. Substituting these values in the criterion, we have

For wheel 3 to left of B

$$60.8 - 37.5 = +$$

For wheel 3 to right of B

$$60.8 - 62.5 = -$$

Therefore wheel 3 satisfies the criterion.

Try wheel 4 at B. Uniform load covers 9 ft. at right end of span. $G = 355 + (9)(2.5) = 377.5$; $\frac{G}{6} = 62.9$; G_2 varies from 62.5 to 87.5. Wheel 4 satisfies the criterion, since $\frac{G}{6}$ lies between the two values of

G_2, thus causing a change in sign when these values are substituted in the criterion. Wheels 2 and 5 did not satisfy the criterion.

To determine which of these wheels gives the maximum shear in the panel, the shears must be calculated and compared. The shear in panel AB is equal to the left reaction R_1 minus the joint load at A due to the loads in panel AB. For wheel 3 at B,

$$R_1 = \left[20{,}455 + (355)(4) + \left(\frac{1}{2}\right)(2.5)(4)^2\right]\frac{1{,}000}{120} = 182{,}460 \text{ lb.}$$

$$\text{Panel load at } A = \frac{287{,}500}{20} = 14{,}375 \text{ lb.}$$

Hence

Shear in panel $AB = 182{,}460 - 14{,}375 = 168{,}085$ lb. For wheel 4 at B,

$$R_1 = \left[20{,}455 + (355)(9) + \left(\frac{1}{2}\right)(2.5)\ (9)^2\right]\frac{1{,}000}{120} = 197{,}930$$

$$\text{Panel load at } A = \frac{600{,}000}{20} = 30{,}000 \text{ lb.}$$

Shear in panel $AB = 197{,}930 - 30{,}000 = 167{,}930$ lb.
Therefore wheel 3 gives the greater shear.

Shear in Panel BC.—Try wheel 3 at C. Wheel 16 is at the right end of the truss. G varies from 306.25 to 322.5 and G_2 varies from 37.5 to 62.5. Substituting these values in the criterion, we have

For wheel 3 to the right of C,

$$\frac{306.25}{6} - 37.5 = +$$

For wheel 3 to the left of C

$$\frac{322.5}{6} - 62.5 = -$$

Therefore wheel 3 satisfies the criterion. Wheels 2 and 4 were also tried but they did not satisfy the criterion.

For wheel 3 at C,

$$R_1 = \frac{15{,}051{,}250}{120} = 125{,}430 \text{ lb.}$$

$$\text{Panel load at } B = \frac{287{,}500}{20} = 14{,}375$$

Shear in panel $BC = 125{,}430 - 14{,}375 = 111{,}055$ lb.

Shear in Panel CD.—Try wheel 2 at D. Wheel 11 is 4 ft. to the left of the right end of the truss; $G = 215$; $\frac{G}{6} = 35.8$; and G_2 varies from

12.5 to 37.5. Hence wheel 2 satisfies the criterion, because the value of $\frac{G}{6}$ lies between the two values of G_2. Try wheel 3 at D. Wheel 12 is 4 ft. to the left of the right end of the span; $G = 240$; $\frac{G}{6} = 40$; G_2 varies from 37.5 to 62.5. Hence wheel 3 also satisfies the criterion. Wheels 1 and 4 were also tried but they did not satisfy the criterion.

When wheel 2 is placed at D,

$$R_1 = [7{,}310 + (215)(4)]\frac{1{,}000}{120} = 68{,}090 \text{ lb.}$$

$$\text{Panel load at } C = \frac{100{,}000}{20} = 5{,}000 \text{ lb.}$$

Shear in panel CD = 63,090 lb.

When wheel 3 is placed at D,

$$R_1 = [8{,}385 + (240)(4)]\frac{1{,}000}{120} = 77{,}785 \text{ lb.}$$

$$\text{Panel load at } C = \frac{287{,}500}{20} = 14{,}375 \text{ lb.}$$

Shear in panel CD = 63,400 lb.

Hence wheel 3 gives the maximum.

Shear in Panel DE.—Try wheel 2 at E. Wheel 9 is at the right end of the span. Hence G varies from 161.25 to 177.5, depending on whether wheel 2 is considered to the right or to the left of E. Therefore $\frac{G}{6}$ varies from 26.9 to 29.6. G_2 varies from 12.5 to 37.5. Substituting these values in the criterion, we have

For wheel 2 to the right of E,

$$26.9 - 12.5 = +$$

For wheel 2 to the left of E.

$$29.6 - 37.5 = -$$

Therefore wheel 2 satisfies the criterion. Wheels 1 and 3 were also tried, but it was found that they did not satisfy the criterion.

For wheel 2 at E,

$$R_1 = \frac{4{,}370{,}000}{120} = 36{,}400$$

$$\text{Panel load at } D = \frac{100{,}000}{20} = 5{,}000$$

and

Shear in panel DE = 36,400 − 5,000 = 31,400 lb.

Shear in Panel EF.—Try wheel 2 at F. Wheel 5 is 5 ft. from the right

end of the span; $G = 112.5$; $\frac{G}{6} = 18.8$; and G_2 varies from 12.5 to 37.5. Hence wheel 2 satisfies the criterion. Wheels 1 and 3 were tried but did not answer.

For wheel 2 at F,

$$R_1 = [1{,}037.5 + (112.5)(5)]\frac{1{,}000}{120} = 13{,}400 \text{ lb.}$$

$$\text{Panel load at } E = \frac{100{,}000}{20} = 5{,}000 \text{ lb.}$$

and

$$\text{Shear in panel } EF = 13{,}400 - 5{,}000 = 8{,}400 \text{ lb.}$$

Shear in Panel FG.—It can readily be seen that the shear in panel FG is zero, for to satisfy the criterion, there can be no load on the truss.

Stresses in Web Members.—The stresses in web members due to positive and negative shear are determined by the methods used on p. 226 for uniform loads. These stresses are given in the following table and are also shown on Fig. 14.

Member	Panel	Positive Shear Stress	Negative Shear Stress
Aa	*AB*	(168,085)(1.3) = 218,510*T*	0
aB	*AB*	(168,085)(1.3) = 218,510*C*	0
Bb	*BC*	(111,055)(1.3) = 144,370*T*	(8,400)(1.3) = 10,920*C*
bC	*BC*	(111,055)(1.3) = 144,370*C*	(8,400)(1.3) = 10,920*T*
Cc	*CD*	(63,400)(1.3) = 82,420*T*	(31,400)(1.3) = 40,820*C*
cD	*CD*	(63,400)(1.3) = 82,420*C*	(31,400)(1.3) = 40,820*T*

2*d*. Maximum and Minimum Stresses.—Maximum and minimum stresses in the members of the truss under consideration are determined by combining the dead and live load stresses calculated in the

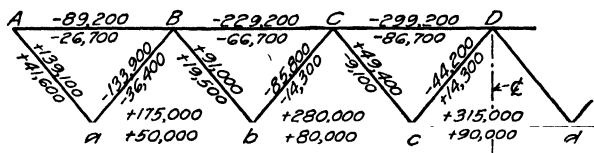



Fig. 15.—Maximum and minimum stresses, deck Warren truss, dead load and uniform live load.

preceding articles. In general the maximum stress in any web member is the sum of the dead load stress and the stress due to positive live load shear; the minimum stress in any web member is the sum of the dead load stress and the stress due to negative live load shear. For chord members of simple trusses, the maximum stress is the sum of the dead and live load stresses and the minimum stress is the dead load only. Figure 15 shows on each member, the maximum and minimum stress in that member for

dead load and uniform live load. The dead and live load values, as calculated in the preceding articles, are taken from Figs. 9 and 11. Figure 16 shows the maximum and minimum stresses for dead load and E-50 live load. Dead load and live load stresses used in making the necessary combinations are taken from Figs. 9 and 14.

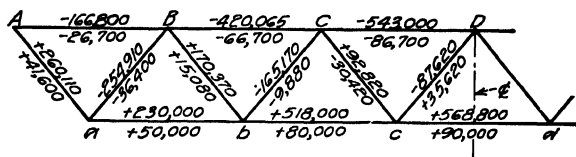



FIG. 16.—Maximum and minimum stresses, deck Warren truss. Dead load and Cooper's E-50 loading.

3. The Pratt Truss.—Figure 17 shows forms of Pratt trusses for highway and railway bridge spans. The forms shown are used for spans from 90 to 200 ft. Most of the Pratt trusses in use are of the pin-connected type. In recent years, however, the riveted type is gaining in favor.

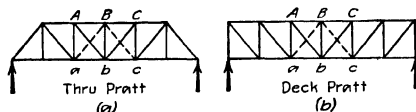



FIG. 17.

Pin-connected Pratt trusses are generally built with *counters* in the panels near the center of the span. The object of these counters is to provide a truss system such that the web members will not be subjected to reversal of stress due to the existence of positive and negative combined shears near the center of the span.

The action of counters is shown in Fig. 18. If any quadrangular frame with hinged joints, shown by the full lines of Fig. 18a, is subjected to a load V (corresponding to a positive shear), the frame will tend to take the form shown by the dotted lines. To prevent this distortion of the frame by means of a member which will take tension only, the tension member must be attached at points A and B, as shown in Fig. 18b. If the load V is applied as shown in Fig. 18c (corresponding to a negative shear), the tension member which will resist distortion must be attached at C and D, as shown in Fig. 18c. In Fig. 18d, members are shown in place across both diagonals of the figure. It is generally assumed that when one member is in action, as AB, the other member, CD, is idle and has no stress.

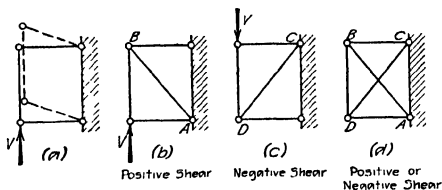



FIG. 18.

The trusses of Fig. 17 are shown with two members in the panels near the center. These members are generally made of eye bars or rods. The full line member is called a *main member* and the dotted line member is called a *counter*.

General methods for the determination of the stresses in Pratt trusses are the same as outlined in Art. **2.** In the following articles, complete calculations will be given for through and deck Pratt trusses with and without counters.

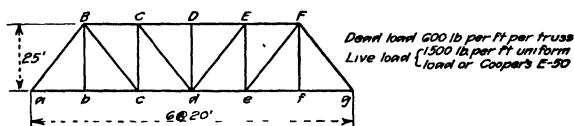



Fig. 19.

4. Stresses in a Through Pratt Truss without Counters.—Figure 19 shows a Pratt truss so constructed that the members are capable of resisting reversals of stress. Let the dead load be taken as 600 lb. per ft. per truss. To illustrate the methods of calculation for uniform and concentrated live load systems, let the uniform loading be taken as 1,500 lb. per ft. per truss, and let the concentrated loading be taken as Cooper's E-50 loading. Required the maximum and minimum stress in all members for the two live load systems.

4*a*. Dead Load Stresses.—Two assumptions are generally made regarding the assumed distribution of the dead load in through

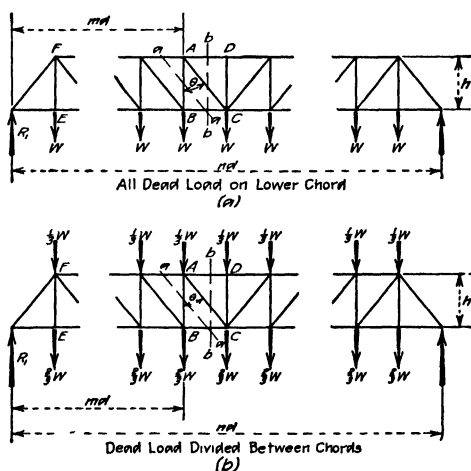



Fig. 20.

bridges. All of the load may be assumed as carried by the lower chord. as shown in Fig. 20*a*; a portion of the dead load, generally one-third, is assumed as carried by the top chord joints, as shown in Fig. 20*b*. In some cases the exact amount of load carried by the upper and lower

chord joints is computed. For all practical purposes the distribution shown in Fig. 20b is sufficiently accurate.

Before proceeding to the calculation of dead load stresses for the truss of Fig. 19, general equations for dead load stresses will be derived for the conditions shown in Fig. 20. A study will also be made as to the effect of distribution of the dead load on the stresses in members and the values of the reactions.

The end reactions are readily seen to be unaffected by the distribution of dead load. For Figs. 20a and b, we have

$$R_1 = \frac{W}{2}(n - 1)$$

Note that in Figs. 20a and b, joint loads are not shown at the end joints. These joint loads have no effect on the stresses in the members and are therefore neglected. If the maximum reaction supported by the piers or abutments is desired, these end joint loads must also be included.

The stress in any chord member, as AD of Figs. 20a and b, is determined from moments about C, the opposite chord point. For the conditions shown

$$\text{Stress in } AD = \frac{\text{Moments about } C}{\text{Height of truss}} = \frac{M_C}{h}$$

For the load distributions shown in Figs. 20a and b

$$M_C = R_1md - W[1 + 2 + \qquad (m - 1)]d$$
$$= \frac{Wd}{2} m(n - m)$$

where n = total number of panels in truss; m = number of panels from left end of truss to moment center; and d = common panel length. Since identical results are obtained for stress in the given chord members of Figs. 20a and b, the chord stresses are seen to be unaffected by the distribution of dead load.

For any diagonal web member, as AC of Figs. 20a and b, the stress is equal to the shear on section b-b multiplied by the secant of the angle which the member makes with the vertical. That is

$$\text{Stress in } AC = (\text{Shear on section } b\text{-}b) \sec \theta$$

The character of stress may be determined from Fig. 2.

The shear on section b-b of Fig. 20a is

$$V = R_1 - mW = \frac{W}{2}[n - (2m + 1)]$$

where m = number of panels from left end of span to the left end of the panel containing section b-b. For Fig. 20b, the shear on section b-b is

$$V = R_1 - \frac{2}{3}Wm - \frac{1}{3}Wm = \frac{W}{2}[n - (2m + 1)]$$

Hence the shears for diagonal web members are unaffected by the dead load distribution.

The stress in a vertical web member, as AB of Fig. 20a is equal to the shear on section a-a. This shear is

$$V = R_1 - Wm = \frac{W}{2}[n - (2m + 1)]$$

For member AB of Fig. 20b, the shear on section a-a is

$$V = R_1 - \tfrac{2}{3}Wm - \tfrac{1}{3}W(m - 1)$$
$$= \frac{W}{2}[n - (2m + 1)] + \frac{W}{3}$$

On comparing the shears for the two-load distributions, it will be found that the shear for the conditions shown on Fig. 20b exceeds the shear for Fig. 20a by one-third of a panel load. Note that when a portion of the panel load is moved to the top chord, the shear on any inclined section, as a-a, is *increased by the amount of load moved to the top chord.*

For a hip vertical, such as member FE, the stress is tension and is equal to the joint load at point E. In Fig. 20a, the stress in FE is equal to W. In Fig. 20b the stress is equal to $\tfrac{2}{3}W$. Moving a portion of the dead load to the top chord has *decreased the stress in the tension vertical by the amount of load moved to the top chord.*

The above analysis suggests the following procedure in calculating dead load stresses when the load is assumed as divided between the two chords: Assume first that all of the dead load is applied at the lower chord joints. Calculate the stresses in all members for this position of the dead load. Then consider a portion of the dead panel load to be transferred to the top chord. As shown above, stresses in chord and diagonal web members are unaffected by this change in dead load distribution and the stresses are as calculated for all dead load on lower chord joints. Stresses in vertical web members are changed by the redistribution of dead load. These changed stresses may be found by applying the following correction: Increase the stresses in all compression verticals by the amount of dead load moved to the top chord; decrease the stresses in tension verticals by the amount of load moved to the top chord.

The dead load stresses in the truss of Fig. 19 will now be determined by the method outlined above. Assuming all dead load applied to the lower chord joints, each panel load is $W = (20)(600) = 12{,}000$ lb.

Stresses in chord members may be obtained directly by moments, as in the case of the Warren truss, or by substitution in the general equation derived above. This equation is

$$M = \frac{Wd}{2}m(n - m)$$

where M = moment at a point m panels from the left end of the span; n = number of panels in the truss; W = panel load; and d = panel length. For the truss under consideration, (Fig. 19), $n = 6$; $d = 20$ ft.; and $W = 12{,}000$ lb. Hence $\frac{Wd}{2} = (\frac{1}{2})(12{,}000)(20) = 120{,}000$ and the moments at the several panel points of Fig. 19 are as follows:

Moment Center	m	$m(n - m)$	Moment (Ft.-lb.)
b	1	5	600,000
c	2	8	960,000
d	3	9	1,080,000

Since the height of the truss is 25 ft. the stresses in the chord members are as follows, noting that moments at top chord moment centers are the same as for the lower chord moment center directly below.

Member	Moment Center	Moment	Stress $= \frac{M}{25}$
$ab.bc$	B	600,000	+24,000
cd	C	960,000	+38,400
BC	c	960,000	−38,400
CD	d	1,080,000	−43,200

+ = tension − = compression

The shears in the several panels may be calculated as in the case of the Warren truss, or they may be calculated by means of the general equation

$$V = \frac{W}{2}[n - (2m + 1)]$$

which was derived above. In this equation the notation is the same as for moments. Note that m is the number of panels from the left end of the truss to the left end of the panel in which the shear is desired (see Fig. 20). The shears in the several panels are as follows:

Panel	m	$[n - (2m + 1)]$	V
ab	0	5	+30,000
bc	1	3	+18,000
cd	2	1	+ 6,000

+denotes positive shear

For diagonal members, $\sec \theta = \frac{[(20)^2 + (25)^2]^{\frac{1}{2}}}{25} = 1.28$. Stresses in diagonals $= V \sec \theta$; stresses in verticals are equal to the shear on section such as a-a, Fig. 20; stresses in hip verticals equal joint load at lower end of member; stress in center vertical is zero, for it has been assumed that all dead load is at the lower chord joints. The stresses in web members are then as follows:

Member	Shear	Stress
aB	30,000	−38,400
Bc	18,000	+23,000
Cd	6,000	+ 7,680
Cc	6,000	− 6,000
Bb	(Joint load at *b*)	+12,000
Dd	(Joint load at *D*)	0

+ = tension — = compression

Stresses in chord and web members are shown on Fig. 21*a*.

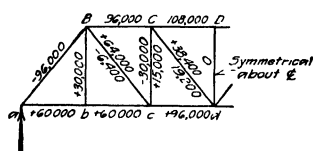
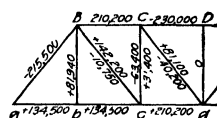
Fig. 21.

Assume now that one-third of the dead load is moved to the upper chord joints. The joint load to be applied at the upper chord joints is $(\frac{1}{3})(12{,}000) = 4{,}000$ lb. Figure 21*b* shows the new load distribution and the resulting stresses in all members. These stresses were obtained from those given on Fig. 21*a* by means of the correction stated on p. 239. It is there shown that only the stresses in the vertical members are affected by the change in load distribution, and that verticals in compression have their stresses increased by the load moved to the top chord, and that verticals in tension have their stresses reduced by this same amount.

Members Cc and Dd are compressing verticals. Their revised stresses are then 10,000 and 4,000 lb. respectively. Member Bb is a tension vertical, and its revised stress is 8,000 lb. All stresses are shown on Fig. 21b.

4b. Live Load Stresses—Uniform Loading.—Panel load = (1,500)(20) = 30,000 lb. These loads are assumed as applied at the lower chord joints.

Maximum chord stresses due to live load occur when all points are fully loaded. The loading conditions are therefore similar to those for dead load and the live load chord stresses may be obtained from those for dead load shown on Fig. 21 by multiplying the dead load stresses by the ratio of the dead and live panel loads. This ratio is $^{30}\!/_{12}$. Hence,

$$\text{Live load stress} = (\text{Dead load stress})\,^{30}\!/_{12}$$

Uniform live load

Cooper's E-50 loading

FIGS. 22 and 23.—Live load stresses, through Pratt truss without counters.

The resulting stresses are shown on Fig. 22.

Since the panel loads, panel lengths and number of panels are the same as for the Warren truss of Art. 3, the shears due to uniform loading are the same as given on p. 226. Hence the shears in the several panels of Fig. 19 are as follows:

PANEL	POSITIVE SHEAR	NEGATIVE SHEAR
ab	+75,000	0
bc	+50,000	− 5,000
cd	+30,000	−15,000

The stresses in the web members as determined from these shears are as follows:

MEMBER	PANEL	SHEAR	STRESS	
aB	ab	+75,000	(75,000)(1.28) =	−96,000
Bc	bc	+50,000	(50,000)(1.28) =	+64,000
		− 5,000	(5,000)(1.28) =	− 6,400
Cd	cd	+30,000	(30,000)(1.28) =	+38,400
		−15,000	(15,000)(1.28) =	−19,200
Cc	cd	+30,000		−30,000
		−15,000		+15,000
Bb	Joint Load at b			+30,000
Dd	Joint Load at D			0

These stresses are shown on the members in Fig. 22.

4c. Live Load Stresses—E-50 Engine Loading.—Since the number and length of panels for the truss under consideration is the same as for the Warren truss of Art. **2,** the moments and shears will be the same as calculated on pp. 227 and 232. These values are as follows:

Moments

Moment Center	Moment
B or *b*	3,362,500 ft.-lb.
C or *c*	5,255,000
d	5,745,600

Shears

Panel	Positive Shear	Negative Shear
ab	+168,085	0
bc	+111,055	− 8,400
cd	+ 63,400	−31,400

Stresses in chord and web members are as follows:

Chord Stresses

Member	Moment Center	Stress $= \frac{M}{25}$
ab bc	*B*	+134,500
cd	*C*	+210,200
BC	*c*	−210,200
CD	*d*	−230,000

Web Stresses

Member	Panel	Shear		Stress
		Positive	Negative	
aB	*ab*	168,085	0	−215,500
Bc	*bc*	111,055	8,400	+142,200 − 10,750
Cd	*cd*	63,400	31,400	+ 81,100 − 40,200
Cc	*cd*	63,400	31,400	− 63,400 + 31,400
Bb	Joint load at *b*. 81,940			+ 81,940
Dd	Joint load at *D*.			0

The joint load at b, known as the floorbeam reaction, is determined by the methods given in Art. **71,** p. 143. It is there shown that

$$R = \frac{2M}{d}$$

where R = floorbeam reaction; M = maximum moment at the center of a beam two panels in length (40 ft. in this case); and d = panel length = 20 ft. From Fig. 142, p. 115, wheel 4 gives maximum moment at the center of a 40-ft. beam. This moment is found to be

$$M = 1{,}000[2{,}693.75 + (145)(1)]^{20}\!/_{40} - 600 = 819{,}400 \text{ lb.}$$

Hence

$$R = \frac{(2)(819{,}400)}{(20)} = 81{,}940 \text{ lb.}$$

The chord and web stresses calculated above are shown on the truss members in Fig. 23.

4*d*. Maximum and Minimum Stresses.—Maximum and minimum stresses for the truss under consideration are determined by

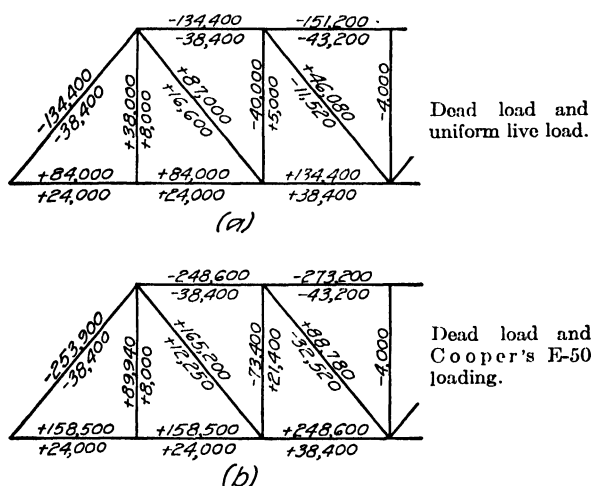



Fig. 24.—Maximum and minimum stresses, through Pratt truss without counters.

combining the dead load stresses shown on Fig. 21*b* and the live load stresses shown on Figs. 22 or 23, depending upon the type of live load in use. Figure 24*a* and *b* show the maximum and minimum stresses for both types of live loading.

5. Stresses in a Through Pratt Truss with Counters.—From Fig. 24 it can be seen that members near the center of the truss are subjected to a reversal of stress. As stated in Art. **3,** reversals of stress may be prevented by placing counters in panels in which the combined shears change sign.

In determining the stresses in a truss with counters, attention must be paid to the form of the truss under any given loading. The form of the truss is dependent upon the shear in the panels of the truss. Assuming the diagonal web members of Fig. 17 to be capable of taking tension only, positive shears in panels *ab* and *bc* will cause members *Ab* and *Bc* to act, while negative shears in these panels will cause members *aB* and *bC* to act. Hence in choosing moment centers for chord stresses or in cutting sections for web stresses, the calculator must determine which members are in action for the given loading conditions before he can proceed with his calculations.

To illustrate the methods of stress calculation for Pratt trusses with counters, consider the truss of Fig. 19 with the additional requirement that web members are not to be subjected to reversals of stress. Uniform live loading and E-50 train load conditions will be considered.

5*a*. Uniform Loading.—As stated above, counters are required in panels where the combined shear changes signs. For the truss under consideration, the combined shears are given in the following table. The shear values are taken from Arts. **4***a* and *b*.

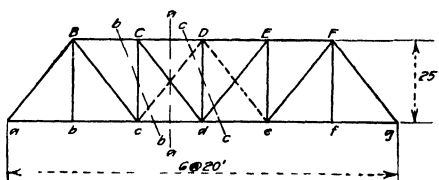
COMBINED SHEARS

Dead Load and Uniform Live Load

Panel	Dead load shear	Positive live load shear	Negative live load shear	Combined shear	
				DL and +*LL*	*DL* and −*LL*
ab	+30,000	+75,000	0	+105,000	+ 30,000
bc	+18,000	+50,000	− 5,000	+ 68,000	+ 13,000
cd	+ 6,000	+30,000	−15,000	+ 36,000	− 9,000
de	− 6,000	+15,000	−30,000	+ 9,000	− 36,000
ef	−18,000	+ 5,000	−50,000	− 13,000	− 68,000
fg	−30,000	0	−75,000	− 30,000	−105,000

From the above table, it can be seen that reversals of shear occur only in panels on either side of the center of the truss. Hence counters are required only in these panels. The dotted lines in Fig. 25 show the required arrangement of counters.

FIG. 25.

Chord stresses in trusses have been shown to have their maximum values when the truss is completely loaded. For such loading conditions, shears on the left of the truss center are positive and shears on the right of the center are negative. The main members, shown by full lines in Fig. 25 are then in action and the

form of the truss is the same as shown in Fig. 19. Hence the maximum and minimum chord stresses have the same values as given in Art. 4d. These values are shown on Fig. 26.

Stresses in web members of a truss occur under partial live loading. Generally it will be found that the maximum stress in a web member located to the left of the truss center is caused by the combined shear due to dead load and positive live load shear. The minimum stress is generally caused by the combined shear due to dead load and negative live load shear. In center verticals of Pratt trusses dead load only often gives a smaller stress than the combination mentioned above.

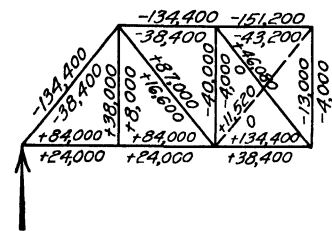



Fig. 26.—Maximum and minimum stresses, through Pratt truss with counters. Dead load and uniform live load.

The stress in members *aB* and *Bc* of Fig. 25 will be the same as for the truss of Fig. 19, for the form of the truss in panels *ab* and *bc* is unaffected by the presence of counters. To determine the stress in member *Cd* of Fig. 25, cut a vertical section *a-a* through panel *cd*. This section cuts member *Cd* and also the counter *Dc*. However, as explained on p. 236, only one of these diagonals can be in action at any time; the diagonal subjected to tension by the existing shear will be in action while the other diagonal will have zero stress. Hence to determine the stress in *Cd*, the form of the truss must be determined for the loadings giving combined positive and negative shears in panel *cd*. When the form of truss is known the stress in the member can readily be determined.

Figure 27*a* shows the position of loads causing maximum positive combined shear in panel *cd*. To the right of section *a-a*, each joint load

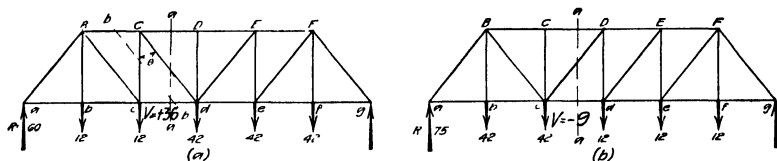



Fig. 27.

is 42,000 lb. (12,000 dead load, 30,000 live load), and to the left of section *a-a*, each load is a dead joint load of 12,000 lb. For this loading, the combined shear in panel *cd*, as given in the table on p. 245, is +36,000 lb. Under positive shear in panel *cd*, member *Cd* is in action and the form of truss is as shown in Fig. 27*a*. Hence, stress in $Cd = (36{,}000)(1.28) = 43{,}080$ lb. tension. This stress in *Cd* is the same as given in Fig. 24*a* for the Pratt truss without counters. For the form of truss shown in Fig. 27*a*, note that the counter is not in action and hence its stress is zero.

The position of loads causing maximum combined negative shear in panel *cd* is shown in Fig. 27*b*. Since the shear in panel *cd* is negative, the counter *Dc* is in action and the form of truss is as shown in Fig. 27*b*. Hence, stress in *Dc* = (9,000)(1.28) = 11,520 lb. tension. Note that the simultaneous stress in *Cd* is zero, for the main member is inactive when the counter is in action.

Figure 26 shows the maximum and minimum stresses in the main member and counter in panel *cd*. The minimum stress in each diagonal is zero. It occurs when the other member in the panel is in action. On comparing the stresses shown in Figs. 24*a* and Fig. 26, it will be found that the maximum stresses shown for member *Cd* on the two figures are the same. The maximum stress shown on Fig. 26 for the counter *Dc* is the same in amount but opposite in character to the minimum stress shown on Fig. 24*a* for member *Cd*. Note that the diagonals are subjected only to tension when counters are present.

To determine the stress in the vertical *Cc* Fig. 25, cut a section *b-b*. This section cuts the vertical *Cc* and also the counter *Dc*. Before the stress in *Cc* may be calculated the form of the truss must be determined.

Placing the loads on the structure which cause maximum positive combined shear in panel *cd*, the form of the truss is as shown in Fig. 27*a*. Hence the stress in *Cc* is equal to the shear on section *b-b*. This shear is 36,000 lb. (positive). Therefore the stress in *Cc* is 36,000 lb. compression. This stress was calculated on the assumption that all of the dead load is applied at the lower chord joints. If one-third of the dead panel load (4,000 lb.) is transferred to the top chord, the stress in *Cc*, determined by means of the correction given on p. 239 is 36,000 + 4,000 = 40,000 lb. compression.

When the loads are placed on the structure so as to give maximum negative combined shear in panel *cd*, the arrangement of loading and form of truss is shown in Fig. 27*b*. For the conditions shown in Fig. 27*b*, the stress in *Cc* is determined by considering conditions at joint *C*. Hence the stress in *Cc* is zero. On transferring one-third of the dead joint load to the top chord, the stress in *Cc* becomes equal to this joint load, or 4,000 lb. compression. Maximum and minimum stresses in *Cc* are as shown on Fig. 26.

The stress in the center vertical *Dd* is determined by methods similar to those used for *Cc*. To determine the stress in *Dd*, cut a section *c-c* as shown in Fig. 25. This section cuts member *Dd* and member *Dc* and *Ed*. Since section *c-c* cuts panel *de*, the combined shears in this panel must be used in determining the form of the truss.

Figure 28*a* shows the loads in position for maximum positive combined shear in panel *de*. For the position of loads shown the shears in panels *de* and *cd* are positive. Hence members *Cd* and *De* are in action and the form of truss is as shown in Fig. 28*a*. Therefore stress

in Dd = shear in panel de = 9,000 lb., compression. On transferring one-third dead panel load to the upper chord points, stress in Dd = 9,000 + 4,000 = 13,000 lb. compression.

The minimum stress in member Dd will occur either under the dead load acting alone, or for the loading causing maximum combined negative shear in panel de. This latter loading is shown in position of Fig. 28b. Since the shear in panel de is negative, member Ed is in action. For the loads shown on Fig. 28b, it can readily be seen that the *simultaneous*

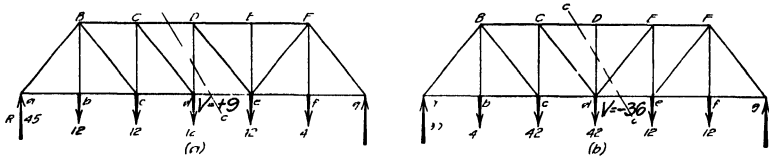



Fig 28

shear in panel cd is +6.0. Hence member Cd is in action in panel cd and the form of truss is as shown in Fig. 28b. Note that under dead load alone, the truss also assumes the form shown in Fig. 28b. Therefore, the minimum stress in Dd = joint load at D. Assuming one-third of the dead joint load transferred to the top chord, stress in Dd = 4,000 lb. compression.

In Pratt trusses where the number of panels is even and greater than six, it will generally be found that the loading causing maximum combined

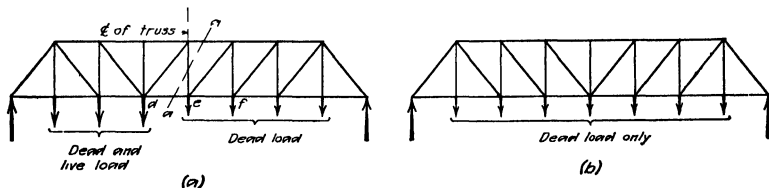



Fig. 29

negative shear in the panel to the right of the center vertical (for example, panel ef Fig. 29) produces a simultaneous shear in de (the panel to the left of the center vertical) which is also negative. The resulting form of the truss is then as shown in Fig. 29a and the stress in the center vertical is determined by shear on section a-a. It will be found that the stress in the center vertical determined for the conditions shown in Fig. 29a is always greater than the stress in this member when the dead load only is acting and the form of the truss is as shown in Fig. 29b. Hence it is best to determine the minimum stress in the center vertical from a consideration of the truss under dead load only. The stress determined for the loading conditions shown in Fig. 29a is neither the maximum nor the minimum stress, but results in a stress somewhere between the desired values.

Maximum and minimum stresses for the truss of Fig. 25 are shown in Fig. 26.

5b. Train Loading.—Maximum and minimum stresses in the truss of Fig. 25 due to dead load and a live load consisting of Cooper's E-50 engine loading are determined by methods exactly similar to those given above. The following table gives the combined shears in the several panels, using values computed in Arts. **4a** and *c*.

Combined Shears
Dead Load and E-50 Train Loading

Panel	Dead load shear	Positive live load shear	Negative live load shear	Combined shear	
				DL and +*LL*	*DL* and −*LL*
ab	+30,000	+168,085	0	+198,085	+30,000
bc	+18,000	+111,055	− 8,400	+129,055	+ 9,600
cd	+ 6,000	+ 63,400	−31,400	+ 69,400	−25,400
de	− 6,000	+ 31,400	−63,400	+ 25,400	−69,400

Maximum and minimum stresses in the end post *aB* and in the diagonal *Bc* are equal respectively to the combined shears given in the above table for panels *ab* and *bc* multiplied by sec θ. The maximum stress in the main diagonal *Cd* is equal to the combined dead load and positive live load shear in panel *cd* times sec θ, and the maximum stress in the counter *Dc* is equal to the combined dead load and negative live load shear in panel *cd* times sec θ. Both stresses are tensile. The minimum stress in these members is zero. Figure 30 shows the resulting stresses in these members.

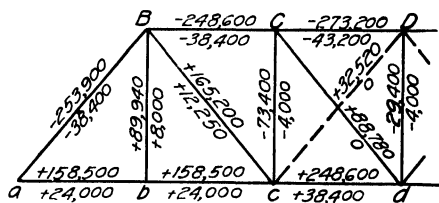



Fig. 30.—Maximum and minimum stresses, Through Pratt truss with counters. Dead load and Cooper's E-50 loading.

Maximum stress in the verticals *Cc* and *Dd* are due respectively to the combined dead load and positive live load shears in panels *cd* and *de*. The resulting stresses, corrected for the portion of the dead load at the top chord, are shown in Fig. 30. Minimum stress in the vertical *Cc* occurs for the loading causing combined dead load and negative live load shear in panel *cd*. The form of the truss is found to be as shown in Fig. 27*b*. Hence the stress in *Cc* is equal to the dead joint load at *C*. The minimum stress in *Dd* occurs when only the dead load is on the structure, form of truss as shown in Fig. 27*a*.

Maximum and minimum stresses in all chord members and in the end vertical *Bb* are the same as for the through bridge without counters. All stresses are shown on Fig. 30.

On comparing the stresses shown in Fig. 24*b* for the truss without counters, and those given on Fig. 30 for the same truss with counters, it will be noted that only the web members in the panel containing the counter have been affected by the introduction of the counter.

6. The Through Pratt Truss with an Uneven Number of Panels.—The general methods employed in the determination of stresses in this form of Pratt truss are the same as given in the preceding article. However, in determining stresses in chord members near the center of the truss, careful attention must be paid to the action of the web members in the center panel. Two general cases will be considered: There are two members in the center panel each designed so that they may be subjected to reversal of stress: Counters are provided in the center and adjacent panels and diagonals are designed to take tension only.

When members capable of resisting reversals of stress are provided in the center panel, the form of truss under unsymmetrical loading is shown in Fig. 31*a*. In determining the stresses in these members it is sufficiently accurate to assume that each member carries one-half the shear in the panel. The stresses are therefore equal, one member being in tension, the other in compression. To determine the stress in chord members *DE* or *dc*, Fig. 31*a*, cut a section *a-b* through the intersection of the diagonal members and remove the portion of the structure to the left of this section, as shown in Fig. 31*b*. For stress in the top chord member take moments about *b*; for stress in the bottom chord member, take moments about *a*. In either case it can readily be seen that the diagonal stresses, S_1 and S_2 do not affect the moment at the given center, for these stresses are equal in amount, opposite in character, and their lines of action are equidistant from the moment center.

When the applied loading is a uniform load, the moment at *b* is equal to the moments at *d* or *e*, for the shear in the center panel is zero. Under train loading, the moment at *a* or *b* must be determined by the methods used in Art. 2*b* for the top chord stresses in the Warren truss of Fig. 5.

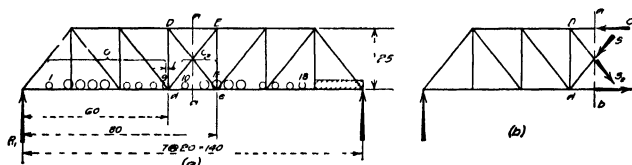



Fig. 31.

Illustrative Problem.—Calculate the stress in chord member *DE* of Fig. 31*a* for Cooper's E-50 loading. Assume that members *De* and *Ed* are both in action.

Moment center at b. The criterion for position of loads for maximum moment at b, as given by Eq. (45), p. 123 is

$$\frac{G}{2} - \left(G_1 + \frac{1}{2} G_2 \right) = 0$$

Try wheel 12 at e. For the position of loads shown in Fig. 31a, $G = 405$, $G_1 = 177.5$, and $G_2 = 37.5$ to 62.5. Substituting these values in the above criterion,

$$(\tfrac{1}{2})(405) - [177.5 + (\tfrac{1}{2})(37.5)] = +$$
$$(\tfrac{1}{2})(405) - [177.5 + (\tfrac{1}{2})(62.5)] = -$$

Wheel 12 satisfies the criterion. By the same process it was found that wheel 13 at e also satisfies the criterion. Wheels were also tried at d, but none satisfied the criterion. The load position shown in Fig. 31a was found to give maximum moment at b.

The moment at b for the loads shown on Fig. 31a is

$$M_b = \tfrac{1}{2}(M_d + M_e)$$

Now

$$R_1 = \left[20{,}455 + (355)(20) + \left(\frac{1}{2}\right)(20)^2(2.5)\right] \frac{(1{,}000)}{(140)} = 200\,400$$

and

$$M_d = (R_1)(60) - [4{,}370 + (177.5)(1)] = 7{,}476{,}500$$
$$M_e = (R_1)(80) - 8{,}385 = 7{,}647{,}000$$

Hence

$$M_b = 7{,}562{,}000 \text{ ft.-lb.}$$
$$\text{Stress in } CD = \frac{7{,}562{,}000}{25} = 302{,}480 \text{ lb. compression}$$

When a Pratt truss with an odd number of panels is provided with counters, the presence of these members complicates the determination of stresses in chord members near the truss center. The general methods to be employed in stress calculation will be illustrated by means of a seven-panel truss, as shown in Fig. 32a. It will generally be found that counters are required in the three panels near the truss center, as shown by the dotted line members.

Under uniform loading, only the full line members in Fig. 32a will be in action. The stresses in all members are readily found by the methods given in the preceding articles.

Under E-50 engine loading, stresses in chord members near the ends of the span and stresses in all web members are determined by methods similar to those used in the preceding articles. Stresses in chord members near the center of the span depend upon the form of truss for the given loading. Hence the simultaneous shears must be calculated and the form of truss determined before the proper moment center can be selected and the bending moment calculated.

In determining the stresses in the top chord members in the center panels where counters are located, that is in members CD, DE and EF

of Fig. 32*a* it will generally be found that when the E-50 train loading is placed in position for maximum moment at joint *d* the shear in all panels to the left of *d* is positive and the shear in all panels to the right of *d* is negative. The form of the truss is then as shown in Fig. 32*b*. Maximum moment at *d* gives the chord stress for member *CDE*. When the loads are placed in position for maximum moment at *e*, it will generally be found that all panels to the left of *e* have positive shear and all panels to the right of *e* have negative shear. The form of the truss is then as shown in Fig. 32*c* and the stress in *DEF* is given by moment at *e*. In designing the top chord section from *C* to *F*, the maximum live load stress is determined by the greater of the moments at *d* and *e*.

In calculating the stress in the bottom chord member *d-e* of Fig. 32*a*, it is to be expected that the moment center will be located at *D* or at *E*. On placing the train load on the truss in position for maximum moment at

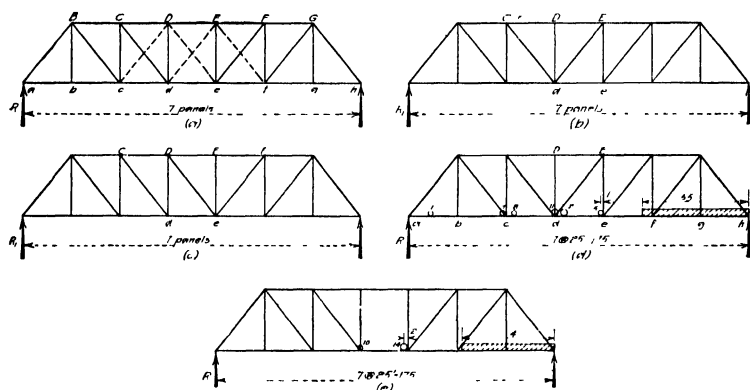



FIG. 32.

D, it generally will be found that the shear in panel *de* is negative. The form of the truss is then as shown in Fig. 32*b*. Hence *D* is not the proper moment center. On placing the train load in position for maximum moment at *E*, it will be found that the shear in panel *de* is positive and the form of the truss as shown in Fig. 32*c*. Again, *E* is not the proper moment center. Hence for the conditions stated above, the maximum stress in *de* may be due to one of the following conditions: (*a*) Moment at *E* for loads in position for maximum moment at *d*; (*b*) moment at *D* for loads in position for maximum moment at *e*.

It will generally be found that a stress in *de* greater than the one given by either of the above load positions is produced when the shear in the center panel is zero. This statement can readily be shown to be true by noting the effect on the stress in *de* of a change in the position of the loads. Thus, when the loads are in position for maximum moment at *d*, Fig. 32*b*, it will be found that moving these loads to the left will cause an increase in the moment at *E*. This increase in moment will be found

to continue as long as member dE remains in action, that is, as long as the shear in panel de remains negative. As soon as the shear in panel de becomes positive, the form of truss changes to that shown in Fig. 32c and it will be found that the stress in de becomes smaller. Similar conditions are found to exist for moment at D, Fig. 32c. Hence we conclude that the greatest stress in de occurs for zero shear in panel de, or when the moments at D and E are equal. This position of loads must be determined by trial, as shown in the problem which follows:

Illustrative Problem.—Assume the truss of Fig. 32a to consist of seven 25-ft. panels and that the height of truss is 30 ft. Calculate the maximum chord stress in members DE and de.

Stress in DE.—As stated above, the moment center for member DE may be at joint d or at joint e. From Fig. 141, p. 114, wheel 11 gives maximum moment at d and wheel 13 gives maximum moment at e.

When wheel 11 is placed at joint d, Fig. 32d, we find for the conditions shown that

$$R_1 = [20{,}455 + (355)(55) + \left(\frac{1}{2}\right)(55)^2(2.5)]\,\frac{1{,}000}{175} = 250{,}100 \text{ lb.}$$

Shear in panel cd =

$$R_1 - \text{(loads 1 to 7 plus joint load at } c \text{ due to loads 8 to 11)}$$

$$\therefore V_{cd} = 250{,}100 - \left(145{,}000 + \frac{701{,}250}{25}\right) = +77{,}000 \text{ lb.}$$

The shear in panel de is

$$V_{de} = R_1 - \text{(loads 1 to 11 plus joint load at } d \text{ due to loads 12 to 15)}$$

$$\therefore V_{de} = 250{,}100 - \left[215{,}000 + \frac{1{,}050{,}000 + (91{,}250)(1)}{25}\right]$$

$$= -10{,}600 \text{ lb.}$$

Hence the form of truss is as shown in Fig. 32d. Moment at d is

$$M_d = (R_1)(75) - 7{,}310 = 11{,}447{,}500 \text{ ft.-lb.}$$

$$\text{Stress in } DE = \frac{11{,}447{,}500}{30} = 381{,}250 \text{ lb., compression}$$

When wheel 13 is placed at e, we find $R_1 = 209{,}460$; $V_{de} = +7{,}960$ lb.; form of truss as shown in Fig. 32c; moment at e = 11,361,000 ft.-lb.; stress in DE = 378,700 lb. Hence maximum stress in DE occurs for wheel 11 at d.

Stress in de.—By trial it was found that zero shear in panel de occurs when wheel 10 is placed at joint d, as shown in Fig. 32e. For the loading shown

$$R_1 = \left[20{,}455 + (355)(47) + \left(\frac{1}{2}\right)(47)^2(2.5)\right]\frac{1{,}000}{175} = 228{,}000 \text{ lb.}$$

$$V_{de} = R_1 - \text{(loads 1 to 10 plus joint load at } d \text{ due to loads 11 to 14)}$$

$$V_{de} = 228{,}000 - \left[190{,}000 + \frac{750{,}000 + (100{,}000)(2)}{25}\right] = 0$$

$$\text{Moment at } d = (R_1)(75) - 5{,}790{,}000 = 11{,}310{,}000 \text{ ft.-lb.}$$

$$\text{Stress in } de = \frac{11{,}310{,}000}{30} = 377{,}000 \text{ lb. tension}$$

As stated on p. 252, maximum stress in de may sometimes be found for the load positions giving maximum moment at D or E. With wheel 9 at d, form of truss as shown in Fig. 32b, the moment at E is found to be 11,180,000 ft.-lb., and with wheel

13 at e, form of truss as shown in Fig. 32c, the moment at D is 11,161,700 ft.-lb. Since both of these moments are less than for the position of loads giving zero shear in the center panel, the maximum stress in de occurs for the load position shown in Fig. 32e.

The general method of procedure followed in the solution of the above problem will be found to apply to all seven-panel Pratt trusses except those with panel lengths from 16 to 20 ft. In these trusses it was found that when the loads were placed in position for maximum moment at point d, Fig. 32, the shear in panel de was positive and the form of truss as shown in Fig. 32c. Hence the moment center for de is at D. It was found that the stress in de for moment at D was greater than for the position of loads giving zero shear in panel de.

7. The Deck Pratt Truss.—The general methods of stress analysis for a deck Pratt truss are exactly the same as for the through truss. Let

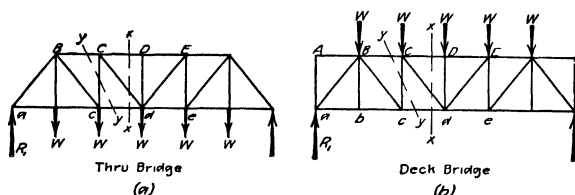



FIG. 33.

Fig. 33 show through and deck Pratt trusses of the same span supporting panel loads of the same magnitude. In the through bridge, Fig. 33a, all loads are assumed as applied at the lower chord and in the deck bridge, Fig. 33b, all loads are applied at the top chord.

For the conditions shown in Fig. 33, it can readily be seen that the reactions and the moments at the several panel points are equal for the through and deck bridge. Hence the chord stresses for the two trusses are equal. Also, the shears on vertical sections, such as x-x, are the same for the two trusses. Since stresses in the diagonals are determined by shears on the vertical sections, it is evident that the stresses in diagonal members of the through and deck trusses are equal.

As shown in Art. **4**a the stress in a vertical of the through bridge, as for example Cc of Fig. 33a, is given by the shear on section y-y. For the conditions shown, this shear occurs in bottom chord panel c-d and its value is $R_1 - 2W$.

To determine the stress in the corresponding member of the deck truss, which is Cc of Fig. 33b, cut a section y-y. The summation of forces to the left of this section, which is the shear in top chord panel BC, is seen to be $R_1 - W$. Hence the stresses in the verticals of a deck truss are in general greater than those in the through bridge.

To avoid the confusion which often occurs in selecting the proper shear for stress determination in verticals of a Pratt truss, the following

rule is useful: Cut a section, as *y-y* Fig. 33, through the member whose stress is desired. To determine the panel in which the shear is to be taken, follow this section to the chord on which the live load is carried (lower chord in through bridge, top chord in deck bridge).

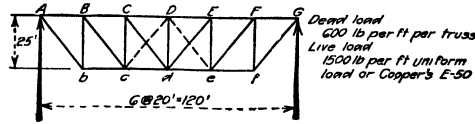



Fig. 34.

On applying the above rule to the through truss of Fig. 33*a*, the section *y-y* leads to the lower chord, panel *c-d*. For the deck bridge of Fig. 33*b*, the section leads to top chord panel *BC*.

The above discussion assumes all of the applied load as carried on one chord. If the dead load is divided between the two chords, it can

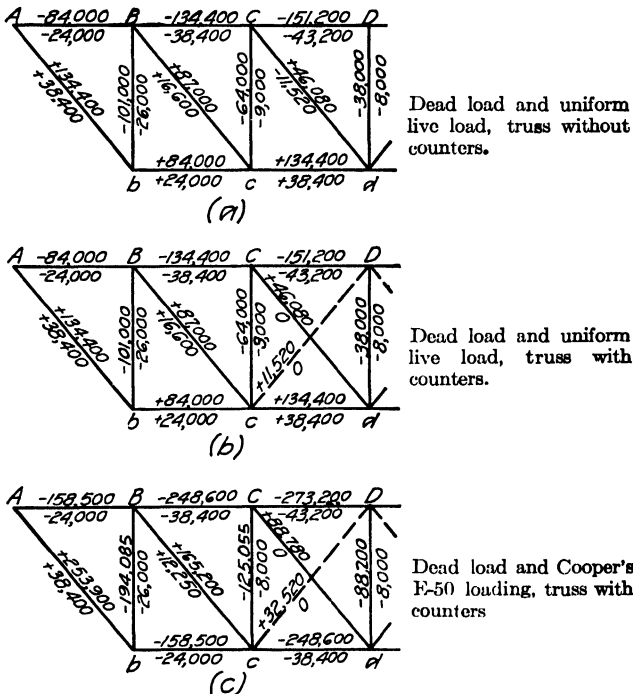



Fig. 35 —Maximum and minimum stresses, deck Pratt truss.

readily be shown by an analysis similar to the one given on p. 237 that the stresses in chord and diagonal web members remain unchanged. The stresses in vertical members are changed on dividing the dead load between the two chords. As in the case of the through bridge, the stresses

in the verticals of the deck bridge due to divided panel loads may be determined from the stresses due to undivided panel loads by means of a correction, which is applied in the following manner: Calculate the stresses in the verticals assuming all of the load as applied at top chord panel points. For verticals in compression, *reduce* their stress by the amount of load moved to the lower chord; for tension verticals, *increase* their stress by the load moved to the bottom chord.

To illustrate the methods of stress calculation for the deck Pratt truss assume that the truss of Fig. 19 is converted into a deck structure by transferring the loads to the top chord and by changing the direction of the end post, as shown in Fig. 34. The dead load will be taken as 600 lb. per ft. per truss. Two-thirds of this load will be assumed as carried by the top chord panel points and one-third as carried by the lower chord points. Live load stresses will be determined for a uniform load of 1,500 lb. per ft. per truss and for E-50 train loading. All of the live load will be applied at top chord joints.

Since the dimensions and loadings for the truss of Fig. 34 are the same as for the truss of Fig. 19, the moments at panel points and the shears in the panels will be the same as calculated in Art. **4**; this work will not be repeated here. Except for members in the end panel of Fig. 34, all chord and diagonal web stresses are the same as given in Art. **4**. Figures 35*a* and *b* show the maximum and minimum stresses in these members for uniform loading for the truss with and without counters.

The general methods for the determination of stresses in the verticals of the truss of Fig. 34 are the same as given in Art. **4**. From p. 242, the combined shears due to dead load and uniform live load are as follows:

Combined Shears

Dead Load and Uniform Live Load

Panel	Dead load shear	Positive live load shear	Negative live load shear	Combined shear	
				DL and +*LL*	*DL* and −*LL*
AB	+30,000	+75,000	0	+105,000	+30,000
BC	+18,000	+50,000	− 5,000	+ 68,000	+13,000
CD	+ 6,000	+30,000	−15,000	+ 36,000	− 9,000
DE	− 6,000	+15,000	−30,000	+ 9,000	−36,000

Consider first the truss without counters, as shown in Fig. 36*a*, and assume all loads as applied at the upper chord joints. The stresses in members *Bb* and *Cc* are determined for shear on sections 1-1 and 2-2 respectively. For section 1-1, shear in panel *AB* determines the stress in *Bb*. Hence for member *Bb*, maximum stress = 105,000 lb. compression; minimum stress = 30,000 lb. compression. For member *Cc*, shear

on section 2-2 in panel *BC*, we have maximum stress = 68,000 lb. compression; minimum stress = 13,000 lb. compression. The stress in vertical *Dd* is due to the joint loads at *D*. Maximum stress = sum of dead and live joint loads = 30,000 + 12,000 = 42,000 lb. compression; minimum stress = dead joint load = 12,000 lb. compression.

On transferring one-third of the dead load to the lower chord points, the above stresses will be *reduced* by the amount of load moved to the

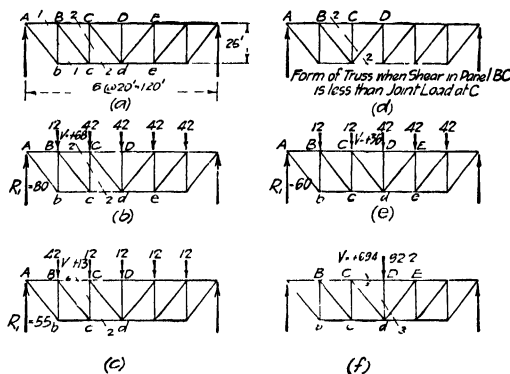



Fig. 36.

lower chord. The corrected stresses, which are shown on Fig. 35*a*, are then as follows:

Member	Stress with all load at top chord		Load moved to lower chord	Corrected stress	
	Max	Min		Max	Min.
Bb	−105,000	−30,000	4,000	−101,000	−26,000
Cc	− 68,000	−13,000	4,000	− 64,000	− 9,000
Dd	− 42,000	−12,000	4,000	− 38,000	− 8,000

− denotes compression

When counters are required, the combined shear table given above, shows that they must be provided in panels *CD* and *DE*. The arrangement of members is then as shown in Fig. 34.

Since there are no counters in panels *AB* or *BC*, the stress in *Bb* is the same as given above for the truss without counters.

In determining the stress in *Cc* consider first the loading which gives maximum positive combined shear in panel *BC*. This loading is shown in position on Fig. 36*b*. For the conditions shown the combined shear in panel *BC* is +68,000 lb. (see table on p. 256). The simultaneous shear in panel *CD* is then 68,000 − 42,000 = +26,000 lb. Therefore member *Cd* is in action in panel *CD*, the counter *Dc* is inactive, and the form of truss is as shown in Fig. 36*b*. The stress in *Cc* is therefore determined by the shear on section 2-2. Hence stress in *Cc* = 68,000 lb. compression.

Figure 36*c* shows the loads in position for minimum combined shear in panel *BC*. For the conditions shown, shear in panel *BC* = +13,000 lb. and the simultaneous shear in panel *CD* = 13,000 − 12,000 = +1,000 lb. Hence member *Cd* is in action in panel *CD*, the form of truss is as shown in Fig. 36*c*, and the stress in *Cc* is 13,000 lb. compression. Note that these stresses are the same as for the truss without counters. Applying the correction to account for the fact that a portion of the dead load is applied at the lower chord joints, the final stresses in *Cc* are as shown on Fig. 35*b*.

In Fig. 36*c*, the shear in panel *BC* is greater than the joint load at *C*. Hence the shear in panel *CD* is also positive and the main member *Cd* is in action. The shear on section 2-2 then determines the stress in *Cc*. In some trusses it will be found that the shear in panel *BC* is less than the joint load at *C*. When this condition exists, the simultaneous shear in panel *CD* will be negative and the counter *Dc* will be in action. The form of the truss will then be as shown in Fig. 36*d*. For the conditions shown in Fig. 36*d* the stress in *Cc* is determined by the joint load at *C* and not by the shear on section 2-2.

The stress in the center vertical is determined in a similar manner. Figure 36*e* shows the loads in position for maximum combined shear in panel *CD* (dead load and positive live load shear). For the conditions shown the simultaneous shear in panel *DE* = 36 − 42 = −6. Since shear in panel *DE* is negative, member *dE* is in action and the form of truss is as shown in Fig. 36*e*. The stress in *Dd* = joint load at *D* = 42,000 lb. compression.

The minimum stress in vertical *Dd* is due to the dead joint load at joint *D*. It occurs under dead load only, form of truss as shown in Fig. 36*a*. When the loads producing minimum combined shear in panel *CD* (dead load and negative live load shear) are placed on the structure, conditions similar to those described in Fig. 29 are found to exist. The resulting stress in the vertical is somewhere between the maximum found above and the true minimum. Figure 35*b* shows the corrected maximum and minimum stresses in all members of the truss under consideration.

Maximum and minimum stresses in the deck Pratt truss of Fig. 34 due to dead load and E-50 train loading are shown in Fig. 35*c*. Combined shears for dead load and E-50 train loading for the truss under consideration are the same as given in the table on p. 249. Maximum and minimum stresses in the diagonal web members are equal to the combined shears in the several panels times sec θ. Note that the resulting values are the same as given on Fig. 30 for the through bridge, except that the character of stress in *aB* is changed due to the difference in slope of the end post member.

Maximum and minimum stresses in the vertical *Bb* of Fig. 35*c* are due to the maximum and minimum combined shears in panel *AB* (panel

ab in the table on p. 249). The form of truss for the loading causing maximum stress in *Cc* is similar to Fig. 36*b*. Figure 36*f* shows the form of truss for the loading causing maximum stress in *Dd*. As shown in Fig. 36*f* and the table on p. 249, the combined dead load and positive live load shear in panel *CD* is +69.4. From the calculations given on p. 233, wheel 3 is placed at joint *D* for maximum positive shear in panel *CD*. Hence the live panel load at joint *D* is 80.2 (calculated by the methods given in Art. **71**, p. 143) and the total dead and live joint load at *D* is 80.2 + 12 = 92.2. Therefore the simultaneous shear in panel *DE* is negative and the form of truss is as shown in Fig. 36*f*. The joint load at *D* therefore determines the maximum stress in *Dd*. Stress values for the vertical members given in Fig. 35 have been corrected for the portion of the dead load at the lower chord joints.

Minimum stress in the vertical *Cc* will be found to occur when the form of truss is as shown in Fig. 36*d*. If wheel 1 is placed at *B*, train headed to the right, the live load shear in panel *BC* is found to be 6,460 lb., negative shear. Since the dead load shear in panel *BC* is +18,000 lb. (see table on p. 249), the combined shear in panel *BC* is +11,540 lb. The dead joint load is 12,000 lb. Hence the simultaneous shear in panel *CD* is negative and the form of truss is as shown in Fig. 36*d*. Therefore the minimum stress in *Cc* is the dead joint load at the top chord joint, which is 8,000 lb. The minimum stress in the center vertical *Dd* is due to the dead load only.

Maximum and minimum chord stresses are the same as for the through bridge, except for members near the end of the truss.

8. The Howe Truss.—Figure 37 shows the general arrangement of members in six-panel through and deck Howe trusses. The Howe truss differs from the Pratt truss in that all diagonal web members in the Howe truss are in compression and all vertical members are in tension. Note that if the Pratt truss of Fig. 34 be rotated about *AG* as an axis so that points *b* to *f* lie above *AG*, the resulting truss is exactly the same as those shown in Fig. 37.

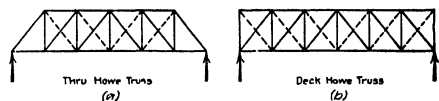



Fig. 37.

The Howe truss was used extensively in the early days of bridge building when timber was plentiful and could readily be obtained at the bridge site. Both top and bottom chord members and all diagonal members were made of wood. Since the diagonals are in compression, connection between these members and the chords was made by means of bearing blocks. Very simple joint details were thus secured. Iron or steel rods were used for the vertical members.

General methods of stress determination for the trusses shown in Fig. 37 are identical with those used for the Pratt truss. On comparing

the through Howe truss of Fig. 37*a* with the deck Pratt truss of Fig. 34, it can be seen that the general arrangement of members is identical. Also, except for members in the end panel, the deck Howe truss of Fig. 37*b* and the through Pratt truss of Figs. 19 or 25 are identical. It will be noted that counters (shown by dotted lines) have been provided in all panels of Fig. 37. This was done in order to secure additional rigidity. Therefore, it can readily be seen that the methods of stress determination for a Howe truss are identical with those of the Pratt truss of similar arrangement as indicated above. Note, however, that all stresses in web members are opposite in character.

TRUSSES WITH INCLINED CHORDS

When the length of span exceeds about 175 ft., it is generally considered advisable to use a variable height of truss in order to secure a gain in economy. As shown in preceding chapters, the bending moment diagram for a truss under full uniform load is a parabola. Hence if the top chord panel points of the truss shown in Fig. 38*a* lie on a parabola which is similar in form to the moment diagram, the horizontal components of top chord are all equal and the simultaneous stresses in the diagonals are zero. When the truss is subjected to partial uniform loading, the diagonal members are all subjected to reversal of stress. Counters are therefore required in every panel, as shown in Fig. 38*a*.

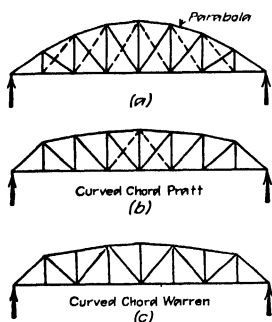



FIG. 38.

In practice it has been found that the use of counters in all but the center panels may be avoided and that a substantial gain in economy may be secured by using a flatter curve as shown in Fig. 38*b*. Also the appearance of the structure is improved by the flatter top chord. Figure 38*b* shows a Pratt truss with a curved chord and Fig. 38*c* shows a curved chord Warren truss.

9. General Methods of Stress Calculation.—Stresses in chord members of a curved chord truss may be determined by means of the general formula

$$S = \frac{M}{t} \tag{1}$$

where S = stress in given member (compression for top chords, tension for bottom chords), M = moment at opposite chord point, and t = perpendicular distance from member in question to moment center.

In Fig. 39*a*, which shows a typical six-panel curved chord Pratt truss, the stresses in bottom chord members are determined from moments about opposite upper chord points as moment centers. The

distance t of Eq. (1) is the vertical height of the truss at the moment center. Note that this is the same as for a truss with horizontal chords.

Moment centers for top chord members are located at lower chord panel points. To determine the stress in any top chord member, as BC of Fig. 39a, cut a section 1-1. Figure 39b shows the portion of the structure to the left of this section with all loads and forces in position. The moment center is at c, the intersection of stresses S_1 and S_2. To

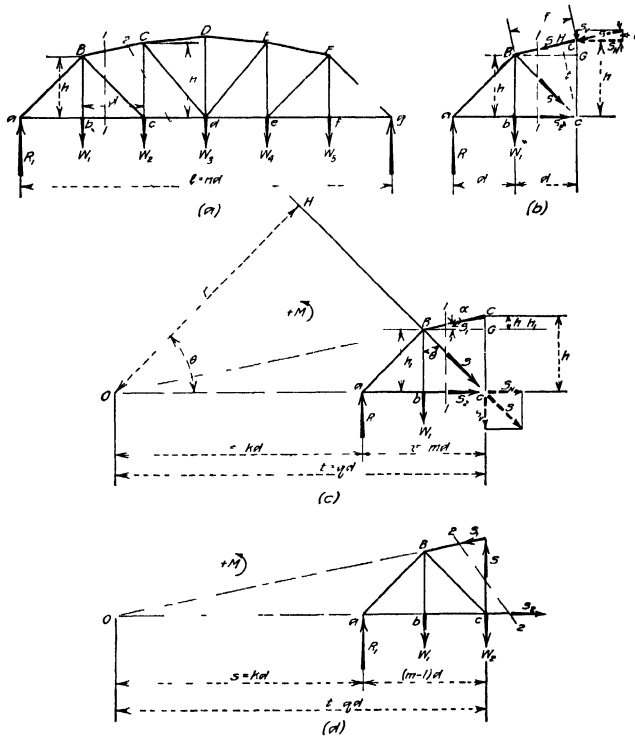



Fig. 39.

determine the lever arm t, consider the similar triangles BGC and CHc, from which we have the proportion $t:d::h:f$. Solving for t we derive

$$t = \frac{d}{f} \cdot h \tag{2}$$

In this equation the several terms indicate distances shown on Fig 39b. Note that $f = [(h - h_1)^2 + d^2]^{1/2}$. Having determined t for the given member and the moment at c, the stress is readily determined by Eq. (1).

The stress in any top chord member, as BC, may also be determined as follows: Let S, the stress in BC, be divided into its vertical and

horizontal components. Denote these components by S_V and S_H respectively. Since S (or its components) may be considered as applied at any point on its line of action, assume that it is so applied that its vertical component S_V passes through the moment center c. Figure 39*b* shows the assumed conditions. Then

$$S_H = \frac{M_c}{h} \tag{3}$$

where M_c = moment at c, and h = vertical height of truss at c. If α = angle between chord member and the horizontal, we have

$$S = S_H \sec \alpha \tag{4}$$

To show that Eqs. (1) and (4) give identical results, note from Eqs. (3) and (4) that

$$S = \frac{M_c}{h \cos \alpha} \tag{5}$$

But from Fig. 39*b*, $h \cos \alpha = t$. Therefore these equations are identical. It is recommended that Eqs. (3) and (4) be used in calculating chord stresses instead of Eq. (1).

The position of loads giving maximum chord stresses in the above equations is evidently the one giving maximum moment at several moment centers. Since the conditions of the supports and the positions of the moment centers are the same as for the trusses considered in the preceding articles, the determination of maximum moments for uniform and train loading are exactly the same as for Warren and Pratt trusses of the same span length.

10. Stresses in Diagonal Web Members.—To determine the stress in any diagonal web member, as *Bc*, Fig. 39*a*, cut a section 1-1 and remove the portion of the structure to the left of this section, as shown in Fig. 39*c*. Two general methods will now be developed for the determination of stress in the web member. The first of these methods is considered preferable as the desired stress may be determined without reference to the stress in any other member.

10*a*. First Method—The Method of Moments.—The stress S in member *Bc*, Fig. 39*c*, may be determined by taking moments about point O, the intersection of the lines of action of chord stresses S_1 and S_2. For the conditions shown in Fig. 36*c* point O is located on the lower chord produced at a distance s (k panels of length d each) to the left of the left support. On taking moments about O, considering moments in the direction of the arrow as positive, we have

$$+R_1 s - W_1(s + d) - Sr = 0 \tag{6}$$

from which

$$S = \frac{R_1 s - W_1(s + d)}{r} \tag{7}$$

Noting that the numerator of the right hand term is the moment of the applied loads to the left of section 1-1 about point O, which we will denote by M_0, we may write

$$S = \frac{M_o}{r} \tag{8}$$

where r is the perpendicular distance from O to the line of action of S.

The character of stress in member Bc may be determined from the sign of Eqs. (7) or (8). If M_0 is positive (*counter-clockwise moments positive*), the sign of Eq. (8) is *plus*, and S is *tension*, as assumed in Fig. 39c. A negative sign in Eq. (8) indicates that S is *compression*. It will be found convenient to assume in every case that S is tension. On adopting the above notation for moment, a plus sign will indicate *tension* and a minus sign will indicate *compression* in member Bc.

The lever arm r may be determined by scale from a layout of the truss, or it may be calculated directly from similar triangles in Fig. 39c. Consider the similar triangles OHc and Bbc, in which

$$OH : Oc :: Bb : Bc$$

Now $OH = r =$ required lever arm; $Oc = t =$ distance from right end of panel containing section 1-1 to moment center O; $Bb = h_1 =$ height of truss at left end of panel in question; and $Bc =$ length of member Bc. In terms of h_1 and panel length d, $Bc = (h_1^2 + d^2)^{1/2}$. Solving the above proportion, we derive

$$r = \frac{th_1}{(h_1^2 + d^2)^{1/2}} \tag{9}$$

The term t in Eq. (9) may be determined from the similar triangles BCG and OCc, from which we readily derive

$$t = \frac{h}{(h - h_1)} d \tag{10}$$

It is often convenient to express t in terms of panel lengths d. Let $q =$ number of panels of length d from O to c. Then

$$q = \frac{t}{d} = \frac{h}{(h - h_1)} \tag{11}$$

Having given q, the value of k (Fig. 39c) is readily determined.

The stress in Bc may also be expressed in terms of its vertical component. Resolve S into its vertical and horizontal components, S_V and S_H respectively, and apply these forces at point c, Fig. 39c. On taking moments about point O, we may write

$$S_V = \frac{M_o}{t} \tag{12}$$

where M_o has the same value as defined for Eq. (8). If θ = angle between member Bc and the vertical,

$$S = S_o \sec \theta \tag{13}$$

Note that Eq. (13) is similar to Eq. (2), p. 218, except that S_v of Eq. (12) replaces the shear on the section.

In substituting in Eq. (12), the value of M_o for the conditions shown in Fig. 36c is

$$M_o = R_1 s - W_1(s + d) \tag{14}$$

which may be written in the form

$$M_o = (R_1 - W_1)s - W_1 d$$

Note that $R_1 - W_1 = V$, the shear on section 1-1 due to the applied loads, and that $W_1 d = M_a$ = moment about a, the left end of the truss, due to the panel loads between point a and section 1-1. Using this notation, we have

$$M_o = V\text{s} - M_a \tag{15}$$

Equation (15) is in convenient form for calculation, as will be shown later (see problem on pp. 272 to 278).

Load positions for maximum stress in diagonal web members may be determined by noting the effect of varying load positions on the value of M_o of Eqs. (8) or (12). Uniform and concentrated loading conditions will be considered.

Stresses due to uniform loading may be determined by two methods. These are: (*a*) Conventional Method, in which it is assumed that for partial loading of the truss, all panel points on one side of a section may be fully loaded with no load on the panel points on the other side of that section, and (*b*) Exact Method, in which the exact panel loads due to partial loading are taken into account. Similar loading conditions were assumed in trusses with horizontal chords.

Consider first the conventional method of loading. For maximum tension in Bc, as given by Eqs. (8) or (12), the value of M_o from Eq. (14) must be *positive* (counter-clockwise, see Fig. 39c) and as great as possible. It can readily be seen from Eq. (14) and Fig. 39c that if any load W_1 exists, it produces negative moment, thereby reducing the positive value of M_o. Therefore, to obtain maximum positive M_o, make R_1 as large as possible without placing any load at panel points to the left of section 1-1. This can be done by placing full panel loads at all points to the right of section 1-1, and none to the left. Note that this is the same as the load position for maximum positive shear on section 1-1. For maximum compression in Bc, the moment M_o must be negative. The maximum negative value of M_o is obtained when the negative moment due to loads

such as W_1 is as great as possible and the positive moment due to R_1 is as small as possible. This may be obtained by loading all possible points to the left of section 1-1, no load on points to the right, for it is evident that loads to the right of 1-1 have no effect on negative moment, but they do increase R_1, thereby reducing the effective negative moment.

General formulas for stress in any diagonal, as *Bc*, may readily be derived from the conditions shown in Fig. 39*c*. Assuming all joints to the right of section 1-1 fully loaded with panel loads *W*, the vertical component of stress in *Bc* is

$$S_V = + \frac{Wk}{2nq}(n - m + 1)(n - m) \tag{16}$$

For all joints to the left of section 1-1 fully loaded, no load to the right

$$S_V = - \frac{Wm}{2nq}(m - 1)(k + n) \tag{17}$$

In these equations plus denotes tension and minus denotes compression. All dimensions are given in terms of panel lengths. The notation adopted is shown on Fig. 39.

The stress in any vertical as *Cc* is determined by cutting a section 2-2, as shown in Fig. 39*d*. General methods of analysis and loading conditions are the same as given above for a diagonal. The general formula for stress in the vertical is as follows:

$$S = -(Vk - M_a)\frac{1}{q} \tag{18}$$

where V = shear on section 2-2; M_a = moment about a of loads between section 2-2 and a; and k and q are respectively the number of panels from the moment center O to R_1 and to the line of action of S. A plus sign in Eq. (18) indicates tension and a minus sign indicates compression. On substituting values of V and M_a in Eq. (18), the stress in *Cc* for loads to the right or left of section 2-2 are found to be

For all loads to right of panel *cd*

$$S = - \frac{Wk}{2nq}(n - m + 1)(n - m) \tag{19}$$

For all loads to left of panel *cd*

$$S = + \frac{W}{2nq}m(m - 1)(k + n) \tag{20}$$

Position of Loads for Exact Stress in Web Members.—The exact position of a uniform load for maximum stress in a diagonal web member may be determined from the conditions shown in Fig. 40. In this case the partial panel load at b is also taken into consideration.

To determine the true position of the uniform load for maximum stress in *Bc*, assume first that the uniform load extends a distance x into panel *bc*. By the methods given above calculate the stress in *Bc*. Then assume that the head of the uniform load is moved forward a distance Δ until it extends a distance $x + \Delta$ into the panel. Again calculate the stress in *Bc*. On subtracting the former stress from the latter, we have an expression for the change in stress in *Bc* due to a small movement of the uniform load. This expression for change in stress may also be derived by considering only the small piece of uniform *d* load of intensity w lb. per ft. covering a length Δ. This load is assumed as placed at a

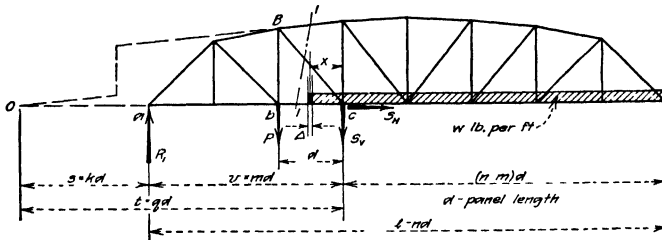



FIG. 40.

distance x to the left of point *c*, Fig. 40. For the conditions shown in Fig. 40, the vertical component of stress in *Bc* due to this small load is

$$S_V' = \frac{M_o}{t} = \frac{R_1 kd - P[kd + (m - 1)d]}{qd}$$

where

$$R_1 = \frac{w\Delta}{nd}[x + (n - m)d] \text{ and } P = \frac{w\Delta x}{d}.$$

If for a given value of x, S_V' in the above equation is positive, we know that the stress is increased when the uniform load is advanced a distance Δ. If a negative result is obtained, we know that the forward movement of the load has decreased the stress in *Bc*. Therefore, the stress in *Bc* is a maximum when, for a certain value of x, S_V' is zero, that is, S_V' is neither increasing nor decreasing. To determine this value of x, place the expression for S_V' equal to zero and solve for x, from which we derive

$$x = \frac{k(n - m)}{n(m - 1) + k(n - 1)}d$$

To determine the maximum value of stress in *Bc*, place the head of the uniform load a distance x to the left of *c* Fig. 40 and calculate the resulting stress by the methods given above. It will be found that S_V, the vertical component of stress in *Bc*, is

$$S_V = \frac{wd}{2n}(n - m)^2\left[1 + \frac{k}{n(m - 1) + k(n - 1)}\right]\frac{k}{q}$$

In practice the exact position of loads is seldom used in determining stresses. The conventional method of loading, given above, yields stresses which are slightly on the side of safety. Moreover, the work of calculation is somewhat simplified by the use of the less complicated conventional method, which will be used in all the work to follow.

Position of E-50 Train Load for Maximum Stress in Web Members.—To determine the position of a system of concentrated loads for maximum stress in any web member in a curved chord Pratt truss, as for example, diagonal Bc of Fig. 41a, we will first derive an expression for the change in vertical component of stress in Bc due to a small forward movement of the load system. It will generally be found that maximum stress in any web member occurs when loads are located in the panel containing the

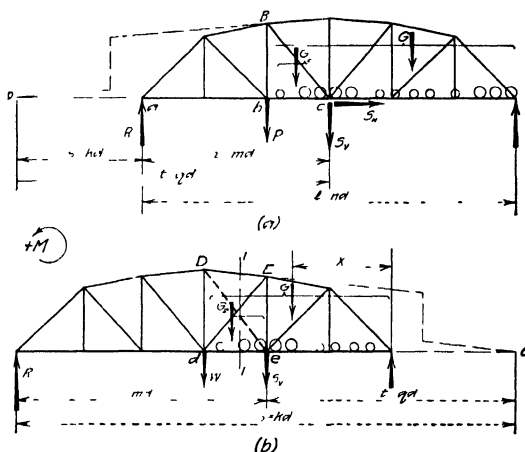



Fig. 41.

member in question and the portion of the truss to the right of that panel. In Fig. 41a let G_2 represent the load in panel bc and let G represent the total load on the structure. When these loads are moved a small distance Δ to the left, no load passing points b, c or the right end of the span, certain changes take place in the values of R_1 and P. These changes may readily be shown to be

$$R_1' = \frac{G\Delta}{nd} \text{ and } P' = \frac{G_2\Delta}{d}$$

From Eqs. (12) and (15), the change in vertical component of stress in Bc, in terms of R_1' and P', is

$$S_V' = \left[\frac{G}{n\left(1 + \frac{m-1}{k}\right)} - G_2\right]\frac{\Delta k}{d} \qquad (21)$$

If the substitution of particular values in Eq. (21) yields a positive result, it is evident that a forward movement of the loads causes an increase in the stress in Bc. A negative result indicates a decrease in the value of S_V. A zero value indicates that the stress in Bc is neither increasing nor decreasing, but has reached its maximum value. Hence, on equating Eq. (21) to zero, the relation between loads and distances for the load position giving maximum stress may be determined.

On equating Eq. (21) to zero, we derive,

$$\frac{G}{n\left(1 + \frac{m - 1}{k}\right)} - G_2 = 0 \tag{22}$$

which is the criterion for position of loads for maximum stress in any web member. In this equation, G = total load on structure; G_2 = load in panel in which the shear is desired; n = number of panels in truss; m = number of panels from left support to right end of panel in which shear is desired; and k = number of panels from moment center O (Fig. 41a) to left support. Note that except for the added term $\frac{m-1}{k}$ in the denominator of the first term, Eq. (22) is similar to Eq. (55), p. 131. When the top chord of the truss of Fig. 41a is horizontal, O is located at infinity. Hence k becomes infinite and Eq. (22) reduces to the form of Eq. (55), p. 131.

By means of analysis similar to the one given on p. 109, it can be shown that the criterion of Eq. (22) is satisfied when some wheel crosses point c. When the critical wheel is located to the right of c, Eq. (22) yields a positive value, and when it is located to the left of c, a negative result is obtained. Two positive results indicate that a further movement to the left is necessary, and two negative results indicate that the loads have been placed too far to the left and must be moved to the right to satisfy the criterion. Having located the proper load position by means of Eq. (22), the stress in the member is determined by means of Eqs. (12) and (15). The criterion of Eq. (22) applies also to a vertical web member.

The live load portion of the minimum stress in the web members, or the maximum stress in the counters is best determined from a consideration of members on the right hand side of the truss. Figure 41b shows the loads in position for maximum positive shear in a panel to the right of the truss center. This load position will give maximum tension in the counter or maximum compression in the main member. For the conditions shown in Fig. 41b, it can readily be shown that the vertical component of stress in either web member cut by section 1-1 is

$$S_V = \pm \frac{M_o}{q} \tag{23}$$

where

$$M_o = Vk + M_a \tag{24}$$

In these equations, M_o = moment about O; V = shear on section 1-1 due to applied loads; M_a = moment about left end of span of loads to left of section 1–1; k = number of panels from left end of truss to moment center O, q = number of panels from line of action of vertical component of web member stress to moment center O. For stress in member *De* of Fig. 41*b*, Eq. (23) carries a *plus* sign; for member *Ed*, a *minus* sign is used. A plus sign in Eq. (23) indicates *tension;* a minus sign indicates *compression*. When the minimum stress in a vertical is desired for a truss without counters we have

$$S = +\frac{M_o}{q} \tag{25}$$

where M_o has the value given by Eq. (24). The sign notation is as above.

By an analysis similar to the one given above it can readily be shown that the criterion for position of loads for maximum stress in Eqs. (23) or (25) is

$$\frac{G}{n\left(1 - \frac{m-1}{k}\right)} - G_2 = 0 \tag{26}$$

Fig. 41*b* shows the notation used.

The above criterion differs from the one given by Eq. (22) only in the minus sign in the denominator of the first term. This difference is due to the fact that the moment center O of Fig. 41*b* is to the right of the truss while the moment center in Fig. 41*a* is to the left. Equation (26) may be obtained from Eq. (22) by changing the sign of k.

The problem given in Art. **13** illustrates the application of these criteria to an actual truss.

10*b*. Second Method—Method of Vertical Components.—On referring to Fig. 39*c*, it can be seen that a summation of vertical forces taken for section 1-1 contains as unknowns, the vertical components of the stresses in the diagonal *Bc* and the top chord member *BC*. It is, therefore, not possible to determine the stress in a web member of a curved chord truss by means of an equation involving only the stress in the diagonal. Before the stress in the diagonal can be determined, the simultaneous stress in the chord member must also be known. In this respect the method of vertical components is not as convenient as the method of moments, for the former method generally calls for additional calculations in order to determine the top chord stress.

From Fig. 39*c*, a summation of vertical forces gives

$$R_1 - W_1 - S\cos\theta - S_1 \sin\alpha_1 = 0$$

Noting that $R_1 - W_1 = V$, the shear in section 1-1, we may write

$$S = (V - S_1 \sin \alpha_1) \sec \theta \tag{27}$$

In this equation $S_1 \sin \alpha_1$ represents the vertical component of *simultaneous* top chord stress.

The stress in the vertical *Cc* may be determined from a summation of vertical forces to the left of Sec. 2-2, Fig. 39*a*. If V = shear on Sec. 2-2, we have

$$S = -(V - S_1 \sin \alpha_1) \tag{28}$$

The character of stress in any web member, either diagonal or vertical, is determined by the sign of the term $(V - S_1 \sin \alpha_1)$ of Eq. (27). This term corresponds to the shear V of Eq. (2), p. 218, and the character of stress may be determined from Fig. 2, as in the flat top truss.

Load positions for maximum stress are the same as given on the preceding pages. In the problem on p. 272, stress calculations are made by the method of moments as well as by the method of vertical components.

11. Stresses in Tension Verticals.—The stress in the center vertical, *Dd*, Fig. 39*a*, is tension. When counters are used, the stress in *Cc* may also be tension for certain load positions. General methods for the determination of these stresses will now be given.

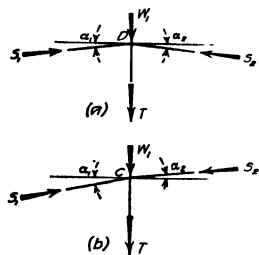



Fig. 42.

The stress in vertical *Dd* may be determined by removing joint D together with all forces acting as shown in Fig. 42*a*. Let S_1 and S_2 represent the top chord stresses meeting at the joint in question; let α_1 and α_2 respectively be the angles which these stresses make with the horizontal, assuming for the time being that these angles are unequal, which of course is not the usual case. The load W_1 represents any portion of the dead load which may be carried at joint D, and T represents the stress in the center vertical. From a summation of vertical forces, we readily derive

$$T = S_1 \sin \alpha_1 + S_2 \sin \alpha_2 - W_1 \tag{29}$$

when α_1 and α_2 are equal, the usual case, then also will S_1 and S_2 be equal, and we have

$$T = 2S_1 \sin \alpha_1 - W_1 \tag{30}$$

Note that $S_1 \sin \alpha_1$ is the vertical component of top chord stress.

When counters are used in a curved chord truss, member *Cd* of Fig. 39*a* drops out and the conditions at joint C are as shown in Fig. 42*b*. The stress in the vertical is equal to the summation of vertical forces.

Since the horizontal components of S_1 and S_2 must be equal, we may write, in terms of S_1,

$$T = S_1 \cos \alpha_1(\tan \alpha_1 - \tan \alpha_2) - W_1 \tag{31}$$

The stresses in these vertical members will generally be found to be tension, for the load W_1 is usually small. Maximum tension in the verticals will exist when the chord stresses have their maximum values. For the conditions shown in Fig. 42*a*, the truss should in general be fully loaded. For the joint shown in Fig. 42*b*, the maximum stress occurs for conditions discussed in the problem on p. 272.

Fig. 43.

12. Special Forms of Curved Chord Trusses.—Curved chord trusses are sometimes constructed with both chords curved, as shown by the sketch of Fig. 43*a*. General methods of stress analysis and load positions for maximum stress are the same as given in the preceding discussion.

Figure 43*b* shows a section cut for the determination of chord stress in member *BC*. The desired stress may be determined by dividing the moment at *b* by the lever arm r_1, or, the chord stress may be resolved into its vertical and horizontal components. On applying these components (shown by S_H and S_V on Fig. 43*b*), the value of S_H may be found, as explained for the truss of Fig. 39. The lever arms may be computed, or they may be scaled from a large size layout of the truss. This latter method will be found sufficiently accurate for all practical cases, and will result in a considerable saving of time.

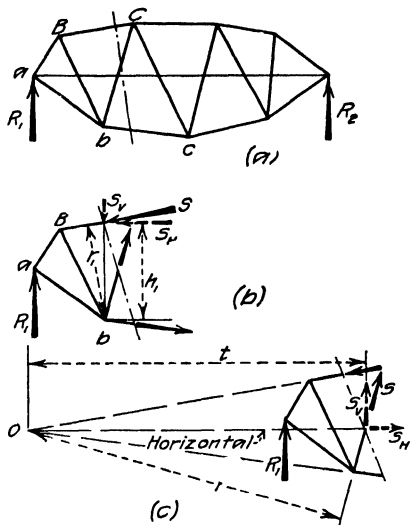
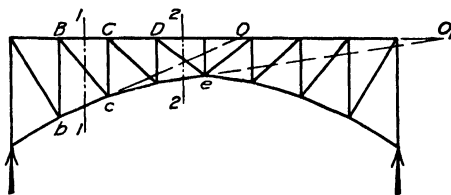



Fig. 44.

Diagonal or web member stresses may be determined from the conditions shown in Fig. 43*c*.

Roof trusses for buildings are sometimes constructed as shown in Fig. 44. This type of truss is seldom used for bridges, except in the form of arches, which are treated elsewhere. It can readily be seen that the truss of Fig. 44 is not economical, for the minimum depth of truss occurs

at the point where the moment is the greatest. This requires very large chord members at the truss center, in which position the moment effect of the dead weight is large. By curving the chord in the opposite direction (as in Fig. 39) greater economy may be secured.

Stresses in chord members of Fig. 44 for live load are a maximum when the truss is fully loaded. For web members, either vertical or diagonal, the loading conditions for moving load depend upon the position of the moment center *O*, Fig. 44. When the moment center *O* for member *Bc*, found by producing chord members *BC* and *bc* to an intersection, meets at a point between the two supports, the maximum stress in *Bc* occurs when the truss is fully loaded. This can readily be seen to be true, for a load at any point on the span causes a positive moment at *O*, and hence a completely loaded truss will give maximum stress in the web member under consideration. When the moment center is outside the supports, as shown by point *O*, for member *De*, the loading conditions are the same as for the truss of Fig. 39.

13. Stresses in a Curved Chord Truss.—To illustrate the application of the general methods of stress analysis for curved chord trusses, as

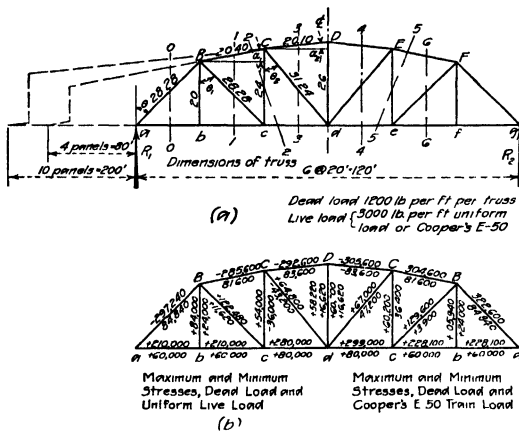



FIG. 45.

given in the preceding article, consider a truss of the dimensions shown in Fig. 45*a*. Let the dead load be taken as 1,200 lb. per ft. per truss. The live load stresses will be calculated for uniform and concentrated load systems. Let the uniform load be taken as 3,000 lb. per ft. and let the concentrated load system be taken as an E-50 train load. The conventional method of calculation will be used for uniform loading. It will be assumed first that the diagonal web members are rigid and capable fo taking both tension and compression. A solution for stresses will also be made assuming that the diagonal web members take tension only and that counters are to be provided where necessary.

13a. Curved Chord Truss without Counters.

Dead Load Stresses.—Dead panel load = (1,200)(20) = 24,000 lb. applied at joints b to f of the lower chord. No load at top chord. Reaction due to dead load = ($\frac{5}{2}$)(24,000) = 60,000 lb.

Chord Stresses.—Moments at all moment centers are as follows:

$$M_B = M_b = (60,000)(1)(20) = 1,200,000 \text{ ft.-lb.}$$
$$M_C = M_c = (1,000)[(60)(2) - (24)(1)](20) = 1,920,000 \text{ ft.-lb.}$$
$$M_d = (1,000)[(60)(30) - 24(1 + 2)](20) = 2,160,000 \text{ ft.-lb.}$$

Stresses in Members.—

$$ab \text{ and } bc = \frac{M_B}{20} = \frac{1,200,000}{20} = 60,000 \text{ lb. tension}$$
$$cd = \frac{M_C}{24} = \frac{1,920,000}{24} = 80,000 \text{ lb. tension}$$

Using Eqs. (3) and (4)

$$BC = \frac{M_c}{24} \sec \alpha_1 = \left(\frac{1,920,000}{24}\right)\left(\frac{20.40}{20}\right) = 81,600 \text{ lb. compression}$$
$$CD = \frac{M_d}{26} \sec \alpha_2 = \left(\frac{2,160,000}{26}\right)\left(\frac{20.10}{20}\right) = 83,600 \text{ lb. compression}$$

Web Members.—Method of moments, using Eqs. (12), (13) and (15). Distance from left reaction to moment center O from Eq. (11).

Sections 1-1 and 2-2

$$q = \frac{24}{24 - 20} = 6 \text{ panels} = 120 \text{ ft.} = t$$
$$k = 6 - 2 = 4 \text{ panels} = 80 \text{ ft.} = s$$

Section 3-3

$$q = \frac{26}{26 - 24} = 13 \text{ panels} = 260 \text{ ft.} = t$$
$$k = 13 - 3 = 10 \text{ panels} = 200 \text{ ft.} = s$$

The dead load shears in the several panels are as follows:

$$ab \quad V = R_1 = 60,000 \text{ lb.}$$
$$bc \quad V = 60,000 - 24,000 = 36,000 \text{ lb.}$$
$$cd \quad V = 60,000 - (2)(24,000) = 12,000 \text{ lb.}$$

Member Bc. (Section 1-1.)—Value of M_o. Use Eq. (15) with V = shear in panel bc = 36,000 lb; k = 4 panels; and M_a = moment of panel load at b about a = (24,000) (1). Unit of length is the panel.

$$M_o = Vk - M_a = [(36,000)(4) - 24,000] = 120,000$$

From Eq. (12), using panel length units,

$$S_V = +\frac{M_o}{q} = \frac{120,000}{6} = +20,000 \text{ lb.}$$

From Eq. (13)

$$\text{Stress in } Bc = S = S_V \sec \theta_1 = (20,000)\left(\frac{28.28}{20}\right) = 28,280 \text{ lb. tension}$$

Member Cc. (Section 2-2.)—Value of M_o. Use Eq. (15) with V = shear in panel cd = 12,000 lb.; k = 4 panels; and M_a = moment of panel loads at b and c about a = (24,000)(1 + 2) = 72,000.

$$M_o = Vk - M_a = [(12,000)(4) - 72,000] = -24,000$$

From Eq. (18)

$$\text{Stress in } Cc = S = -\frac{M_o}{q} = -\frac{(-24{,}000)}{6} = +4{,}000 \text{ lb. tension}$$

Member Cd. (Section 3-3.)—$V = 12{,}000$ lb.; $k = 10$ panels; M_a = moment of panel loads at b and c about $a = 24{,}000\ (1 + 2) = 72{,}000$.

$$M_o = Vk - M_a = (12{,}000)(10) - 72{,}000 = 48{,}000$$
$$S_V = \frac{M_o}{q} = \frac{48{,}000}{13} = +3{,}690 \text{ lb.}$$

$$\text{Stress in } Cd = S = S_V \sec \theta_2 = +(3{,}690)\left(\frac{31.24}{24}\right) = +4{,}800 \text{ lb. tension}$$

Member aB. (Section O-O.)—Stress in $aB = R_1 \sec \theta = (60{,}000)\left(\frac{28.28}{20}\right) = 84{,}840$ lb. compression.

Member Dd. (Joint at D.)—Figure 42a, and Eq. (30). Stress in Dd = (2)(stress in CD)(sin α_2) $= (2)(83{,}600)\left(\frac{2}{20.10}\right) = 16{,}620$ lb. tension.

Member Bb.—Joint load at b = 24,000 lb. tension.

These stresses appear on the left half of Fig. 45b as the minimum stresses except for members Bc, Cc and Cd. If it is desired to transfer one-third of the dead load (8,000 lb.) to the top chord, add a compression of 8,000 lb. to stress in verticals Cc and Dd, and reduce the tension in Bb by 8,000 lb. Stress in $Cc = +4{,}000 - 8{,}000 = -4{,}000$ lb. compression; Stress in $Dd = 16{,}620 - 8{,}000 = 8{,}620$ lb. tension; stress in $Bb = 24{,}000 - 8{,}000 = 16{,}000$ lb. tension. All other stresses remain unchanged.

Live Load Stresses—Uniform Loading.—Panel load = (3,000)(20) = 60,000 lb.

Chord Stresses.—These may be determined by ratio from the dead load. Live load stress = ($^{60}/_{24}$) dead load stress. The chord stresses are as follows:

ab	bc = 150,000 lb. tension
cd	200,000 lb tension
BC	204,000 lb. compression; CD = 209,000 lb. compression.

Web Members.—Stresses in aB, Bb, and Dd may be determined by ratio from the dead load, as in the case of chord stresses. These stresses are as follows: $as = B$ 212,400 lb. compression; Bb = 60,000 lb. tension; Dd = 41,600 lb. tension.

Stresses in the remaining web members are calculated by the general methods used for dead load stress in the same members. The shears in the several panels are as follows:

Panel ab

Positive shear (all joints loaded) $= R_1 = \frac{5}{2}\ (60{,}000) = 150{,}000$ lb.

Negative shear = 0

Panel bc

Positive shear (joints c to f loaded)

$$V = R_1 = \frac{60{,}000}{6}(1 + 2 + 3 + 4) = 100{,}000 \text{ lb.}$$

Negative shear (joint b loaded)

$$V = -R_2 = -\frac{60{,}000}{6}(1) = -10{,}000 \text{ lb.}$$

Panel cd

Positive shear (joints d to f loaded)

$$V = R_1 = \frac{60{,}000}{6}(1 + 2 + 3) = 60{,}000 \text{ lb.}$$

Negative shear (joints b and c loaded)

$$V = -R_2 = -\frac{60{,}000}{6}(1 + 2) = -30{,}000 \text{ lb.}$$

Stress in Bc.—For maximum tension in Bc load all joints to right of section 1-1 (see p. 264). From Eq. (15), with V = positive shear in panel bc = 100,000; $s = k$ panels = 4 panels; and $M_a = 0$ (no load to left of section 1-1), we have

$$S_V = \frac{M_o}{q} = \frac{(100{,}000)(4)}{6} = +66{,}660$$

and

$$S = S_V \sec \theta_1 = (66{,}660)\left(\frac{28.28}{20}\right) = 94{,}200 \text{ lb. tension.}$$

For maximum compression in Bc, load all joints to left of section 1-1. Then V = negative shear in panel $bc = -10{,}000$; M_a = moment of load at b about $a = (60{,}000)(1)$.

$$S_V = \frac{M_o}{q} = \frac{(-10{,}000)(4) - 60{,}000}{6} = -16{,}660$$

and

$$S = S_V \sec \theta_1 = (16{,}660)\left(\frac{28.28}{20}\right) = 23{,}600 \text{ lb. compression.}$$

Stress in Cc. (Section 2-2.).—Maximum compression in Cc. Load joints d to f. V = positive shear in panel cd = 60,000 lb.; $M_a = 0$; k = 4 panels; and q = 6 panels. From Eq. 18,

$$S = -\frac{M_o}{q} = -\frac{(60{,}000)(4)}{6} = -40{,}000 \text{ lb. compression.}$$

Maximum tension in Cc. Load joints b and c. V = negative shear in panel cd = $-30{,}000$ lb.; M_a = moment of loads at b and c about a = 60,000(1 + 2) = 180,000.

$$S = -\frac{M_o}{q} = -\frac{(-30{,}000)(4) - 180{,}000}{6} = +50{,}000 \text{ lb. tension.}$$

Stress in Cd. (Section 3-3.)—Shears taken in panel cd. k = 10 panels; q = 13 panels. For maximum tension in Cd,

$$S = \frac{M_o}{q} \sec \theta_2 = +\frac{(60{,}000)(10)}{(13)} \cdot \frac{(31.24)}{(24)} = +60{,}000 \text{ lb. tension.}$$

For maximum compression in Cd

$$S = \frac{M_o}{q} \sec \theta_2 = +\frac{[(-30{,}000)(10) - (60{,}000)(1 + 2)]}{(13)} \cdot \frac{(31.24)}{(24)}$$

$$= -48{,}000 \text{ lb. compression}$$

Maximum and Minimum Stresses. *Dead Load and Uniform Live Load.*—These stresses are determined by adding the dead and live load stresses. Maximum stress is generally found by adding the dead load stress and the live load stress of the same kind. Minimum stress is generally found by adding the dead load stress and the live load stress of the opposite kind The table below gives the detail work. Maximum and minimum stresses taken from this table are shown in the left half of Fig. 45b.

Maximum and Minimum Stresses
Dead Load and Uniform Live Load
Curved Chord Truss without Counters, Fig. 45

Member	Dead load stress	Live load tension	Live load compression	Maximum stress	Minimum stress
ab *bc*	+60,000	+150,000	. . . . .	+210,000	+60,000
cd	+80,000	+200,000	. . . . . .	+280,000	+80,000
BC	−81,600	. . . . . .	−204,000	−285,600	−81,600
CD	−83,600	. . . .	−201,000	−292,600	−83,600
aB	−84,840	. .	−212,400	−297,240	−84,840
Bc	+28,280	+ 94,200	− 16,660	+122,480	+11,620
Cc	+ 4,000	+ 50,000	− 40,000	+ 54,000	−36,000
Cd	+ 4,800	+ 60,000	− 48,000	+ 64,800	−43,200
Dd	+16,620	+ 41,600	. . . . . .	+ 58,220	+16,620
Bb	+24,000	+ 60,000	. . . . . .	+ 84,000	+24,000

\+ tension — = compression

Live Load Stresses. E-50 Train Loading.

Stresses in Chord Members.—Since the length and number of panels is the same as for the truss of Fig. 6, p. 219, the moments calculated on pp. 227 and 228 of Art. **2** for top chord moment centers may also be used for the truss under consideration in the present article. These moments for the several moment centers of Fig. 45 are as follows:

$$M_B = M_b = 3,362,500 \text{ ft.-lb.}$$
$$M_C = M_c = 5,255,000 \text{ ft.-lb.}$$
$$M_d = 5,745,600 \text{ ft.-lb.}$$

By the same methods as used on p. 273 for dead load stresses, the live load chord stresses for train loading are readily found to be as follows:

$$ab \text{ and } bc = \frac{M_B}{20} = \frac{3,362,500}{20} = 168,100 \text{ lb., tension.}$$
$$cd = \frac{M_C}{24} = \frac{5,255,000}{24} = 219,000 \text{ lb., tension.}$$
$$BC = \frac{M_c}{24} \sec \alpha_1 = \left(\frac{5,255,000}{24}\right)\left(\frac{20.40}{20}\right) = 223,000 \text{ lb. compression}$$
$$CD = \frac{M_d}{26} \sec \alpha_2 = \left(\frac{5,745,600}{26}\right)\left(\frac{20.10}{20}\right) = 222,000 \text{ lb. compression.}$$

Stresses in Web Members.

Member aB.—Shear on section *O-O*, Fig. 45*a*. Load position and shear same as for truss with horizontal chords. See p. 232 for calculation of shear in panel *AB*. $V = 168,085$ lb.

$$\text{Stress in } aB = (168,085)\left(\frac{28.28}{20}\right) = 237,700 \text{ lb. compression.}$$

Member Bc.—Maximum tension in *Bc*. Section 1-1, Fig. 45*a*. Criterion for position of loads (Eq. (22) with $n = 6$, $k = 4$, and $m = 2$):

$$\frac{G}{7.5} - G_2 = 0$$

Try wheel 3 at c. Wheel 16 at right end of span; $G = 306.25$ to 322.5; $\frac{G}{7.5} = 40.8$ to 43.0; $G_2 = 37.5$ to 62.5. Wheel 3 satisfies criterion, for values of $\frac{G}{7.5}$ lie between values of G_2. Wheels 2 and 4 were tried but did not answer. $R_1 = \frac{15,051,250}{120} =$ 125,400 lb. Panel load at $b = \frac{287,500}{20} = 14,400$ lb. Shear in panel $bc = 111,000$ lb. From Eq. (15) $M_o = (111,000)(4) - (14,400)(1) = 429,600$. Stress in $Bc = \frac{M_o}{q} \sec\theta = \left(\frac{429,600}{6}\right)\left(\frac{28.28}{20}\right) = 101,300$ lb. tension.

Maximum Compression in Bc.—Conditions same as for member Fe, load headed left, section 6-6. Criterion for position of loads (Eq (26) with $n = 6$; $m = 5$; $k = 10$); $\frac{G}{3.6} - G_2 = 0$. Try wheel 2 at f. Wheel 5 on the span by 5 ft. $G = 112.5$; $\frac{G}{3.6} = 31.25$; $G_2 = 12.5$ to 37.5. Wheel 2 satisfies criterion. Wheels 1 and 3 tried but did not answer. $R_1 = \frac{[1,037.5 + (112.5)(5)]}{120} = 13,330$ lb. Joint load at $e = \frac{100}{20} = 5,000$ lb. Shear in panel $fe = 13,330 - 5,000 = 8,330$ lb. From Eq. (24), $M_o = Vk + M_a = (8,330)(10) + (5,000)(4) = 103,300$. (Note that M_a = moment about a of joint load at e.) From Eq. (23).

$$S = -\frac{M_o}{q}\sec\theta = \left(\frac{103,300}{6}\right)\left(\frac{28.28}{20}\right) = 24,400 \text{ lb. compression.}$$

Member Cc.—Maximum compression in Cc. Section 2-2, shear in panel cd, Fig. 45a. Criterion (Eq. (22) with $n = 6$; $m = 3$; $k = 4$); $\frac{G}{9} - G_2 = 0$. Try wheel 2 at d. Wheel 11 is on span by 4 ft. $G = 215$; $\frac{G}{9} = 23.9$; $G_1 = 12.5$ to 37.5. Wheel 2 satisfies criterion. Wheels 1 and 3 did not answer. $R_1 = 1,000\left[\frac{7,310 + (215)(4)}{120}\right] =$ 68,100 lb. Panel load at $c = \frac{100,000}{20} = 5,000$ lb. Shear in panel $cd = 68,100 - 5,000 = 63,100$ lb. From Eq. (18)

$$S = -(Vk - M_a)\frac{1}{q} = -[(63,100)(4) - (5,000)(2)]\frac{1}{6}$$
$$= -40,400 \text{ lb. compression.}$$

Maximum Tension in Cc.—Consider member Ec, section 5-5. Criterion (Eq. (26) with $n = 6$; $m = 4$; $k = 10$); $\frac{G}{4.2} - G_2 = 0$. Try wheel 3 at e. Wheel 9 is on span by 5 ft. $G = 177.5$; $\frac{G}{4.2} = 42.2$; $G_2 = 37.5$ to 62.5. Wheel 3 satisfies criterion. Wheels 2 and 4 do not answer.

$$R_1 = 1,000\left[\frac{4,370 + (177.5)(5)}{120}\right] = 43,800 \text{ lb.}$$

Panel load at $d = \frac{287,500}{20} = 14,400$ lb. Shear in panel $de = 43,800 - 14,400 =$ 29,400 lb. From Eqs. (24) and (25)

$$S = +\left(\frac{Vk + M_a}{q}\right) = +\frac{(29,400)(10) + (14,400)(3)}{6}$$
$$= +56,200 \text{ lb. tension.}$$

Member Cd.—Maximum tension in Cd. Section 3-3, Fig. 45a. Criterion (Eq. (22) with $n = 6, m = 3, k = 10$); $\frac{G}{7.2} - G_2 = 0$. Try wheel 2 at d. Wheel 11 is on span by 4 ft. $G = 215$; $\frac{G}{7.2} = 29.9$; $G_2 = 12.5$ to 37.5. Wheel 2 satisfies criterion. Wheels 1 and 3 do not answer. $R_1 = 1{,}000\left[\frac{7{,}310 + (215)(4)}{120}\right] = 68{,}100$ lb. Panel load at $d = \frac{100{,}000}{20} = 5{,}000$ lb. Shear in panel $cd = 68{,}100 - 5{,}000 = 63{,}100$ lb. From Eq. (15), $M_o = (63{,}100)(10) - (5{,}000)(2) = 621{,}000$. Stress in $Cd = +\frac{M_o}{q}\sec\theta = +\left(\frac{621{,}000}{13}\right)\left(\frac{31.24}{24}\right) = 62{,}200$ lb. tension.

Maximum Compression in Cd.—Consider member Ed, section 4-4. Criterion (Eq. (26) with $n = 6$, $m = 4$, $k = 16$); $\frac{G}{4.88} - G_2 = 0$. Try wheel 2 at e. Wheel 9 is at right end of span. $G = 161.25$ to 177.5; $\frac{G}{4.88} = 33.0$ to 36.4; $G_2 = 12.5$ to 37.5. Wheel 2 satisfies criterion. Wheels 1 and 3 do not answer. $R_1 = \frac{4{,}370.000}{120} = 36{,}400$ lb. Panel load at $d = \frac{100{,}000}{20} = 5{,}000$ lb. Shear in panel $de = 36{,}400 - 5{,}000 = 31{,}400$ lb. From Eq. (24), $M_o = Vk + M_a = (31{,}400)(16) + (5{,}000)(3) = 518{,}000$. $S = \frac{M_o}{q}\sec\theta = \left(\frac{518{,}000}{13}\right)\left(\frac{31.24}{24}\right) = 52{,}000$ lb. compression.

Member Dd.—Joint at D. From Eq. (29), stress in Dd = (2) (Stress in CD) ($\sec\alpha_2$) $= (2)(222{,}000)\left(\frac{2}{20.10}\right) = 44{,}100$ lb. tension.

Member Bb.—Stress in Bb = floorbeam reaction for 20-ft. panel = 81,940 lb. See table on p. 142.

Maximum and Minimum Stresses. *Dead Load and Uniform Live Load.*—These stresses, which are given in the following table and on the right half of Fig. 45b, are determined by the methods used on p. 275 for dead load and uniform live load.

MAXIMUM AND MINIMUM STRESSES
DEAD LOAD AND E-50 TRAIN LOAD
CURVED CHORD TRUSS WITHOUT COUNTERS, FIG. 45

Member	Dead load stress	Live load tension	Live load compression	Maximum stress	Minimum stress
ab bc	+60,000	+168,100	. . . . .	+228,100	+60,000
cd	+80,000	+219,000	. . . .	+299,000	+80,000
BC	−81,600	. . . .	−223,000	−304,600	−81,600
CD	−83,600	. . .	−222,000	−305,600	−83,600
aB	−84,840	. .	−237,700	−322,600	−84,840
Bc	+28,280	+101,300	− 24,400	+129,600	+ 3,900
Cc	+ 4,000	+ 56,200	− 40,000	+ 60,200	−36,400
Cd	+ 4,800	+ 62,200	− 52,000	+ 67,000	−47,200
Dd	+16,620	+ 44,100	.	+ 60,700	+16,620
Bb	+24,000	+ 81,940	. . . . .	+105,940	+24,000

+ = tension — = compression

Stress analysis by the method of vertical components will be illustrated by detail calculations for stress in member *Cc* of Fig. 45*a*.

Dead Load Stress.—Section 2-2, Fig. 45*a*. $V = 12{,}000$ lb. (see p. 269). Simultaneous stress in top chord $BC = 81{,}600$ lb. From Eq. (28)

$$\text{Stress in } Cc = -(V - S_1 \sin \alpha_1)$$
$$= -\left[12{,}000 - (81{,}600)\frac{4}{20.40}\right] = +4{,}000 \text{ lb. tension.}$$

Live Load Stress—Uniform Live Load.—Maximum compression in *Cc*. Loads at joints *d* to *f*. $V = +\ 60{,}000$ (see p. 275). Vertical component of simultaneous top chord stress in $BC = \frac{M_c}{24}\left(\frac{4}{20}\right) = \frac{(60{,}000)(2)(20)(4)}{(24)(20)} = 20{,}000$ lb. $= S_1 \sin \alpha_1$ of Eq. (28).

$$\text{Stress in } Cc = -\ (V - S_1 \sin \alpha_1) = -\ (60{,}000 - 20{,}000)$$
$$= -40{,}000 \text{ lb. compression.}$$

Maximum Tension in Cc.—Loads at joints *b* and *c*. $V = -\ 30{,}000$. Vertical component stress in $BC = \frac{(30{,}000)(4)(20)(4)}{(24)(20)} = 20{,}000.$

$$\text{Stress in } Cc = -(V - S_1 \sin \alpha_1) = -\ (-\ 30{,}000 - 20{,}000)$$
$$= +50{,}000 \text{ tension.}$$

Live Load Stress—E-50 Train Loading.—Load positions and shears may be taken from p. 277. Maximum compression in *Cc*; Section 2-2. With wheel 2 at *d*, $R_1 =$ 68,100 lb.; panel load at $c = 5{,}000$ lb.; shear in panel $cd = 63{,}100$ lb. Vertical component of simultaneous chord stress in $BC =$

$$\frac{(R_1)(2)(20)}{(24)} \tan \alpha_1 = \frac{(68{,}100)(2)(20)}{(24)}\left(\frac{4}{20}\right) = 22{,}700 \text{ lb.}$$

From Eq. (28)

$$\text{Stress in } Cc = -(63{,}100 - 22{,}700) = -\ 40{,}400 \text{ compression.}$$

Maximum Tension in Cc.—Consider member *Ee* and section 5-5. With wheel 3 at *e*, $R_1 = 43{,}800$ lb.; panel load at $d = 14{,}400$ lb.; and $V = 29{,}400$ lb. Vertical component of simultaneous stress in $EF =$

$$[(43{,}800)(4) - (14{,}400)(1)]\ {}^{20}\!/_{24} \cdot {}^{4}\!/_{20} = 26{,}800$$

For conditions on section 5-5, Eq. (28) must be written in the form

$$S = +(V + S \sin \alpha_1)$$

Then

$$\text{Stress in } Cc = (29{,}400 + 26{,}800) = 56{,}200 \text{ lb. tension.}$$

Note that these results check the values given by the method of moments.

13*b*. Curved Chord Truss with Counters.—From the tables of maximum and minimum stresses on pp. 276 and 278 it will be noted that the diagonal *Cd* and the vertical *Cc* are subjected to reversals of stress. The presence of counters in panels *cd* and *de* will prevent a reversal of stress in the diagonal *Cd*, although the vertical *Cc* will still be subjected to a stress reversal. However, it will generally be found that the presence of a counter will reduce the range of stress in the vertical.

The presence of counters in a curved chord truss affects only the stresses in the web members of the panel containing the counter. Stresses in the remaining members are calculated by the methods given in Art. **13*a***.

In calculating stresses in members located in panels containing counters, the form of truss for any given loading must first be determined. The stresses may then be determined by the usual methods. It will often be found that the dead load stress in a member for the loading causing maximum stress is not the same as that for the loading causing minimum stress. This is due to the change in form of truss for the different loading conditions.

It will now be assumed that counters are provided in the center panels of the truss considered in Art. **13a.** Detail calculations will be given only for those members whose stresses are affected by the presence of counters.

Member Cd.—Maximum stress in *Cd*, same as for truss without counters. See p. 275 for calculations. Minimum stress in *Cd* is zero; it occurs when the counter *Dc* is in action.

Member Dc.—Minimum stress in counter is zero; it occurs when the main member is in action.

Maximum Tension in Dc.—Consider member *De*, section 1-1, Fig. 46c.

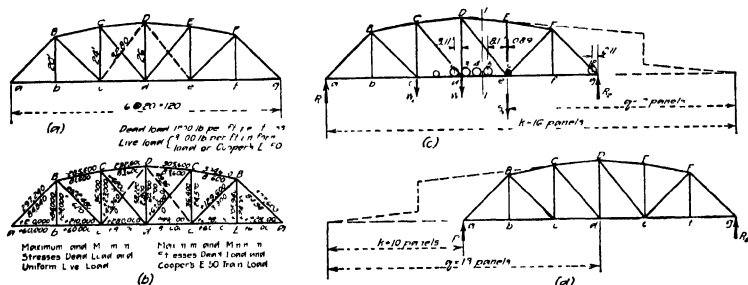



FIG. 46.

Dead Load Stress.—Use Eqs. (23) and (24). Joint loads of 24,000 at all points $R_1 = 60{,}000$ lb. Shear in panel *de*; $V = -12{,}000$. From Eq. (24) $M_o = Vk + M_a = (-12{,}000)(16) + 24{,}000(1 + 2 + 3) = -48{,}000$ (M_a = moment about *a* of loads between *a* and section 1-1). From Eq. (23)

$$S_V = +\frac{M_o}{q} = +\frac{(-48{,}000)}{12} = -4{,}000$$

Hence

$$S = S_V \sec\theta = (-4{,}000)\left(\frac{32.80}{26}\right) = -5{,}040 \text{ lb. compression.}$$

Live Load Stress—Uniform Live Load.—Panel loads of 60,000 lb. at *e* and *f*. Shear in panel $de = R_1 = \frac{60{,}000}{6}(1 + 2) = 30{,}000$ lb. From Eq. (24), $M_o = (30{,}000)(16) = 480{,}000$. Note that $M_a = 0$ for this case. Then

$$S = +\frac{M_o}{q}\sec\theta = +\frac{(480{,}000)}{(12)}\left(\frac{32.80}{26}\right) = +50{,}400 \text{ lb. tension.}$$

Live Load Stress E-50 Train Loading.—Load position same as for maximum compression in *Cd* (see p. 278). Hence $R_1 = 36{,}400$ lb.; panel load at $d = 5{,}000$ lb.; shear in panel $de = 31{,}400$ lb. From Eq. (24), $M_o = (31{,}400)(16) + (5{,}000)(3) = 518{,}000$.

$$S = +\frac{M_o}{q}\sec\theta = \left(\frac{518{,}000}{12}\right)\left(\frac{32.80}{26}\right) = +54{,}400 \text{ lb. tension.}$$

Maximum tension in Dc for dead load and uniform live load is $-5{,}040 + 50{,}400 = 45{,}360$ lb. For dead load and E-50 train loading, maximum tension $= -5{,}040 + 54{,}400 = 49{,}360$ lb.

Member Cc.—Maximum compression in Cc same as for truss without counters (see p. 273).

Maximum Tension in Cc.—Consider member Ee, form of truss as shown in Fig. 46c. Tension in Ee depends upon top chord stress (conditions shown in Fig. 42b). It will be maximum when chord stresses are as great as possible for the form of truss shown in Fig. 46c. True live load position must be determined by trial (see below).

Dead Load Stress.—Use Eq. (31) with $S_1 = 81{,}600$ lb. (see p. 273); $\cos \alpha_1 = \frac{20}{20.4}$ (see Fig. 45a); $\tan \alpha_1 = 4/20$; $\tan \alpha_2 = 2/20$; and $W_1 = 0$ (all dead load on lower chord). Then

$$\text{Stress in } Cc = (81{,}600)\left(\frac{20}{20.4}\right)\left(\frac{4}{20} - \frac{2}{20}\right) = 8{,}000 \text{ lb. tension.}$$

Live Load Stress, Uniform Live Load.—Again consider member Ee, form of truss shown in Fig. 46c. For maximum tension in Ee load as many points with live load as is possible without causing compression in member De. We must first find total stress in De for several load positions. Try panel loads of 60,000 lb. at d, e and f. $R_1 = \frac{60{,}000}{6}(1 + 2 + 3) = 60{,}000$ lb. Shear in panel $de = 60{,}000 - 60{,}000 = 0$. From Eqs. (23) and (24)

$$S_V = \frac{Vk + M_a}{q} = \frac{(0)(16) + (60{,}000)(3)}{12} = 15{,}000$$

and $S = S_V\left(\frac{32.8}{26}\right) = 18{,}800$ lb. tension. From p. 280, dead load stress in $Dc = De = 5{,}040$ lb. compression. Total stress in $De = 18{,}800 - 5{,}040 = 13{,}760$ tension. Hence form of truss still as shown in Fig. 46c. Try loads at joints c to f. $R_1 = 100{,}000$; shear in panel $de = -20{,}000$. Then

$$S_V = \frac{(-20{,}000)(16) + (60{,}000)(2 + 3)}{12} = -1{,}670$$

The minus sign indicates that the assumed loading causes compression in De. Hence maximum tension in Ee occurs for joints d, e, and f loaded, for which

$$\text{Stress in } EF = \frac{[R_1(4) - (60)(1)](20)}{24} \times \sec \alpha_1 = 150{,}000 \sec \alpha_1$$

From Eq. (31), with S_1 = stress in EF and $W_1 = 0$,

$$\begin{aligned}\text{Stress in } Ee &= (150{,}000 \sec \alpha_1)(\cos \alpha_1)(\tan \alpha_1 - \tan \alpha_2)\\ &= (150{,}000)(4/20 - 2/20) = 15{,}000 \text{ lb. tension.}\end{aligned}$$

Maximum and minimum stresses in Cc for dead load and uniform live load are then as follows: Compression, 36,000 lb. (member Cd in action; stress same as for truss without counters, see p. 273). Tension: Counter Dc in action, form of truss shown in Fig. 46c (member Ee replaces Cc). Dead load stress, 8,000 lb. tension; live load stress 15,000 lb. tension; total stress, 23,000 lb. tension.

Live Load Stress, E-50 Loading.—The exact position of live load for zero stress in De due to dead load and E-50 train loading must be determined by trial. From p. 280, dead load stress in $De = 5{,}040$ lb. compression. Hence the loads must be placed in position for a live load tension of 5,040 lb. in member De. This stress is given by the live load position shown on Fig. 46c. For the conditions shown, $R_1 = 74{,}090$ lb.; W_1

= 75,830 lb.; W_2 = 10,830 lb.; and shear in panel $de = V = -12{,}570$. (Methods for calculating W_1 are given on p. 143.) Then

$$S = S_V\left(\frac{32.8}{26}\right) = \frac{(-12{,}570)(16) + (2)(10{,}830) + (3)(75{,}830)}{12}\left(\frac{32.8}{26}\right)$$
$$= 5{,}040 \text{ lb. tension.}$$

The simultaneous stress in EF is

$$EF = \frac{(74{,}090)(4)(20) - [1{,}037.5 + (112.5)(8.11)]}{24} \sec\alpha_1 = 165{,}700 \sec\alpha_1$$

From Eq. (31)

$$\text{Stress in } Ee = (165{,}700 \sec\alpha_1)(\cos\alpha_1)(\tan\alpha_1 - \tan\alpha_2)$$
$$= 165{,}700(\tfrac{4}{20} - \tfrac{2}{20}) = 16{,}570 \text{ lb. tension.}$$

Maximum and minimum stress in member Cc for dead load and E-50 train load are then as follows. Compression: 36,400 lb. (Member Cd in action. Stress same as for truss without counters, see p. 278.) Tension: Counter Dc in action. Form of truss shown in Fig. 46c. (Member Ee replaces Cc.) Dead load stress, 8,000 lb. tension; live load stress, 16,570 lb. tension; total tension 24,570 lb.

Member Dd.—Maximum tension in Dd is the same as for the truss without counters (see p 273).

Maximum Compression in Dd.—When counters are provided, Dd becomes a compression member for certain loading conditions. Form of truss shown in Fig. 46d.

Dead Load Stress.—Panel loads of 24,000 lb. at each joint. Shear in panel de = – 12,000 lb. From Eq. (18)

$$S = -\frac{(Vk - M_a)}{q} = -\frac{[(-12{,}000)(10) - (24{,}000)(1 + 2 + 3)]}{13}$$
$$S = 20{,}300 \text{ lb. tension.}$$

Live Load Stress,—Uniform Live Load.—Panel loads of 60,000 lb. at e and f. $R_1 = \frac{60{,}000}{6}(1 + 2) = 30{,}000$ lb. = shear in panel de. From Eq. (18)—(Note that M_a = 0)

$$S = -\frac{(30{,}000)(10)}{13} = 23{,}100 \text{ lb. compression.}$$

Hence, total stress, dead load and uniform live load = +20,300 – 23,100 = –2,800 lb. compression.

Live Load Stress, E-50 Train Load.—Load position determined by Eq. (22) with $n = 6$, $m = 4$, and $k = 10$. $\frac{G}{7.8} - G_2 = 0$. Try wheel 2 at e, Fig. 46d. Wheel 9 at right end of span. G = 161.25 to 177.5; $\frac{G}{7.8}$ = 20.7 to 22.8; G_2 = 12.5 to 37.5. Wheel 2 satisfies criterion. Wheels 1 and 3 were tried but did not answer. $R_1 = \frac{4{,}370{,}000}{120} = 36{,}400$ lb. Panel load at $d = \frac{100{,}000}{20} = 5{,}000$ lb. Shear in panel de = 31,400 lb. From Eq. (18)

$$\text{Stress in } Dd = -\frac{(Vk - M_a)}{q} = -\frac{(31{,}400)(10) - (5{,}000)(3)}{13}$$
$$= -23{,}000 \text{ lb. compression.}$$

Hence, total stress, dead load and E-50 train load = +20,300 – 23,000 = 2,700 lb compression.

Maximum and minimum stresses, as calculated above, are shown on Fig. 46b.

TRUSSES WITH MULTIPLE WEB SYSTEMS

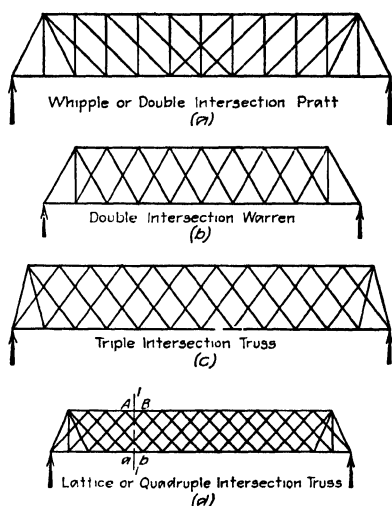
Typical examples of trusses with multiple web systems are the Whipple, Fig. 47*a* and the Double Intersection Warren, Fig. 47*b*. The Whipple truss can be seen to be a combination of Pratt trusses. It is therefore sometimes known as a Double Intersection Pratt. Figure 47*c* is composed of three Warren trusses and Fig. 47*d* is composed of four Warren trusses.

Trusses of the type shown in Fig. 47 are seldom constructed at present. They were very popular from 1870 to 1880 but in later years have been replaced by trusses of the type shown in Fig. 55, p. 290. The type of truss shown in Fig. 47 was devised to secure in long span bridges a short panel length combined with an economical ratio of height to span length (about 1 in 6) and an economical slope for diagonal members (about 45 deg.). From Fig. 47 it can be seen that this has been accomplished, except in the case of the slope of a few of the diagonals near the end of the truss.

Fig. 47.

Although multiple intersection trusses are not in general use at present, the engineer in his practice may find occasion to check the stresses in an existing structure in regard to its safety under present day loads. For this reason a brief discussion on methods of stress determination will be given in the articles which follow.

14. General Methods of Stress Analysis.—On cutting a section through any one of the trusses of Fig. 47, it will be noted at once that in general it is impossible to find any section which will not intersect more than three members. Hence the stresses on any section cannot be determined by the methods of statics and the structure is statically indeterminate. Two general methods of analysis may be used for the determination of stresses in the structure under consideration. These are the Exact Method, and the Approximate Method.

The exact method of analysis is based on the theory of redundant members as given in Sec. 5. A solution by the exact method for stresses in a truss of the type shown in Fig. 47*a* is given on p. 395. The exact method of solution is long and tedious, and, as shown on p. 398, the difference in stresses determined by the exact method and the

approximate method is so small that the increased labor of an exact solution is hardly warranted in the cases usually encountered in practice.

The approximate method of solution is based on the assumption that the trusses shown in Fig. 47 may be separated into two or more independent stable simple trusses. Thus the double intersection trusses may be decomposed into two trusses; the triple intersection into three trusses, etc. Each simple truss thus formed is then analyzed for the portion of the applied loads tributary to its joints. After the stresses in each component truss have been determined, the several trusses are again joined to form the original composite truss. Where a member of the composite truss is common to more than one of the independent truss systems, it is assumed that the total stress in that member is the *sum of the stresses in the members of the independent trusses which coincide to form the member of the complete truss.*

FIG. 48

To illustrate: Let Fig. 48*a* show a double intersection Warren truss carrying panel loads W at the lower chord joints. It can readily be seen that this truss may be divided into the two independent stable structures shown in Figs. 48*b* and *c*. In forming these independent trusses, it is assumed that the chord and end post members of the complete truss are divided into two parts. These parts are assumed to form the chords and end posts of the two independent trusses. For the truss under consideration it was not necessary in forming the independent trusses to divide any of the web members of the complete truss. In any case care must be taken when dividing any structure into independent trusses to make certain that the resulting trusses are stable and statically determinate. For convenience in distinguishing the two systems which form the truss of Fig. 48*a*, the web members of one truss are shown in full lines and those of the other are shown in dotted lines. These systems are referred to as the *full line* and *dotted line* systems.

In Fig. 48*a*, panel loads are shown at each lower chord joint. When the truss is separated into independent systems, it is assumed that any panel load is carried by the system which contains its point of application. Thus the load at joint *b* Fig. 48*a* is carried by the dotted line system Fig. 48*c*, and the load at joint *c* is carried by the full line system Fig. 48*b*.

After the truss of Fig. 48*a* has been divided into two independent truss systems, the stresses in these systems for fixed and moving loads are determined by the methods given in the preceding pages. When all

stresses have been computed the two independent trusses are again joined to form the original truss. The stress in any chord member, as *CD*, Fig. 48*a* is the sum of the stresses found for member *BD* of the full line system and member *CE* of the dotted line system, since these members of the independent trusses are joined to form a member of the complete truss. For web member *De* of Fig. 48*a*, the stress is as computed for member *De* of the full line system. Also, for *Cd* of Fig. 48*a* the stress is the same as for *Cd* of the dotted line system.

Stresses in the lattice, or quadruple intersection truss of Fig. 47*d* may be determined by the method outlined above. A somewhat shorter solution may be made by cutting any section, as 1-1, Fig. 47*d*, through the intersection of the diagonal web members. In calculating web stresses it is assumed that the shear on this section is equally divided between the members cut by the section. To determine the stresses in

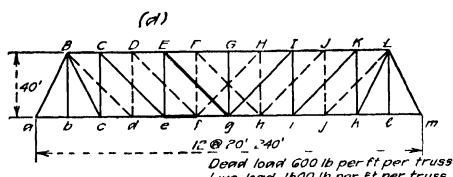
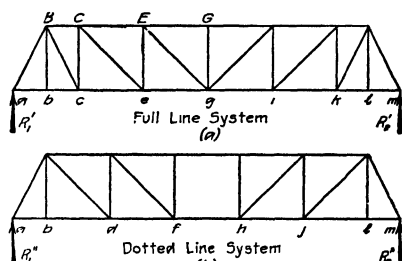



FIG. 49. FIG. 50.

chord member *AB*, take moments about the point where section 1-1 cuts the bottom chord. It can readily be seen that the moment of web stresses about this point will be zero and that the stress in *AB* is the moment about the given center divided by the height of truss. While this method is somewhat more approximate than the one outlined above, the results obtained are accurate enough for all practical cases.

When the trusses under consideration are subjected to uniform loading, the panel loads are each equal to the load per foot times a panel length, which is to be taken as the distance between successive panel points, as shown on Fig. 48*a*. Load positions for maximum stress are determined by the same methods as used for Warren or Pratt trusses.

When a concentrated load system such as E-50 train loading is used, the panel concentrations are determined by the method given in Art. 71, p. 140. Load positions for maximum stress must in general be determined by cut and try methods. It is possible to derive criteria for load positions, but the resulting expressions are so cumbersome and difficult of application that it will usually be found best to resort to a direct comparison method.

In the problems which follow, the general methods explained above will be applied to the determination of stresses in certain members of a double intersection truss.

Illustrative Problem.—Required the maximum and minimum stress in all members of the through truss shown in Fig. 49 due to a dead load of 600 lb. per ft. per truss and a uniform live load of 1,500 lb. per ft. per truss.

Figure 50 shows the given truss divided into two independent truss systems. Member *Bb*, and the corresponding member on the right side of the truss, has been divided into two parts, one part going to each system. It will be assumed here that any load at joint *b* is divided equally between the two systems. The exact division of load at this point is indeterminate, but the above assumption is probably reasonably correct.

Since the general methods of stress analysis for the independent trusses shown in Fig. 50 differ only in minor details from those given in Art. 4, the complete calculations will be given here only for chord member *ef* and for web member *Eg*. Maximum and minimum stresses for all members are shown on Fig. 51.

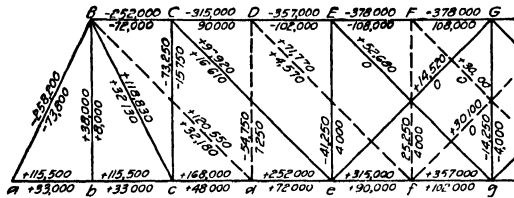



FIG. 51.—Maximum and minimum stresses, dead load and uniform live load.

Panel loads are to be determined for the complete truss of Fig. 49. Dead panel load = (600)(20) = 12,000 lb.; live panel load = (1,500)(20) = 30,000 lb.

Member ef.—Member *df* of the dotted line system and member *eg* of the full line system form member *ef*.

Dead Load Stress.—Panel loads of 12,000 lb. at all lower chord joints of Fig. 50*a* and *b* except *b* and *l*, where 6,000 lb. loads are placed.

Member *eg*, full line system. Moment center at *E*.

$$R_1' = (\tfrac{6}{2})(12{,}000) = 36{,}000 \text{ lb.}$$

$$\text{Stress in } eg = \frac{M_E}{h} = \left[(36)(4) - (12)\left(2 + \frac{3}{2}\right)\right](1{,}000)\left(\frac{20}{40}\right) = 51{,}000 \text{ lb. tension.}$$

Member *df*, dotted line system, moment center at *D*.

$$R_1'' = (\tfrac{5}{2})(12{,}000) = 30{,}000 \text{ lb.}$$

$$\text{Stress in } df = \frac{M_D}{h} = [(30)(3) - (6)(2)](1{,}000)\frac{20}{40} = 39{,}000 \text{ lb. tension.}$$

Total dead load stress in *ef* = 51,000 + 39,000 = 90,000 lb. tension.

Live Load Stress.—Load position for maximum stress same as for dead load. Hence live load stress may be determined by ratio from dead load. Live load stress = Dead load stress $(\tfrac{30}{12})$ = (90,000)$(\tfrac{30}{12})$ = 225,000 lb. tension.

Maximum and Minimum Stress in ef.—Maximum stress = dead plus live load stresses = 315,000 lb. tension; Minimum stress = dead load stress = 90,000 lb. tension.

Member Eg.—This member forms a part of the full line system. Loads on the dotted line system are assumed to have no effect on stress in *Eg*. It will be assumed that *Eg* is capable of taking tension only; a counter must be provided in case of a possible reversal of stress.

Dead Load Stress.—Panel loads of 6,000 at *b* and *l*, Fig. 50*a*; loads of 12,000 at *c* to *k* inclusive. R_1' = 36,000 lb. Shear in panel *eg* = +6,000 lb.

$$\text{Stress in } Eg = V \sec \theta = (6{,}000)(1.414) = 8{,}480 \text{ lb. tension.}$$

Live Load Stress.—Positive shear in panel *eg*; loads of 30,000 lb. at *g*, *i*, and *k* load of 15,000 lb. at *l*.

$$+V_{eg} = R_1' = \frac{30,000}{12}\left(\frac{1}{2} + 2 + 4 + 6\right) = +31,250$$

Negative shear in panel *eg*; loads of 30,000 lb. at *c* and *e*; load of 15,000 lb. at *b*

$$-V_{eg} = -R_2' = -\frac{30,000}{12}\left(\frac{1}{2} + 2 + 4\right) = -16,250$$

Stress due to positive shear $= V \sec \theta = (31,250)(1.414) = 44,200$ lb. tension.
Stress due to negative shear $= (16,250)(1.414) = 23,000$ lb. compression.

Maximum and Minimum Stress in Eg.—Maximum stress = dead load and positive live load $= 8,480 + 44,200 = 52,680$ tension. Minimum stress: Compression due to negative live load shear is greater than dead load stress. Hence counter acts, and minimum stress in $Eg = 0$.

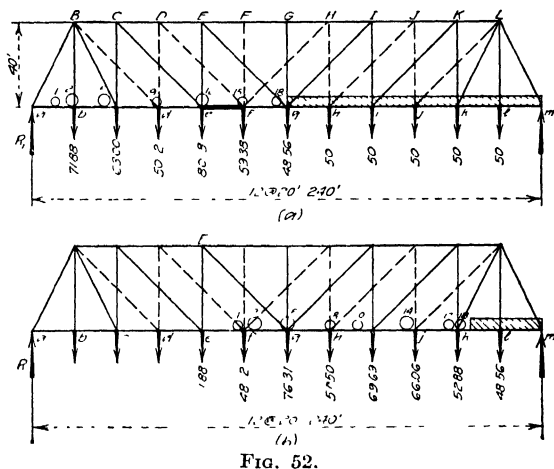



FIG. 52.

Illustrative Problem.—Required the live load stresses in members *ef* and *Eg* of the truss shown in Fig. 49 for E-50 train loading.

Two solutions will be given for this problem.

First Solution. *Panel Concentration Method.*—The general method of solution is exactly the same as given in the preceding problem. However, the panel loads must be determined by the method given in Art. 71, p. 140. To determine the maximum stress in a given member, several positions of the train load must be assumed. Panel concentrations must be determined for each load position; the truss must be divided, as shown in Fig. 50, and the resulting stress calculated as in the preceding problem. This process must be repeated until some load position is found which gives a greater stress than any other possible load position. The greatest obtainable value is the desired maximum stress. In the work which follows, the calculations are given in condensed form for the load position found to yield a maximum result.

Member ef.—Wheel 12 at joint *e* was found to give maximum stress in *ef*. Load position and resulting panel concentrations are shown on Fig. 52*a*. These concentrations were calculated by the method used in the problem given on p. 141.

Divide the truss into two systems as shown in Fig. 50, assuming that half the loads at joints *b* and *l* go to each system. For the full line system, $R_1 = 190,500$ lb.; and

stress in $eg = \frac{M_E}{40} = 263{,}800$ lb., tension. For the dotted line system, $R_1 = 140{,}500$ lb. and

$$\text{Stress in } df = \frac{M_D}{40} = 174{,}800 \text{ lb. tension}$$

Total stress in $ef = 438{,}600$ tension.

Member Eg.—Load position yielding maximum stress and panel concentrations due to that loading are shown on Fig. 52*b*. The stress in Eg is determined by considering the full line system (Fig. 50*a*) and only those panel concentrations which are applied at the panel points of this system.

For the assumed conditions

$R_1 = 73{,}480$ lb.; shear in panel $eg = 73{,}480 - 1{,}880 = 71{,}600$ lb.

Stress in $Eg = (71{,}600)(1.414) = 101{,}200$ lb. tension.

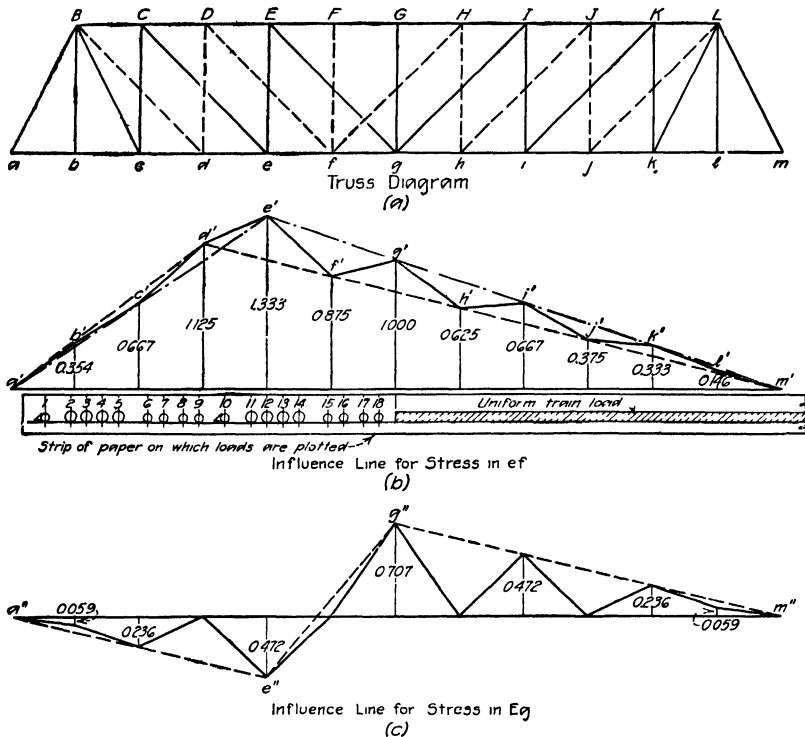



FIG. 53.

Second Solution. *Influence Line Method.*—This method of stress determination will be found very useful when the determination of load position by means of a criterion is impossible or inadvisable because of the amount of detail work required.

To apply the influence line method to the determination of stress in any member of a structure, the truss is first laid out to scale. The influence line for stress in the given member is then calculated and constructed to the same distance scale as used for the truss. Methods for calculating influence lines are given in Sec. 1. After the influence line is drawn, a load position is assumed and the loads are located on the truss drawing. The influence line ordinate for each load is scaled and the product of

load and ordinate obtained for each load. The sum of all such products is the required stress. On repeating this process for several load positions, the maximum stress may readily be determined. This process will be carried out for the assigned members.

Member ef.—The influence line for member *ef* is a combination of the influence lines for stress in member *eg* of the full line system and member *df* of the dotted line system. Figure 53*a* shows the truss diagram. By methods given in Art. **73,** p. 146, the influence line for stress in *eg* of the full line system is represented by the triangle *a'e'm'*. It is assumed that a full load is used at joints *b* and *l*. The influence line for stress in *df* of the dotted line system is given by the triangle *a'd'm'*. In plotting these diagrams, the horizontal scale is the same as the distance scale for the truss and the vertical, or ordinate scale, was taken as 1 in. = 0.5 lb Noting that a unit load moving across the structure is applied alternately at the panel points of the two systems, it is evident that the influence line for the combined structure may be obtained by connecting alternate points on the two influence lines. At joints *b* and *l*, where the load is assumed as equally divided between the two systems, an average of the two ordinates must be taken. The final influence line is *a'b'c'd'e'f'g'h'i'j'k'l'm'* of Fig. 53*b*.

The influence line of Fig. 53*b* shows that loads in the vicinity of joint *e* have maximum effect on the stress in *ef*. Hence the train load should be placed with a group of heavy loads near this point. Try wheel 12 at *e*. Since several positions must be tried it will be best to plot the wheel load on a strip of paper which may be moved at will. This is shown in Fig. 53*b* by the narrow rectangular strip on which the loads are plotted. A convenient form for tabulating loads and influence ordinates is given below. As stated on p. 153, the stress due to uniform load is equal to the product of the area of the influence line covered by uniform load and the load per foot.

STRESS IN *ef*
INFLUENCE LINE METHOD
Concentrated Loads

	Wheel loads					
	Wheel No.	load 12 5	Wheel No	load 25 0	Wheel No.	load 16 25
Wheel loads and ordinates	1	0 198	2	0 335	6	0 737
	10	1 200	3	0 423	7	0.853
			4	0 497	8	0.987
			5	0.578	9	1.103
			11	1.288	15	0.903
			12	1 333	16	0.903
			13	1 220	17	0.937
			14	1.105	18	0 965
Sum of ordinates.		1 398		6 779		7.388
Product ordinate and load . . .	17,475		169,450		120,055	
Stress from wheel loads.	306,980					

Uniform Load

Area influence line under uniform load =

$$20[(\tfrac{1}{2})(1{,}000) + 0.625 + 0.667 + 0.375 + 0.333 + 0.146 + 0] = 52.92$$

Stress from uniform load = (2,500)(52.92) = 132,300. Total stress in ef = 439,280 lb. tension. This stress was found to be the maximum value.

Member Eg.—The influence line for stress in this member is shown in Fig. 53*c*. To construct this influence line, draw the influence line for stress in member Eg of the full line system, which is shown by the line $a''e''g''m''$. Since loads at the panel points of the dotted line system are assumed to have no effect on the stress in Eg, the influence line ordinates at panel points of the dotted line system must be zero, This may be represented by the construction shown in Fig. 52*c*. At points b and l the load is assumed as equally divided between the two systems. Hence the influence line ordinate at this point is taken as one-half the value given by the line $a''g''e''m''$. It was found by trial that wheel 5 at g gave the maximum stress. The maximum stress was found to be 101,240 lb. tension. Note that the results obtained by the two methods of calculation are in close agreement.

TRUSSES WITH SUBDIVIDED PANELS

In modern types of long-span simple trusses, the proper relation between slope of diagonal web members and panel length for maximum

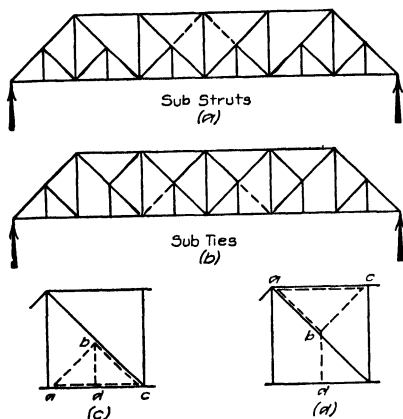
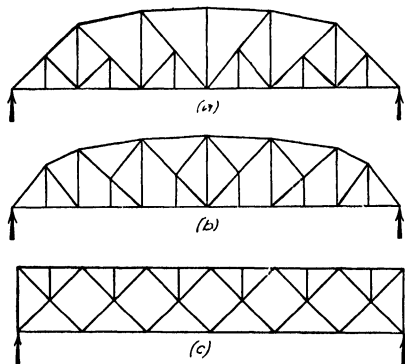



FIG. 54. FIG. 55.

economy in material is secured by means of subdivided panels. Figures 54*a* and *b* show a six panel Pratt truss with subdivided panels. In this form, the truss is known as a Baltimore truss.

Two methods of subdividing a panel of a truss are shown in Figs. 54*c* and *d*. In each figure a small auxiliary truss *abc*, shown by dotted lines, divides the main panel *ac* into two equal shorter panels. The truss of Fig. 54*a* is formed by placing auxiliary trusses of the form shown in Fig. 54*c* in each panel. Since *ab* and *bc* of Fig. 54*c* are in compression, the resulting truss arrangement shown in Fig. 54*a* is known as a Baltimore truss with sub-struts. In Fig. 54*d*, members *ab* and *bc* are in tension, and the resulting truss of Fig. 54*b* is known as a Baltimore truss with sub-ties.

Where members of the auxiliary and main trusses coincide, a single structural unit forms the combined member.

The arrangement shown in Fig. 54 permits the use of comparatively short panels and at the same time, favorable inclinations for the diagonals are secured. In most cases, eyebars and pin connections are used extensively in these trusses. The truss of Fig. 54*a* is somewhat more rigid than the one shown in Fig. 54*b*, while the latter is probably somewhat less costly because of the greater use of eyebar tension members.

Figures 55*a* and *b* show a curved chord truss with subdivided panels, known as a Pennsylvania truss. It is also called a Pettit truss. In Fig. 55*a* sub-struts are used and the end post extends over two panels. In Fig. 55*b* sub-ties are used and the framing in the first two panels is the same as for an ordinary curved chord Pratt truss. Figure 55*c* shows a double intersection Warren truss with subdivided top chord panels. This truss is statically indeterminate and is not well adapted for use with pin connections. It is seldom used at present.

15. General Methods of Stress Analysis.

15*a*. The Baltimore Truss with Sub-struts.—General methods of stress calculation will now be developed for chord and web members in trusses of the form shown in Fig. 54*a*. Maximum and minimum stresses may be determined by combining dead and live load stresses by the methods explained for Warren and Pratt trusses in the preceding articles.

STRESSES IN CHORD MEMBERS

Stresses in top chord members of Fig. 56*a* are determined from moments about opposite lower chord points. For example: Stress in EG = moment at g divided by height of truss. General methods of calculation and position of loads for maximum stress are therefore the same as for a Pratt truss.

Stress in any lower chord member, as fg, Fig. 56*a*, may be determined by cutting section 1-1. The moment center is located at E, the intersection of EG and $f'g$. Assuming that all joints are loaded with panel loads W, and noting that the load at f is to the left of section 1-1 and to the right of the moment center E, we have

$$\text{Stress in } fg = \frac{R_1md - W[1 + 2 + \ldots (m - 1)]d + W_fd}{h}$$

In this expression $R_1md - W_f[1 + 2 + \ldots (m - 1)]d$ = moment at E due to applied loads = M_E, and W_f = floorbeam reaction at f due to loads in panels ef and fg. For panels of equal length, $W_f = \frac{2M_1}{d}$ (see Eq. (4) p. 143) where M_1 = moment at f in a beam of length eg due to loads on that beam. Hence, in general

$$\text{Stress in } fg = \frac{M_E + 2M_1}{h} \qquad (1)$$

in which M_E and M_1 have the values defined above. Thus for a truss with equal panels, the *moment at any top chord center is equal to the bending moment at that center plus twice the center moment in the beam formed by the two panels to the right of the given center.*

When the panels are of unequal length, the floorbeam reaction is given by Eq. (3), p. 143. If d_1 and d_2 = lengths of successive panels to right of the moment center,

$$\text{Stress in } fg = \frac{M_E + M_1\left(\dfrac{d_1 + d_2}{d_2}\right)}{h} \tag{2}$$

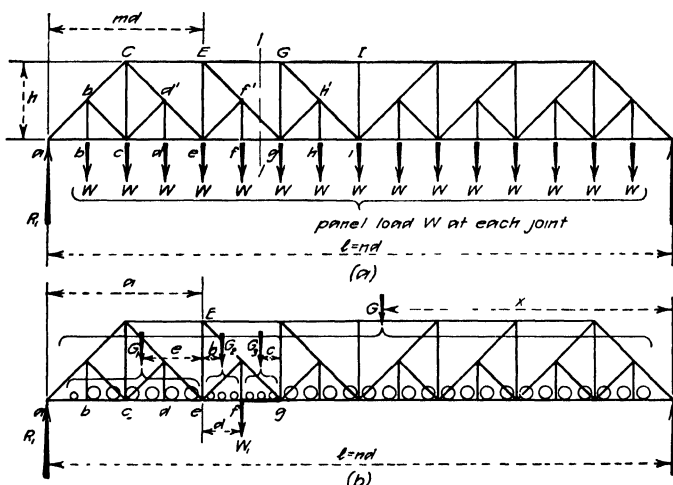



FIG. 56.

Load positions for maximum live load moment are determined by the methods used in the preceding articles. For uniform load, all points are to be fully loaded.

Figure 56*b* shows the loading conditions for a concentrated load system. Assume the loads to be divided into groups as shown, and let W_1 = floorbeam reaction at f due to loads in panels ef and fg. For the conditions shown

$$M_E = R_1a - G_1e + W_1d = \frac{Gx}{l}a - G_1e + \frac{G_2b + G_3c}{d}\cdot d$$

Assume that the load groups are all moved a short distance Δ to the left. On placing equal to zero the change in M_E due to the assumed movement, we have

$$\frac{Ga}{l} - (G_1 + G_2 - G_3) = 0 \tag{3}$$

Equation (3) is the criterion for position of loads for maximum stress in any lower chord member of the truss of Fig. 56. To satisfy this criterion, a load must pass point f (or corresponding points in other panels).

Having determined the proper load position from Eq. (3), the stress in the chord member may be calculated from Eqs. (1) or (2).

Illustrative Problem.—Calculate the stresses in member fg of the truss shown in Fig. 57 due to a dead load of 1,200 lb. per ft. per truss; a uniform live load of 3,000 lb. per ft. per truss; and for E-50 train loading.

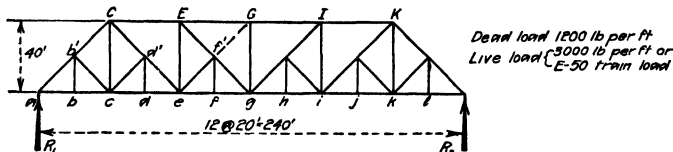



Fig. 57.

Dead Load Stress.—Panel load = (1,200)(20) = 24,000 lb. Use Eq. (1), p. 291.

$$\text{Stress in } fg = \frac{M_E + 2M_1}{h}$$

$R_1 = (^{11}/_2)(24{,}000) = 132{,}000$ lb. Moment center at E.
$M_E = (1{,}000)[(132)(4) - 24(1 + 2 + 3)](20) = 7{,}680{,}000$ ft.-lb.
M_1 = moment at center of 40 ft. span under a center load of 24,000 lb. = $(\frac{1}{2})$ (24,000)(20) = 240,000 ft.-lb.

$$\text{Stress in } fg = \frac{7{,}680{,}000 + 480{,}000}{40} = 204{,}000 \text{ lb., tension.}$$

Live Load Stress, Uniform Loading.—Live load stress due to uniform loading may be determined by ratio from dead load. Live panel load = (3,000)(20) = 60,000 lb.

$$\text{Stress in } fg = (^{60}/_{24})(204{,}000) = 510{,}000 \text{ lb., tension.}$$

Live Load Stress, E-50 Train Load.—Criterion $\frac{G}{3} - (G_1 + G_2 - G_3) = 0$ (Eq. (3)

Fig. 58.

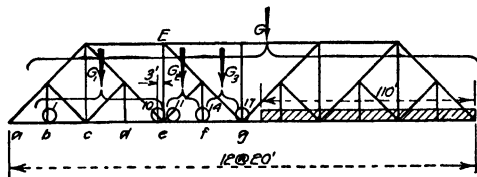
with a = 4 panels = 80 ft.). Try wheel 14 at f, Fig. 57. Load position shown in Fig. 58. $G = 630$; $G_1 + G_2$ = loads 1 to 13 or 1 to 14 = 265 to 290; G_3 = loads 14 to 16 or 15 to 17 = 57.5 to 48.75. From Eq. (3)

Load 14 to right of f

$$^{630}/_3 - (265 - 57.5) = +$$

Load 14 to left of f

$$^{630}/_3 - (290 - 48.75) = -$$

Wheel 14 satisfies the criterion. Wheel 13 also satisfies the criterion, but wheels 12 and 15 do not.

Stress in fg is given by Eq. (1). For the load position shown in Fig. 58

$$M_E = [20{,}455 + (355)(110) + (\tfrac{1}{2})(2.5)(110)^2] - 5{,}790 + (190)(3)$$
$$= 18{,}516.7 \text{ thousand ft.-lb.}$$

and M_1 = moment at f in beam of span, eg = ½(3,026.25) − 750 = 763.125 thousand ft.-lb.

$$\text{Stress in } fg = 1{,}000\frac{18{,}516.7 + 2(763.125)}{40} = 517{,}070 \text{ lb. tension.}$$

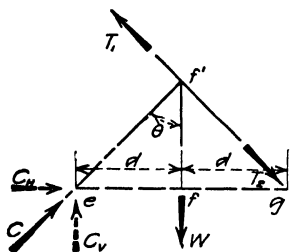
When wheel 13 is placed at f the stress in fg is 501,020 lb. tension. Hence, wheel 14 at f gives the maximum stress.

STRESSES IN WEB MEMBERS, TRUSS WITHOUT COUNTERS

It will be assumed first that all members are designed for both tension and compression. On p. 299 the effect of counters will be considered.

Stresses in diagonal web members ab', $d'e$, $f'g$, and $h'i$ of Fig. 56a are determined by the same methods as used for the Pratt truss. Thus for member $f'g$, cut section 1-1 and determine the combined shears in panel fg.

Fig. 59.

Members bb', dd', etc. are hangers which support the joint loads at the lower chord joints. For concentrated live load systems, the stress in these hangers is equal to the floorbeam reaction as calculated from Eqs. (3) or (4), p. 143, or as given directly by the table on p. 144.

Stresses in Sub-struts.- -The stress in a sub-strut, as for example $f'e$, may be determined by considering the forces acting on joint f'. Figure 59 shows joint f' and the system of concurrent forces which acts at f'. Let W = joint load at f, and C = stress in sub-strut, assumed as compression. Divide C into its vertical and horizontal components and assume these component forces applied at joint e. On taking moments about g, we find that

$$C_V = \tfrac{1}{2}W \tag{4}$$

That is: Stress in a sub-strut is a compression whose vertical component is one-half the joint load at the lower end of the adjacent hanger. The stress in the sub-strut is $C_V \sec \theta$, where θ = angle between sub-strut and the vertical. Hence the stress in the sub-strut depends on the stress in the adjacent hanger. Stresses in members $b'c$, cd', and ef' are all equal.

Illustrative Problem.—Determine the maximum and minimum stresses in sub-strut ef' of Fig. 57. Loading conditions as given on Fig. 57.

Dead joint load = (1,200)(20) = 24,000 lb.

Stress in ef' = (½)(24,000) sec 45° = 16,970 lb. compression. The uniform live load stress in ef' may be determined by ratio of dead to live load. Thus

$$\text{Stress in } ef' = \left(\frac{3{,}000}{1{,}200}\right)(16{,}970) = 42{,}430 \text{ lb. compression.}$$

For E-50 train loading, the joint load at f, as given by the table on p. 144, is 81,900 lb. Hence

Stress in ef' = $(\frac{1}{2})(81{,}900)(1.414)$ = 57,900 lb. compression.

For dead load and uniform live load, maximum stress = 16,970 + 42,430 = 59,400 lb. compression; minimum stress = 16,970 lb. compression.

For dead load and E-50 train load, maximum stress = 16,970 + 57,900 = 74,870 lb. compression; minimum stress = 16,970 lb. compression.

Stress in Upper Portion of Diagonal Web Members.—To determine the stress in member Ef', Fig. 60*a*, cut a section 1–1. Remove the portion of the structure to the left of this section as shown in Fig. 60*b*. Let $V = R_1 - \Sigma W$ = shear on section 1–1; T = stress in Ef'; C = stress in sub-

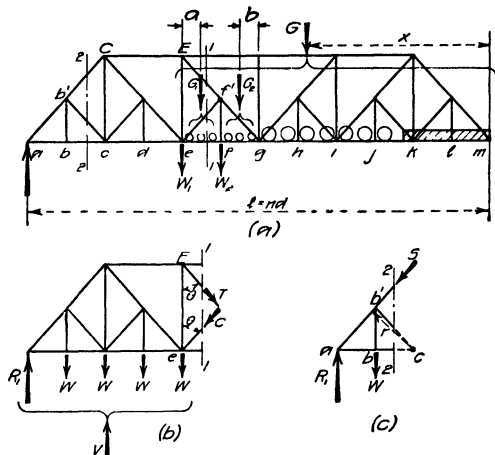



Fig. 60.

strut ef'; and T_V and C_V = vertical components of these stresses. From Fig. 60*b*

$$T_V = V - C_V \tag{5}$$

That is: Vertical component of stress in Ef' = shear on section 1-1 minus vertical component of stress in the sub-strut cut by section 1-1. In Eq. (5) V and C_V must be determined subject to the given loading conditions.

Maximum tension in Ef' under uniform live load occurs when joint f, Fig. 60*a*, and all joints to the right are fully loaded (conventional method of loading, see p. 93). It is quite evident that joints g to l should be loaded. To show that joint f should also be loaded, consider the effect of a single joint load W at f. For the 12-panel truss of Fig. 60*a*, $V = R_1 = \frac{7}{12}W$. From Eq. (4), $C_V = \frac{1}{2}W$. Hence

$$T_V = +\frac{7}{12}W - \frac{1}{2}W = +\frac{1}{12}W$$

A joint load at f increases the tension in Ef'. In the same manner, it can be shown that for minimum stress in Ef', joints b to e should be loaded.

Illustrative Problem.—Determine the maximum and minimum stress in Ef' of Fig. 57 (no counters) due to a dead load of 1,200 lb. per ft. and a uniform live load of 3,000 lb. per ft.

Dead panel load = (1,200)(20) = 24,000 lb.

Live panel load = (3,000)(20) = 60,000 lb.

Dead load shear, panel $ef = R_1 - (4)(24,000) = (1\frac{1}{2})(24,000) - (4)(24,000) =$ 36,000 lb. For dead load $C_V = (\frac{1}{2})(24,000) = 12,000$ lb. From Eq. (5)

$$T_V = 36,000 - 12,000 = 24,000 \text{ lb}$$

Dead load stress $= T_V \sec 45° = (24,000)(1.414) = 33,940$ lb. tension.

For maximum live load tension in Ef', load joints f to l, Fig. 57.

$$V = R_1 = \frac{60,000}{12}(1 + 2 + 3 + 4 + 5 + 6 + 7) = 140,000 \text{ lb.}$$

$C_V = (\frac{1}{2})(60,000) = 30,000$; $T_V = 140,000 - 30,000 = 110,000$. Then $T =$ (110,000) (1.414) = 155,500 lb. tension.

For maximum live load compression in Ef' (minimum stress condition) load joints b to e. $V = -R_2 = -\frac{60,000}{12}(1 + 2 + 3 + 4) = -50,000$ lb. $C_V = 0$ (no load at f); $T = (-50,000)(1.414) = 70,700$ lb. compression.

Maximum stress in $Ef' = 33,940 + 155,500 = 189,440$ lb. tension.

Minimum stress in $Ef' = 33,940 - 70,700 = 36,760$ lb. compression. For minimum stress in Ef' when counter $f'G$ is in action, see problem on p. 300.

To determine the position of a concentrated load system for maximum stress in Ef', divide the load into groups as shown in Fig. 60a. Let W_1 and W_2 = panel concentrations at e and f. For the conditions shown

$$T_V = V - C_V = R_1 - W_1 - \tfrac{1}{2}W_2 \tag{6}$$

In terms of the several load groups,

$$T_V = \frac{Gx}{nd} - G_1\frac{(d - a)}{d} - \frac{1}{2d}(G_1a + G_2b)$$

On equating to zero the change in this expression due to a small forward movement Δ of the loads, we derive

$$\frac{G}{n} - \frac{1}{2}(G_1 + G_2) = 0 \tag{7}$$

which is the criterion for position of loads for maximum stress in Ef'. To satisfy this criterion, a load must pass point g. Note that this is similar to the criterion for shear in panel eg of a 6-panel Pratt truss without the subdivided panels.

A simplified expression for T_V may be derived by letting M_f and M_g = the moments respectively about f and g of the loads in front of these

points. Then $W_1 = \frac{M_f}{d}$ and from Eq. (2), p. 139, $W_2 = \frac{1}{d}(-2M_f + M_g)$. On substituting these values in Eq. (6) we derive

$$T_V = R_1 - \frac{M_g}{2d} \tag{8}$$

It can be shown that the criterion of Eq. (7) and Eq. (8) apply also to minimum live load stress in Ef'.

Illustrative Problem.—Calculate the maximum live load tension in Ef' of Fig. 57 due to E-50 train loading.

The criterion of Eq. (7) may be written $\frac{G}{6} - (G_1 + G_2) = 0$ where $G_1 + G_2$ = load in panel *eg*.

Try wheel 4 at g. The uniform load covers 29 ft. of the right end of the truss and $G = 355 + (2.5)(29) = 427.5$. $\frac{G}{6} = 71.25$. $G_1 + G_2 = 62.5$ to 87.5.

On substituting in Eq. (7), we have

For 4 to right of g

$$71.25 - 62.5 = +$$

For 4 to left of g

$$71.25 - 87.5 = -$$

Wheel 4 satisfies the criterion. Wheels 3 and 5 were tried but did not answer.

For wheel 4 at g, R_1 =

$$\tfrac{1}{240}[20{,}455 + (355)(29) + (\tfrac{1}{2})(2.5)(29)^2] = 132.5,$$

and $M_g = 600$. From Eq. (8)

$$T_V = 132.5 - \frac{600}{(2)(20)} = 117.5 = 117{,}500 \text{ lb.}$$

Hence

$$T = T_V \sec 45 \text{ deg.} = 166{,}200 \text{ lb. tension.}$$

Stress in Interior Verticals.—The stress in a vertical, such as Ee, depends upon the stress in the diagonal Ef' entering at joint E. Stress in Ee = vertical component of stress in Ef' plus joint load at E.

Stress in Upper End of End Post.—To determine the stress in $b'C$ of Fig. 60*a* cut a section 2-2. Remove the portion of the structure to the left of this section as shown in Fig. 60*c*. Let S = stress in Cb'. The value of S may be determined either by summation of vertical forces on section 2-2, or by moments about c, the intersection of $b'c$ and bc. Since the latter method is the simpler, it will be followed.

From moments about c

$$S = \frac{M_c}{r}$$

where M_c = moment at c and r = distance from c to line of action of S. Since the stress in Cb' depends upon moment at c, load positions are determined as for moment at point c in a span of length l.

Illustrative Problem.—Calculate the stress in member Cb' of Fig. 57 due to dead and live load.

Dead Load Stress.—Panel load = 24,000 lb. R_1 = 132,000 lb. M_c = (1,000) [(132)(2) − (24)(1)]20 = 4,800,000 ft.-lb. Lever arm of Cb' = (20)(sec 45 deg.) = 28.28′. Stress in $Cb' = \frac{4,800,000}{28.28}$ = 169,600 lb. compression.

Live Load Stress Uniform Live Load.—Live panel load = 60,000 lb. By ratio from panel load

$$\text{Stress in } Cb' = (169,600)(^{60}/_{24}) = 424,000 \text{ lb.}$$

Live Load Stress, E-50 Train Load.—From Fig. 140, p. 113, wheel 5 at c gives maximum moment, which is found to be 11,824,200 ft.-lb. Then

$$S = \frac{11,824,200}{28.28} = 418,500 \text{ lb. compression.}$$

Maximum and minimum stresses may be found in the usual manner.

Stress in Vertical Cc.—The stress in vertical Cc of Fig. 61a may be determined by a consideration of the system of concurrent forces acting

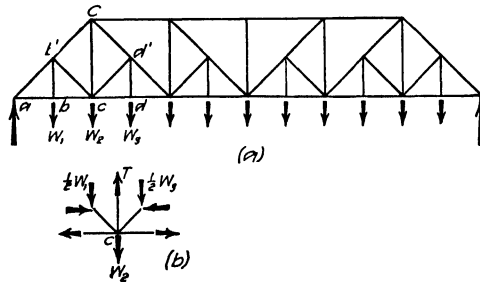



Fig. 61.

at joint c. Let W_1, W_2, and W_3 = joint loads respectively at b, c and d. Figure 61b shows joint c removed with all forces acting. The stresses in $b'c$ and $d'c$ have been resolved into their vertical and horizontal components. As shown on p. 294, the vertical components of these stresses are equal to one-half the joint loads at b and d, as indicated in Fig. 61b.

Let T = stress in vertical Cc. For the conditions shown in Fig. 61b

$$T = \tfrac{1}{2}W_1 + W_2 + \tfrac{1}{2}W_3 \tag{9}$$

It can readily be seen that loads on other parts of the truss have no effect on the value of T.

When the applied load is uniform and the panels are of equal length $W_1 = W_2 = W_3 = W$, and

$$T = 2W \tag{10}$$

For a concentrated load system, the several loads are in general not equal. Let M_1, M_2, M_3 and M_4 represent respectively the moments at b, c, d,

and e, Fig. 61a, due to the loads to the left of these points, and let d = the common panel length. Then from Eq. (2), p. 141: $W_1 = \frac{-2M_1 + M_2}{d}$; $W_2 = \frac{M_1 - 2M_2 + M_3}{d}$; and $W_3 = \frac{M_2 - 2M_3 + M_4}{d}$.

On substituting these values in Eq. (9) and reducing, we derive

$$T = \frac{\frac{1}{2}M_4 - M_2}{d} \tag{11}$$

But from Fig. 61a it can readily be seen that $(\frac{1}{2}M_4 - M_2)$ is the center moment for a beam four panels in length. Let M_c represent this moment. Then

$$T = \frac{M_c}{d} \tag{12}$$

where T = stress in vertical Cc; M_c = center moment for a beam four panels in length; and d = common panel length.

The stress in Cc will be a maximum when M_c has its maximum value. The proper load position may be determined as for a simple beam of the same dimensions.

Illustrative Problem.—Determine the stress in member Cc of Fig. 57 for the given loading conditions.

Dead Load Stress.—Panel load = 24,000 lb. From Eq. (10)

$$T = 2W = 18,000 \text{ lb. tension.}$$

Live Load Stress Uniform Load—Panel load = 60,000 lb.

$$T = 2W = 120,000 \text{ lb. tension.}$$

Live Load Stress, E-50 Train Loading.—Use Eq. (12). M_c = center moment in a beam four panels in length = (4)(20) = 80 ft. From Fig. 142, p. 115, wheel 13 at the center of an 80-ft. beam gives maximum moment, which is found to be 2,699, 375 ft.-lb. From Eq. (12), with d = 20

$$T = \frac{2,699,375}{20} = 132,968.75 \text{ lb. tension.}$$

STRESSES IN WEB MEMBERS. TRUSS WITH COUNTERS

To prevent reversal of stress in a main member, as $f'g$ of Fig. 57, a counter, shown by the dotted line $f'G$, is placed in the panel. Assuming that members $f'g$ and $f'G$ carry tension only, $f'g$ is in action when the combined shear in panel fg is positive, and $f'G$ is in action when the combined shear is negative. These combined shears and the resulting stresses in $f'g$ and $f'G$ are determined by the methods used for the Pratt truss.

The presence of counter $f'G$ effects only the minimum stresses in the web members in panels ef and fg. Maximum stresses in these members are the same as for the truss without counters. The determination of these minimum stresses for the several members will now be considered in detail.

Member Ef'.—Figure 62 shows the form of the truss for panels *ef* and *fg* when the shear in panel *fg* is negative and counter *f'G* is in action. By an analysis similar to that used for stress in the sub-struts (p. 294), it can be shown that the stress in *Ef'* is a tension whose vertical component is equal to one-half the joint load at *f*.

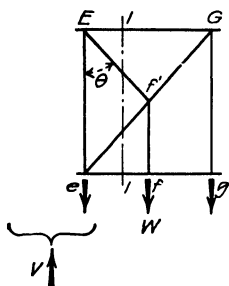



Fig. 62.

The minimum stress in *Ef'* is $\frac{1}{2} W \sec \theta$, tension, where W is a dead joint load. This stress occurs when the live load is so placed as to cause negative shear in panel *fg*, dead joint load only at joint *f*. The maximum stress in *Ef'* is also tension; it occurs for the loading conditions causing maximum positive shear in panel *ef*, as discussed on p. 295.

Member ef'.—Let T_V and C_V respectively denote the vertical components of stress in *ef'* and *Ef'*, assumed to be tension, and let V = shear on section 1-1 Fig. 62, assumed as a positive shear. From summation of vertical forces on section 1-1, we have

$$T_V = -V + C_V \tag{13}$$

A plus sign in Eq. (13) indicates tension in *ef'*.

Under uniform loading, member *ef'* has maximum tension when live panel loads are placed at joints from the left end of the truss up to panel *e*. If a joint load be placed at *f*, shear in panel *ef* (12 panel truss Fig. 57) $= +\frac{7}{12}W$ and C_V (for member *Ef'*) $= +\frac{1}{2}W$. From Eq. (13) $T_V = -\frac{1}{12}W$, indicating that the load at *f* causes compression in *ef'*. Hence no load should be placed at *f*.

Illustrative Problem.—Determine the maximum and minimum stress in *ef'* of Fig. 57 for the given uniform load conditions, assuming the counter *f'G* in position.

The maximum stress in *ef'* is 59,400 lb. compression, as calculated on p. 294. For minimum stress in *ef'*, load joints *b* to *e* of Fig. 57 with live panel loads of 60,000 lb. each. Shear in panel $ef = -R_2 = -\frac{60{,}000}{12}(1 + 2 + 3 + 4) = -50{,}000$ lb

From Eq. (13), with $C_V = 0$ (no live load at joint *f*)

$$\begin{aligned} \text{Stress in } ef' &= T_V \sec 45 \text{ deg} \\ &= -(-50{,}000)(1.414) = 70{,}700 \text{ lb., tension.} \end{aligned}$$

The dead load stress in *ef'* must be determined for the form of truss shown in Fig. 62. With dead panel loads of 24,000 lb. at each joint, shear in panel $ef = 24{,}000(1\frac{1}{2} - 4) = +36{,}000$ lb. and C_V for member $Ef' = (\frac{1}{2})(24{,}000) = 12{,}000$ lb. From Eq. (13)

$$\begin{aligned} T = T_V \sec 45 \text{ deg.} &= (-36{,}000 + 12{,}000)1.414 \\ &= 33{,}940 \text{ lb. compression.} \end{aligned}$$

Hence, minimum stress in *ef'*

$$= +70{,}700 - 33{,}940 = 36{,}760 \text{ lb. tension.}$$

For E-50 train loading, it can be shown by an analysis similar to the one used on p. 296 for load position for *Ef'*, that the criterion for load posi-

tion for maximum tension in ef' is the same as given by Eq. (7), p. 296. To satisfy the criterion for member ef', the train is to be headed to the right and some wheel must pass point e, the left end of panel ef, Fig. 57.

The stress in ef' is given by the equation

$$T_V = R_2 - \frac{M_e}{2d}$$

where M_e = moment of loads in panels efg about e. This equation is derived by the same methods as used on p. 297 for Eq. (8).

Illustrative Problem.—Determine the maximum and minimum stress in ef', Fig. 57, due to dead load and E-50 train loading. Assume counter $f'G$ in action.

As given above, the dead load stress in ef' = 33,940 lb. compression. On substituting in the criterion for load position (Eq. (7), p. 296), it was found that wheel 3 was the only one which answered. From the above equation,

Stress in $ef' = T_V$ sec 45 deg. = 54,200 lb. tension.

The minimum stress in $ef' = -33,940 + 54,200 = 20,260$ lb. tension.

Stresses in the verticals Ee and Gg are equal to the vertical components of stress in Ef' or Gf', modified by the presence of any joint load which may exist at E or G.

15b. The Baltimore Truss with Sub-ties.—Figure 63 shows a 12-panel Baltimore truss with sub-ties. The position of counters is shown by the dotted members ef' and $h'i$.

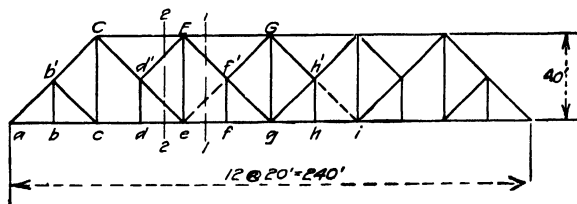



FIG 63.

General methods for the determination of stresses in the members of the truss of Fig. 63 are quite similar to those explained in Art. **15a** for the truss with sub-struts. In the present article, special attention will be given to members for which the methods of stress analysis differ from those given in the preceding articles. Wherever possible, reference will be made to existing formulas.

Stresses in lower chord members are determined for moment centers at top chord points as in the Pratt truss. For stress in any top chord member, as EG, Fig. 63, cut a section 1-1 (Ef' in action in panel ef) and take point g as the moment center. By the methods used on p. 291 for stress in fg, it can be shown that

$$\text{Stress in } EG = \frac{M_g + 2M_1}{h}$$

where M_g = moment at g due to applied loading, M_1 = moment at f in a beam of span eg due to loads in panels ef and fg; and h = height of

truss. For member CE, moment center is at e and M_1 = moment at d in beam ce.

Maximum live load stress for uniform live load occurs when the truss is fully loaded. The criterion for load position for maximum stress in a top chord member for E-50 train loading is

$$\frac{Ga}{l} - (G_1 + 2G_2) = 0 \tag{14}$$

For member EG, Fig. 63, a = distance from left end of span to moment center at g; l = span length, G = total load on span; G_1 = load to left of joint e; and G_2 = load in panel ef. To satisfy this criterion some load must be placed at point f. Stress calculations for a top chord member are practically the same as given in the problem on p. 293.

Stresses in ab', $b'C$, and $b'c$ are calculated as for the Baltimore truss with sub-struts.

The stress in $d'E$ is a tension whose vertical component is equal to one-half the joint load at d. This result is obtained by the same general method as used for sub-struts on p. 294. Maximum stress in $d'E$ occurs under dead and live joint loads at d; minimum stress under dead load only.

Maximum and minimum stresses in member Cd' are determined from the combined shears in panel cd. These shears are calculated by the methods used for a corresponding panel in a 12-panel Pratt truss.

Stress in member $d'e$ is determined from a summation of vertical forces on section 2-2, Fig. 63. If T_V and C_V respectively denote vertical components of stress in $d'e$ and $d'E$, and V = shear on section 2-2, we have

$$T_V = V + C_V$$

For uniform train loading, it can be shown by means of the above equation that the maximum stress in $d'e$ occurs when the truss is loaded from d to the right end, and the minimum stress occurs when b and c are loaded. For E-50 train loading, the criterion for load position for maximum stress is the same as Eq. (7), p. 296, and the value of T_v is given by Eq. (8), p. 297. For the member under consideration, M_e, the moment of the loads to the left of joint e replaces M_g of Eq. (8).

Maximum and minimum stresses in the web members of panels ef and fg are affected by the presence of the counter ef'. Maximum stress in Ef' occurs when the combined shear in panel ef is positive. For uniform loading joint f and all joints to the right are loaded. For E-50 train loading, the shear is determined as for the corresponding panel in a 12-panel Pratt truss. Member ef', the counter, is inactive when the shear in panel ef is positive, and its stress is zero, its minimum value. The minimum stress in Ef' occurs when the counter ef' is in action, shear on section 1-1 negative. When ef' is in action, the form of truss for panels ef and fg is the same as shown in Fig. 62. The stress in Ef' is

then a tension whose vertical component is equal to one-half the joint load at f. For minimum stress in Ef', a dead joint load only should be placed at f. Maximum tension in ef' is determined by the same methods given in p. 296 for ef' of Fig. 62.

The maximum stress in $f'G$, Fig. 63, is a tension which occurs when the counter ef' is in action, form of truss as shown by Fig. 62. Loading conditions for maximum negative combined shear in panel fg prevail. Minimum stress in $f'G$ occurs when the counter ef' is not in action; member $f'G$ is then acting as a sub-tie and its stress is determined as for $d'E$.

Maximum stress in $f'g$, Fig. 63, occurs when the shear in panel fg is positive. Methods of calculation and load position for maximum value are determined as for member $d'e$. The minimum stress in $f'g$ is zero; it occurs when the counter ef' is in action, form of truss as shown in Fig. 62.

Stress in the vertical Cc depends upon the joint loads at b and c, Fig. 63. Let W_1 and W_2 respectively represent these joint loads, and let T = stress in Cc. On considering the system of concurrent forces acting on joint c, taken as a free body, we have from summation of vertical forces,

$$T = \tfrac{1}{2}W_1 + W_2$$

For stress in Cc due to dead load, or uniform live load, full panel loads should be placed at b and c. If W = panel load, $T = \tfrac{3}{2}W$. For E-50 train loading, the panel concentrations at b and c may be determined from Eq. (2), p. 111. If M_1, M_2 and M_3 denote respectively the moments at b, c and d, Fig. 64, due to the loads to the left of these points, it can readily be shown that

$$T = \frac{3}{2d}\left(\frac{2}{3}M_3 - M_2\right) = \frac{3}{2}\frac{M_c}{d}$$

where M_c = moment at point c in a beam of span ad (three panels of d each). For maximum stress in Cc, the train load is to be placed in position for maximum moment at c in the span ad. The minimum stress is due to the dead load only.

The stress in Ee depends upon the vertical components of stress in $d'E$ and Ef' and that portion of the dead load which is applied at E. This latter load will be neglected for the present. The vertical component of stress in $d'E = \tfrac{1}{2}W_d$, where W_d = joint load at d, and the vertical component of stress in $Ef' = V_{ef}$ = shear in panel ef. If C = stress in Ee, we have

$$C = \tfrac{1}{2}W_d + V_{ef} \text{ (compression)}$$

The proper position of a uniform live load for maximum stress in Ee must be determined by trial. Two load positions are possible: First, joint f and all points to the right are loaded. Then $V_{ef} = {}^{28}\!/_{12}W$,

where W = a joint load, and $W_d = 0$. From the above equation, $C = {}^{28}/_{12}W$. Second, joint d and all points to the right are loaded. Then $V_{ef} = {}^{45}/_{12}W - 2W = {}^{21}/_{12}W$, and $W_d = \frac{W}{2}$. From the above equation, $C = {}^{21}/_{12}W + {}^{6}/_{12}W = {}^{27}/_{12}W$. The former load position gives the greater stress. (For a 16-panel truss, the second load position gives maximum stress.) For E-50 train loading two load positions are possible for maximum stress in Ee. First, shear in panel ef is maximum, no load to left of joint e. The criterion for position of loads is the usual criterion for shear in any panel of a truss. Second, the load extends into panel cd causing a joint load at d. The criterion for load position can be shown to be

$$\frac{G}{n} - \left(\frac{G_1}{2} - \frac{G_2}{2} + G_3\right) = 0$$

where G = total load on truss; n = number of panels; G_1 = load in panel cd; G_2 = load in panel de; and G_3 = load in panel ef. To satisfy this criterion some load must pass joint d or joint f.

For the truss of Fig. 63 it was found that the maximum stress in Ee occurred when wheel 3 was placed at f. The resulting stress was 146,100 lb. tension.

The minimum stress in Ee will occur when loads are applied in positions which will stress the counter ef' in panel ef without placing a live joint load at d. The stress in Ee is then due to the dead load stresses in $d'E$ and Ef' = a dead joint load. When a portion of the dead load is applied at E, the stress in Ee is found from the above equation for C by adding to the result a compression equal to the load moved to the top chord, and noting that the load W_d is reduced in value.

Maximum and minimum stresses in Gg, Fig. 63, are determined by similar methods.

15c. The Pennsylvania Truss.—Figure 64 shows a Pennsylvania or Pettit truss. The dotted horizontal and vertical members at joints d' and f' act as supports for the top chord and web members. They have no definite direct stresses but may be subjected to secondary stresses due to the deformation of the truss as a whole. The dotted member ef' is a counter.

General methods of stress analysis for the truss of Fig. 64 are based on a combination of the methods used in Art. **15b** for the Baltimore truss with sub-ties and those given in Arts. **9, 10** and **11** for the curved chord Pratt truss.

The arrangement of members in the two panels at the end of the truss of Fig. 64 is exactly the same as for the truss of Fig. 45. Methods of analysis are the same as given in Art. **13** for corresponding members.

Stresses in chord members are determined by the same methods as used in Art. **15b** for the Baltimore truss with sub-ties. The criterion

for load position for maximum stress in top chord members is given by Eq. (14), p. 302. In calculating top chord stresses, it will be found convenient to resolve the stress in chord members into vertical and horizontal components. The procedure is then as explained in Art. **13** for the curved chord Pratt truss.

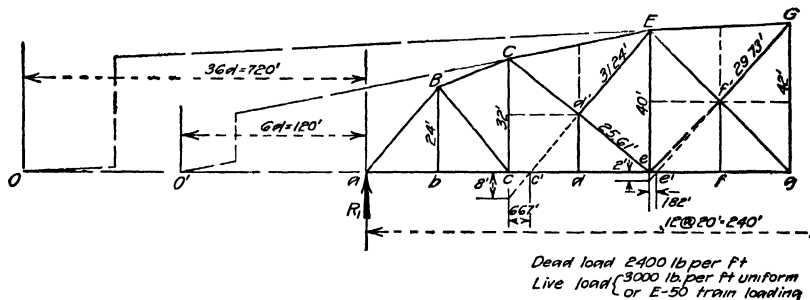



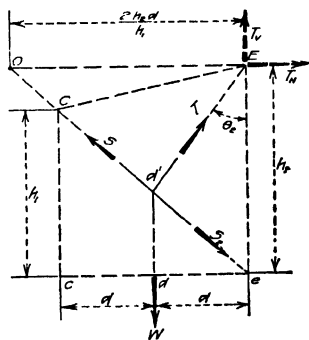
Fig. 64.

Stress in Cd'.—Maximum and minimum stress in Cd' of Fig. 64 is determined by the methods used in Art. **13** for the calculation of stress in diagonal Bc of Fig. 45. Combined shears are determined for panel cd of Fig. 64. For moving loads, joint d and all joints to the right are fully loaded for maximum tension. For maximum compression, load joints b and c. The stress in Cd' due to these loads may be determined by means of Eq. (17), p. 265, with O' as a moment center. For train loading, the criterion for load position for maximum stress and the methods of calculation are the same as given on p. 276 for member Bc of Fig. 45.

Fig. 65.

Stress in d'E.—Member $d'E$ of Fig. 64 is a sub-tie. The stress in $d'E$ may be determined from Fig. 65, which shows the system of concurrent forces acting at joint d'. Let W = load at joint d, and let T = stress in $d'E$. Resolve T into its vertical and horizontal components, and assume these components to be applied at joint E. To determine T_V, take moments about O, the intersection of the line of action of S_1 and S_2 with a horizontal through E. From similar triangles it can readily be shown that the horizontal distance from O to the line of action of T_V is $\frac{2h_2d}{h_1}$. From moments about O, we have

$$T_V = \frac{W}{2h_2}(2h_2 - h_1) \tag{1}$$

The stress in $d'E$ is then $T = T_V \sec \theta_2$.

In some cases the load W is divided between joints d, d', and a top chord joint directly over d'. The value of W in Eq. (1) is to be taken as the sum of the three loads.

Maximum stress in Cd' occurs when dead and live joint loads are applied at d. Minimum stress occurs when only a dead joint load is applied at d. For train loading, load W is equal to the floorbeam reaction at d for a beam ce.

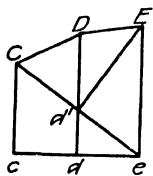



Fig. 66.

If the top chord is not straight between joints C and E, but is curved, as shown by CDE of Fig. 66, the stress in Dd' due to the top chord stresses must be determined and added to W, proper attention being paid to the character of stress in Dd'. Methods for determining the stress in Dd' are given on p. 270.

Illustrative Problem.—Determine the maximum and minimum stress in the sub-tie $d'E$ of Fig. 64 due to the loadings given on the figure.

Dead Load Stress.—Dead panel load = (2,400)(20) = 48,000 lb. From Eq. (1) with $h_2 = 40$ ft., $h_1 = 32$ ft., and $\sec\theta_2 = \frac{31.24}{40 - (1/2)(32)} = 1.30$ (see Figs. 64 and 65), we have

$$T_V = \frac{W}{80}(80 - 32) = 0.6W = 28{,}800 \text{ lb.}$$

and

$$T = 1.30T_V = (1.30)(28{,}800) = 37{,}400 \text{ lb. tension}.$$

Uniform Live Loading.—$W = (3{,}000)(20) = 60{,}000$ lb.

$$T = (0.6)(60{,}000)(1.3) = 46{,}800 \text{ lb. tension.}$$

Train Loading.—From the table on p. 144 the floorbeam concentration at d = 81,900 lb.

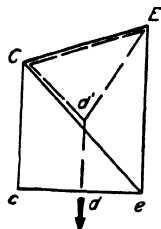
Then

$$T = (0.6)(81{,}900)(1.3) = 64{,}900 \text{ lb. tension.}$$

Maximum and Minimum Stress.—Minimum stress = dead load stress = 37,400 lb. tension. Maximum stress, uniform live load = 37,400 + 46,800 = 84,200 lb. tension. Maximum stress, train loading = 37,400 + 64,900 = 102,300 lb. tension.

Stress in $d'e$.—The stress in $d'e$ may be determined by cutting a vertical section through panel de of Fig. 64 and taking moments about O'. To simplify the resulting moment equations, resolve the stresses in $d'E$ and $d'e$ into vertical and horizontal components applied at c' and e respectively.

Fig. 67.

The stress in $d'e$ may also be determined by considering the framing of panel ce, Fig. 64, to be composed of the auxiliary truss $Cd'E$, shown by dotted lines in Fig. 67, and the ordinary curved chord Pratt truss framing $CEce$, as shown by the full lines. It is evident from Fig. 67 that the load at d is transferred to C and E by the auxiliary truss. The stress in $d'e$ will remain unchanged if the load

at d is assumed as transferred to c and e by the beam cde. Hence, to determine the stress in $d'e$, transfer the panel load at d to joints c and e, placing at these points loads which are equal to the reactions at c and e for the load at d. Then calculate the stress in Ce of the curved chord Pratt truss formed by removing the auxiliary frame $Cd'E$ from Fig. 64.

It can readily be shown that for maximum tension in $d'e$ due to uniform live load (conventional method of loading), joint d and all joints to the right should be fully loaded. For minimum live load stress due to uniform live loading, joints c and all joints to the left are to be fully loaded. For train loading, the criteria for load position for maximum and minimum stress are the same as used in the problem of Art. **13** for the diagonal web members.

Illustrative Problem.—Calculate the maximum and minimum stress in member $d'e$ of Fig. 64 for the loadings given on the figure.

Dead Load Stress.—Dead panel load = 48,000 lb. $R_1 = (11/2)(48) = 264$. Distribute load at d to joints c and e. Hence panel loads at c and e are $(3/2)(48) = 72$. All other joint loads = 48. Shear in panel $ce = 264 - 48 - 72 = 144$. Use Eqs. (12), (13), and (15), p. 263. From Eqs. (12) and (15), in which M_a = moment of loads at b and c about a; t = distance in panel lengths from O' to c; and s = distance in panel lengths from O' to a, we have

$$S_V = \frac{(144)(6) - (48)(1) - (72)(2)}{10} = 67.2$$

If θ = angle between $d'e$ and the vertical, $\sec\theta = \dfrac{25.61}{16} = 1.6$.

From Eq. (13)

$$S = (67{,}200)(1.6) = 107{,}500 \text{ lb. tension.}$$

which is the dead load stress in $d'e$.

Live Load Stress, Uniform Loading.—Live panel load = 60,000 lb. For maximum live load tension in $d'e$, load joint d and all joints to the right. On distributing the load at d to joints c and e, joint load at $c = 30$; joint load at $e = 90$; all other joint loads = 60. Hence $R_1 = 225$. Shear in panel $ce = 225 - 30 = 195$. $M_a = (30)(2) = 60$. Then

$$S_V = \frac{(195)(6) - 60}{10} = 111$$

and

$$S = (111{,}000)(1.6) = 177{,}500 \text{ lb. tension.}$$

For maximum live load compression in $d'e$, load joints b and c with panel loads of 60 $R_1 = 105$. Shear in panel $ce = 105 - 120 = -15$. $M_a = (60)(1 + 2) = 180$. Then

$$S_V = \frac{(-15)(6) - 180}{10} = -27$$

and

$$S = (27{,}000)(1.6) = 43{,}200 \text{ lb. compression.}$$

Live Load Stress E-50 Train Loading.—The criterion for position of loads for maximum live load tension in $d'e$ is given by Eq. (22), p. 268, with $m = 2$, $k = 3$, and $n = 6$, from which $\frac{G}{8} - G_2 = 0$. Wheel 4 at e, train headed to the left, answers the criterion. $R_1 = 216.5$. Load at d due to loads in panel $cde = 30$. Hence panel load at $c = 15$. Shear in panel $ce = 216.5 - 15 = 201.5$.
Then

$$S_V = \frac{(201.5)(6) - (15)(2)}{10} = 117.9$$

and

$$S = (117{,}900)(1.6) = 188{,}400 \text{ lb. tension}$$

The criterion for maximum live load compression in $d'e$ is given by Eq. (26), p. 269, from which we have $0.3G - G_2 = 0$. Wheel 3 at c, train headed to the right, answers the criterion. Right reaction $= 21.9$, and load at d due to loads in panel $cde = 14.4$. Hence load at $e = (\frac{1}{2})(14.4) = 7.2$. Shear in panel $ce = -21.9 + 7.2 = -14.7$. To determine the value of M_a, note that for the given load position, wheel 9 is 5 ft. to the right of a, the left support, and that wheels 1 and 2 are located in panel cd. The joint load at c due to wheels 1 and 2, and to the joint load of 14.4 at d is $23.1 + 7.2 = 30.3$. Hence, M_a = moment about a of wheels 9 to 3 plus moment of load at c about a = [2,770 + (140)(5)] + (30.3)(40) = 4,690
Then

$$S_v = \frac{(-14.7)(6)(20) - 4{,}690}{(10)(20)} = -32.27$$

and

$$S = (32{,}270)(1.6) = 51{,}700 \text{ lb. compression.}$$

Maximum and Minimum Stresses.—For uniform loading: Maximum stress in $d'e$ = 107,500 + 177,500 = 285,000 lb. tension; minimum stress = 107,500 − 43,200 = 64,300 lb. tension. For E-50 train loading: maximum stress = 107,500 + 188,400 = 295,900 lb. tension; minimum stress = 107,500 − 51,700 = 55,800 lb. tension.

Maximum and minimum stresses in the members of panels ef and fg, Fig. 64, are affected by the presence of the counter ef'. Methods of analysis for the several members will now be given.

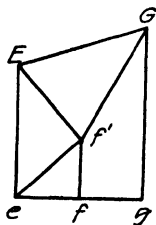



FIG. 68.

Stress in Ef'.—Maximum stress in Ef' occurs when the shear in panel ef is positive. The counter ef' is not in action and Ef' carries the shear. For uniform loading, joint f and all joints to the right are fully loaded. For train loading, the criterion of Eq. (22), p. 268 is to be used. The method of calculation is illustrated by the problem which follows.

Minimum stress in Ef' occurs when the shear in panel fg is negative. The form of truss is then as shown in Fig. 68. Member $f'g$ is not in action and Ef' acts as a sub-tie.

To determine the stress in Ef' when acting as a sub-tie, consider the system of concurrent forces acting at joint f', Fig. 69. Let T = stress in Ef'; Q = stress in $f'G$; S = stress in ef'; and W_f = joint load at f.

Figure 69 shows the relative position of the several forces. Since *ef′* and *f′G* are not in the same straight line, an expression for stress in *Ef′* will be a function of W_f and either S or Q.

To derive an expression for stress in *Ef′* in terms of W_f and Q, resolve T and Q into their vertical and horizontal components, and take moments about A, Fig. 69, a point located at the intersection of the line of action of S and the vertical Gg. We then have

$$T_V = \frac{W_f}{2} - Q_H\left(\frac{h_2 - h_1}{2d}\right)$$

But

$$Q_H = Q_V \tan \theta_1 = Q_V \frac{2d}{2h_2 - h_1}$$

Hence

$$T_V = \frac{W_f}{2} - Q_V\left(\frac{h_2 - h_1}{2h_2 - h_1}\right) \qquad (2)$$

FIG. 69.

By the methods given on p. 262 for stress in member Bc, Fig. 45, it can be shown that

$$Q_V = -\frac{Vs - M_a}{t} \qquad (3)$$

in which V = shear on section 1-1, Fig. 69 (panel *fg*, Fig. 64); s = distance from O, Fig. 64, to left reaction; M_a = moment about left reaction of loads to left of section 1-1; and t = distance from O to e', the intersection of the bottom chord and member *f′G*, Fig. 64. A positive result in Eq. (3) indicates tension; a negative result indicates compression. Placing the value of Q_V given by Eq. (3) in Eq. (2), we have

$$T_V = \frac{W_f}{2} + \frac{Vs - M_a}{t} \cdot \frac{h_2 - h_1}{2h_2 - h_1} \qquad (4)$$

Since any load placed to the left of section 1-1, Fig. 69 (panel *fg*, Fig. 64) will cause negative shear on section 1-1, it is evident that the least stress in *Ef′* due to uniform live load will occur when joint *e* and all joints to the left carry live panel loads in addition to the dead panel loads. A similar loading condition occurs for train loading. The exact position of the train loading is best determined by trial, as explained in the following problem.

Illustrative Problem.—Determine the maximum and minimum stress in member *Ef′* of Fig. 64 for the loadings shown on the figure.

Maximum Stress in Ef′ (Counter *ef′* not in action).—Dead panel load = 48. $R_1 = (1\frac{1}{2})(48) = 264$. Use Eqs. (12), (13), and (15), p. 263. Dead load shear in

panel $ef' = V = +72$. $M_a = (48)(1 + 2 + 3 + 4) = 480$. $S_V = \frac{(72)(36) - 480}{42} =$ 50.3. Member Ef' makes an angle of 45 deg. with the vertical. Hence

$$S = (50{,}300)(1.41) = 70{,}900 \text{ lb. tension}$$

which is the dead load stress in Ef'.

Panel load for uniform live load = 60. Load joints f and all joints to the right.

$$V = R_1 = {}^{60}\!/_{12}(1 + 2 + \ . \ . \ . + 7) = 140, \text{ and } M_a = 0.$$

Hence

$$S_V = \frac{(140)(36)}{42} = 120,$$

and

$$S = (120{,}000)(1.41) = 169{,}200 \text{ lb. tension.}$$

Maximum stress for dead load and uniform live load = 70,900 + 169,200 = 240,100 lb. tension.

The criterion for position of train loading is given by Eq. (22), p. 268, with $n = 12$, $m = 5$, and $k = 36$, from which $\frac{G}{13\frac{1}{3}} - G_2 = 0$. Wheels 2 and 3 at f, train headed to the left, answer the criterion. With 2 at f, $R_1 = 150.7$; load at $e = 5.0$; V = shear in panel $ef = 150.7 - 5.0 = 145.7$, and M_a = moment about a of load at e. Then

$$S_V = \frac{(145.7)(36) - (5)(4)}{42} = 124.5$$

and

$$S = (124{,}500)(1.41) = 175{,}600 \text{ lb. tension.}$$

Maximum stress due to dead load and E-50 train load = 70,900 + 175,600 = 246,500 lb. tension. Wheel 3 at f gives a somewhat smaller value.

Minimum Stress in Ef'.—As stated on p. 308, the minimum stress in Ef' occurs under dead load and live panel loads at joint e and joints to the left. Use Eq. (4), p. 305. W_f = dead joint load = 48. V = shear in panel fg. For dead load, $V = 24$, and for uniform live panel loads of 60 at joints b to e, $V = -R_2 = -{}^{60}\!/_{12}(1 + 2 + 3 + 4) = -50$. For dead load, $M_a = 48(1 + 2 + 3 + 4 + 5) = 720$, and for live load, $M_a = 60(1 + 2 + 3 + 4) = 600$. From Fig 64, $s = 36$ panels; t = distance from O to e' = 40 panels plus 1.82 ft. = 40.09 panels; $h_1 = 40$ ft.; and $h_2 = 42$ ft.

Then for dead load

$$T_V = 24 + \frac{(24)(36) - 720}{40.09} \cdot \frac{2}{44} = 24.2$$

and

$$T = (24{,}200)(1.41) = 34{,}100 \text{ lb. tension.}$$

For uniform live load

$$T_V = 0 + \frac{(-50)(36) - 600}{40.09} \cdot \frac{2}{44} = -2.7$$

and

$$T = (2{,}700)(1.41) = 3{,}800 \text{ lb. compression.}$$

Total stress in Ef', dead load and uniform live load = 34,100 − 3,800 = 30,300 lb. tension.

The position of the E-50 train loading which will give minimum stress in Ef' can be seen from Eq. (4), p. 309 to be that load position for which the shear in panel fg, Fig.

64, is negative and as large as possible, at the same time keeping the load W_f at f as small as possible. The proper load position is best determined by trial. Try wheel 1 at e, Fig. 64, train headed to the right. Shear in panel $de = -R_2 = -46.6$; $M_a = 11{,}200$; $W_f = 0$.
Then

$$T_V = 0 + \frac{(-46.6)(36)(20) - 11{,}200}{(40.09)(20)} = -5.6$$

and

$$T = (5{,}600)(1.41) = 7{,}900 \text{ lb. compression.}$$

Wheel 2 was found to give tension in Ef'. Minimum stress in Ef' due to dead load and E-50 train load = 34,100 − 7,900 = 26,200 lb. tension.

Stress in f′g.—Maximum stress in member $f'g$, Fig. 64, is determined by the same methods as used for member $d'e$. Minimum stress in $f'g$ is zero. It occurs when the shear in panel fg is negative, form of truss as shown in Fig. 68.

Stress in f′G.—Maximum stress in $f'G$, Fig. 64, occurs when the shear in panel fg is negative, form of truss as shown in Fig. 68. Equation (3), p. 309, gives the value of the vertical component of stress in $f'G$. For uniform live load, joint f and all joints to the left are to be fully loaded. For train loading, the position of loads for maximum stress is given by Eq. (26), p. 269.

Minimum stress in $f'G$ occurs under dead load, counter ef' not in action. The stress in $f'G$ may be determined from Eq. (1), p. 305.

Illustrative Problem.—Determine the maximum and minimum stress in member $f'G$, Fig. 64 for the loadings given in the figure

Maximum Stress in f′G.—Form of truss for panels ef and fg must be as shown in Fig. 68. Dead panel load = 48. Shear in panel $fg = +24$. From Eq. (3), p. 309, with $s = 36$ panels, and t = distance from O to e', Fig. 64 = 40.09 panels,

$$Q_V = -\frac{(24)(36) - (48)(1 + 2 + 3 + 4 + 5)}{40.09} = -3.59$$

Then

$$Q = (3{,}590)\left(\frac{29.73}{22}\right) = 4{,}860 \text{ lb. compression,}$$

which is the dead load stress in $f'G$ for the form of truss shown in Fig. 68.

For uniform live load, panel loads of 60 are to be placed at joints f to b. $V = -R_2 = -{}^{60}\!/_{12}(1 + 2 + 3 + 4 + 5) = -75$.
Then

$$Q_V = -\frac{(-75)(36) - (60)(1 + 2 + 3 + 4 + 5)}{40.09} = +89.9$$

and

$$Q = (89{,}900)\left(\frac{29.73}{22}\right) = 121{,}800 \text{ lb. tension.}$$

Maximum stress in $f'G$ due to dead load and uniform live load = 121,800 − 4,860 = 116,940 lb. tension.

The criterion for position of E-50 train load for maximum stress in $f'G$ is given by Eq. (26), p. 269, with $n = 12$, $m = 7$, and $k = 48$, from which $\frac{G}{10.5} - G_2 = 0$. Con-

sidering the portion of the structure to the left of panel fg, Fig. 64, it will be found that wheel 2, train headed to the right, answers the criterion. V = shear in panel fg = $-(R_2$ − joint load at $g)$ = −(83.8 − 5.0) = −78.8. M_a = moment of wheels 18 to 2 about a, plus moment about a of joint load at f due to wheel 1 = 18,750 + (7.5)(100) = 19,500. Then

$$Q_V = -\frac{(-78.8)(36)(20) - 19{,}520}{(40.09)(20)} = +95.1$$

and

$$Q = (95{,}100)\left(\frac{29.73}{22}\right) = 128{,}900 \text{ lb. tension.}$$

Maximum stress in $f'G$ due to dead load and E-50 train load = 128,900 − 4,860 = 124,040 lb. tension.

Minimum Stress in $f'G$.—Minimum stress in $f'G$ occurs under dead load only, form of truss as shown by full lines in Fig. 64. From Eq. (1), p. 305, with W = 48, h_2 = 42, and h_1 = 40,

$$T_V = \frac{(48)}{(2)(42)}(84 - 40) = 25.1$$

and

$$T = (25{,}100)\left(\frac{29.73}{22}\right) = 33{,}900 \text{ lb. tension.}$$

Stress in ef'.—Member ef' acts as a counter when the shear in panel ef of Fig. 64 is negative. The form of truss is then as shown in Fig. 68.

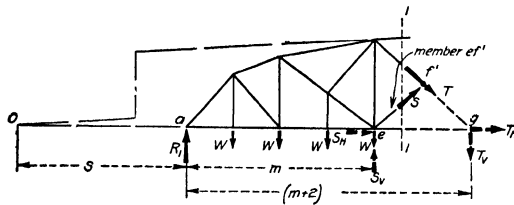



Fig. 70.

To determine the stress in ef', cut a section through panel ef of Fig. 64 and remove the portion of the structure to the left of this section, as shown in Fig. 70. Resolve S and T into horizontal and vertical components applied at joints e and g respectively. From moments about O,

$$S_V = \frac{-Vs + M_a + T_V(s + m + 2)}{s + m}$$

in which V = shear on section 1-1, assumed as positive, and M_a = moment about a of loads between section 1-1 and the left end of the truss. On taking moments about point A' of Fig. 69, it can readily be shown that

$$T_V = \frac{h_1}{2h_2}W_f - S_V\frac{(h_2 - h_1)}{h_2}$$

in which W_f = load at joint f. Substituting this value of T_V in the above equation, we derive

$$S_V = \frac{-Vs + M_a + \frac{h_1}{2h_2}(s + m + 2)W_f}{(s + m) + \frac{(h_2 - h_1)}{h_2} \cdot (s + m + 2)} \tag{5}$$

A positive result in Eq. (5) indicates tension and a negative result indicates compression. Since ef' can take tension only, a negative result in Eq. (5) indicates that the shear on section 1-1 is positive, which is contrary to the assumed conditions.

Illustrative Problem.—Calculate the maximum and minimum stress in member ef' of Fig. 64 for the loadings shown on the figure.

Minimum stress in ef' = 0, for the member is a counter.

Maximum stress in ef' may be determined from Eq. (5). For the dimensions shown on Fig. 64, Eq. (5) takes the form

$$S_V = \frac{-36V + M_a + 20W_f}{42} \tag{6}$$

Dead panel load = 48. V = shear in panel ef = +72; $M_a = 48(1 + 2 + 3 + 4) = 480$; $W_f = 48$. Then from Eq. (6)

$$S_V = \frac{-(36)(72) + 480 + (20)(48)}{42} = -27.4$$

and

$$S = (27,400)(1.41) = 38,600 \text{ lb. compression}$$

which is the dead load stress in ef', form of truss as shown in Fig. 68.

Live panel load = 60. For live load tension in ef', Eq. (6) shows that the shear in panel ef must be negative. Hence, load all joints to the left of e. To determine if a load should be placed at f, we find on substituting in Eq. (6) values for a single load at f that the stress in ef' is compression. Hence a live panel load should not be placed at f.

For live panel loads of 60 at joints b to e, Fig. 64, V = shear in panel ef = $-R_2 = -\frac{60}{12}(1 + 2 + 3 + 4) = -50$; $M_a = 60(1 + 2 + 3 + 4) = 600$; and $W_f = 0$. From Eq. (6)

$$S_V = \frac{-(36)(-50) + 600}{42} = +57.1$$

and

$$S = (57,100)(1.41) = 80,600 \text{ lb. tension.}$$

Maximum stress in ef' due to dead load and uniform live load = $-38,600 + 80,600$ = 42,000 lb. tension.

For E-50 train loading it was found by trial from Eq. (6) p. 313, that wheel 3 at e, train headed to the right, gave the maximum stress in ef'. For the given loading conditions $R_2 = 62.7$; panel load at f = 14.4; shear in panel $ef = -(62.7 - 14.4) = -48.3$; and $M_a = \frac{1}{20}[11,763.75 + (23.1)(80)]$. Substituting these values in Eq. (6), we find that $S_V = 64.5$. Then $S = (64,500)(1.41) = 91,100$ lb. tension. Maximum stress in ef' due to dead load and E-50 train load = $-38,600 + 91,100 = 42,500$ lb. tension.

Stress in Ee.—The stress used in designing member *Ee*, Fig. 64, will generally be the maximum compressive stress in the member. Under certain loading conditions it is possible that a small tension may exist in *Ee*. This tensile stress will seldom influence the design of the cross section of the member. In designing the end connections for *Ee*, most specifications take into account any reversal of stress, however small it may be. Methods for the determination of tension in *Ee* will therefore be given.

The dead load stress in *Ee* may be determined by removing the auxiliary trusses *Cd'E* and *Ef'G* of Fig. 64, and distributing the load at joint *d* to joints *C* and *E* and the load at joint *f* to joints *E* and *G*. For live load stress the form of truss under the given load position will generally suggest the method of analysis to be adopted. This will be explained in the problem which follows.

Illustrative Problem.—Determine the maximum compression and tension in member *Ee* of Fig. 64 due to uniform loading.

Maximum Compression in Ee.—Form of truss shown in Fig. 64, counter *cf'* not in action.

To determine dead load stress in *Ee* place panel loads of 48 at all joints. Remove auxiliary trusses and transfer joint load at *d* to *C* and *E*, and joint load at *f* to *E* and *G*. Hence joints *b*, *c*, and *e* of Fig. 64 carry loads of 48, joints *C* and *G* carry loads of 24, and joint *E* carries a load of 48. Cut an inclined section through members *CE*, *Ee*, and *ef*. Take moments about O', or use Eq. (18), p. 265. For the given loading conditions, $R_1 = 264$, shear on inclined section through panel $ef = V = 264 - 3(48) - 24 = +96$; and $M_a = (48)(1 + 2 + 4) + (24)(2) = 384$.

From Eq. (18)

$$S = -\frac{(96)(6) - 384}{10} = -19.2$$

Dead load stress in *Ee* = 19,200 lb. compression.

Maximum compression in *Ee* for uniform live load occurs when joint *f* and all joints to the right are loaded with panel loads of 60. Since joint *d* is not loaded, member *d'E* has zero stress. For the given loading conditions, $R_1 = V_{ef} = \frac{60}{12}(1 + \ldots + 7) = 140$ and $M_a = 0$. From Eq. (18)

$$S = -\frac{(140)(6)}{10} = 84{,}000 \text{ lb. compression.}$$

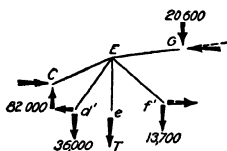



Fig. 71.

Maximum compression in *Ee* = 19,200 + 84,000 = 103,200 lb. compression.

Maximum tension in *Ee* due to uniform live load will occur when the counter *ef'* is in action. The live load should cover the left side of the truss and should extend as far to the right as possible without causing compression in *ef'*. It was found by trial, using Eq. (6) p. 313, that joints *b* to *h* are to be loaded. For the given loading conditions, the vertical components of live load stress in members at joint *E* are found to be as shown on Fig. 71. From summation of vertical forces, $T = 11{,}700$ lb. tension. The dead load stress in *Ee* when the counter *ef'* is in action is found to be 8,100 lb. tension. Hence, total tension in *Ee* = 11,700 + 8,100 = 19,800 lb. If one-third of the dead joint load be assumed as applied at the upper chord joints, the net tension in *Ee* = 19,800 − (⅓)(48,000) = 3,800 lb. tension.

Stresses due to train loading are determined by similar methods.

Stress in Gg.—The stress in member Gg, the center vertical, is calculated by similar methods. Maximum live load compression in Gg will occur for loads at all panel points to the right of g, no load at g. Maximum live load tension in Gg will occur when the truss is fully loaded.

15*d*. Stresses in a Double Intersection Warren Truss with Subdivided Panels.—Stresses in the members of the truss of Fig. 55c may be determined by the application of the methods outlined on p. 233 for the analysis of structures with multiple web systems. Figure 72 shows the manner of dividing a truss of this type into independent systems. The effect of sub-panelling is represented by auxiliary trusses, such as $cd'e$. Dotted and full line web systems represent the independent main truss systems. Where any member is common to more than one of these independent systems, the total stress in that member is the sum of the stresses for the several independent systems.

Chord stresses for the truss of Fig. 72 may be determined by removing the auxiliary trusses $ab'c$, $cd'e$, etc., assuming that the loads at joints

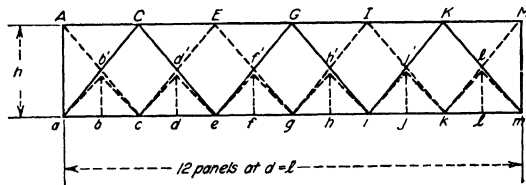



Fig. 72.

b, d, etc. are distributed to the main truss system. If W = panel load at each joint, loads of $2W$ are to be applied at joints c, e, g, i, and k. For any top chord member, as CE, the stress as a member of the dotted line system is $6W\frac{d}{h}$, and as a member of the full line system, its stress is $8W\frac{d}{h}$. Hence the total stress in CE is a compression of $14W\frac{d}{h}$. For a lower chord member, as cde, the stress as a member of the dotted line system is $\frac{1}{h}[(3W)(4d) - (2W)(2d)] = 8W\frac{d}{h}$; as a member of the full line system its stress is $4W\frac{d}{h}$; and as the lower chord member of the auxiliary truss cde for a load W at d, its stress is $W\frac{d}{h}$. Hence the total stress in $cde = 13W\frac{d}{h}$ tension. For uniform live load, all joints are to be fully loaded and the resulting stresses are directly proportional to those given above. For train loading, the panel concentration or the influence line method explained in Sec. 3, Art. **14**, may be used, modified by the presence of the auxiliary trusses.

The dead load stress in the upper half of any web member, as Ef', is due to the shear on a section cutting member Eg of the dotted line system. This shear is due to loads of $2W$ at joints c, e, g, i, and k. Hence, dead load stress in $Ef' = (3W - 2W) \sec \theta = W \sec \theta$, tension. For the lower end of any web member, as $d'e$, the dead load stress is due to the shear on a section cutting member Ce of the full line system plus the vertical component of stress in $d'e$ as a member of the auxiliary truss $cd'e$. Hence, dead load stress in $d'e = \left(2W - \frac{W}{2}\right) \sec \theta = \frac{3}{2} W \sec \theta$. This stress is tension. Note that the shear $2W$ tends to cause tension in $d'e$ while the vertical component of stress in $d'e$ tends to cause compression.

Maximum live load tension in Ef' will occur when joint f and all joints to the right are fully loaded. On removing the auxiliary trusses, live joint loads of $2P$ are then to be placed at joints g and k, where P = live joint load for a panel length d. Hence stress in $Ef' = \frac{2P}{6}(1 + 3) \sec \theta = \frac{4}{3} P \sec \theta$, tension. For stress in $f'g$, the lower end of member Ef', any load P at f will cause a compression of $\frac{P}{2} \sec \theta$ in member $f'g$ of the auxiliary truss $ef'g$. Considered as a member of the dotted line system, the stress in $f'g$ due to a load P at f is $\left(\frac{1}{2}P\right)\left(\frac{1}{2}\right) \sec \theta = \frac{P}{4} \sec \theta$, tension. Hence a load at f causes compression in $f'g$ and this point should not be loaded for maximum live load tension. Placing loads of P at joints g to l, removing the auxiliary trusses and distributing their panel loads to the main truss systems, we have panel loads of $\frac{3}{2} P$ at g and $2P$ at k. The stress in $f'g$ is then $\frac{P}{6}\left[\left(\frac{3}{2}\right)(3) + (2)(1)\right] \sec \theta = \frac{13}{12} P \sec \theta$, tension. For maximum compression load joint f and all joints to the left. Similar methods of analysis are used for other members.

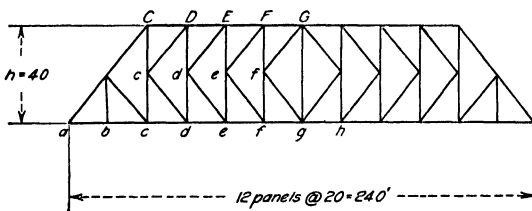



FIG. 73.

16. The K-truss.—The K-truss of Fig. 73 is not, strictly speaking, a subdivided panel truss. However, it will be included under this head because the altered arrangement of members makes possible the use of

short panels in long span trusses. It may therefore be considered as a structure similar in nature to one with subdivided panels.

To determine the stresses in any chord member, as *EF*, Fig. 73, cut a vertical section and remove the portion of the structure to the left of that section, as shown in Fig. 74. Let C and T represent the top and bottom chord stresses and let S_1 and S_2 represent the stresses in the web members cut by the section. Resolve S_1 and S_2 into vertical and horizontal components and consider these forces as applied at e'. On taking moments about e', we have

$$Ch_1 + Th_2 = M_e$$

in which M_e = moment of applied loads about e. From a summation of horizontal forces,

$$C + S_{1H} - S_{2H} - T = 0$$

Since joint e' is in equilibrium, $S_{1H} = S_{2H}$. Hence $T = C$ and

$$C = \frac{M_e}{h_1 + h_2} = \frac{M_e}{h}$$

In a similar manner it can be shown that

$$T = \frac{M_e}{h}$$

Hence, the stresses in top and bottom chord members in any panel are equal, (compression in the top chord member, and tension in the bottom chord member). The amount of stress is equal to the moment at the left end of the panel divided by the height of truss.

Fig. 74.

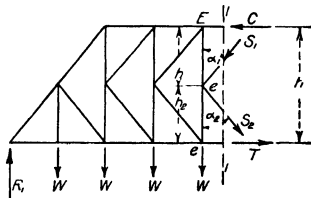
Load positions for uniform loading and criteria for position of train loading are the same as for trusses with horizontal chords.

To determine the stresses in web members Fe' and fe', Fig. 73, consider the conditions shown in Fig. 74. From a summation of vertical forces,

$$S_1 \cos \alpha_1 + S_2 \cos \alpha_2 = V$$

in which V = shear on section 1-1, considered as positive, and α_1 and α_2 are the angles which S_1 and S_2 make with the vertical. From a summation of horizontal forces at e', $S_1 \sin \alpha_1 = S_2 \sin \alpha_2$. On solving these simultaneous equations we derive

and

$$\left.\begin{aligned} S_1 &= V \frac{\sin \alpha_2}{\sin (\alpha_1 + \alpha_2)} \\ S_2 &= V \frac{\sin \alpha_1}{\sin (\alpha_1 + \alpha_2)} \end{aligned}\right\} \quad (1)$$

When $\alpha_1 = \alpha_2$, the usual case in practice,

$$S_1 = S_2 = \tfrac{1}{2}V \sec \alpha \tag{2}$$

Hence when the angle α is equal for the two web members, their stresses are equal and each member takes one-half the shear on the section.

For positive shear, S_1 is compression and S_2 is tension. Load positions for uniform loading and criteria for position of train loading are the same as for trusses with horizontal chords.

The stress in the upper end of the verticals Dd', Ee', and Ff' of Fig. 73 is equal to the vertical component of stress in the diagonal entering the joint at the top of the vertical. As soon as the stress in the diagonal is known, the stress in the vertical may readily be determined.

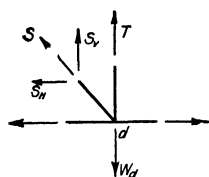



FIG. 75.

To determine the stress in the lower end of any vertical, as dd', Fig. 73, remove joint d and indicate the system of concurrent forces acting at that joint. Figure 75 shows joint d with all forces in position. Let W_d = panel load at d; S = stress in the diagonal entering joint d; and T = stress in dd'. From a summation of vertical forces, $T = W_d - S_V$. On substituting the value of S_V given by Eq. (1),

$$T = W_d - V\frac{\sin \alpha_1 \cos \alpha_2}{\sin (\alpha_1 + \alpha_2)} \tag{3}$$

When $\alpha_1 = \alpha_2$, Eq. (3) becomes

$$T = W_d - \tfrac{1}{2}V \tag{4}$$

In these equations, V = shear in the panel to the left of dd'.

Load positions for maximum stress in dd' will be determined, assuming that Eq. (4) gives the stress in the member. On substituting in Eq. (4) for a single load at d or any joint to the left of d, it will be found that the stress in dd' is tension, and that loads to the right of d give compression in dd'. Hence, for uniform live load, maximum tension in dd' occurs when joint d and all joints to the left are fully loaded. Maximum compression occurs when joint e and all joints to the right are fully loaded.

For train loading, maximum tension in dd' occurs when the train load extends from the left end of the truss into panel de, train headed to the right. The criterion for load position for maximum stress can readily be shown to be

$$\frac{1}{2}\left(\frac{G}{n} + G_2\right) - G_1 = 0 \tag{5}$$

in which G = total load on span; G_1 = load in panel to the right of dd'; G_2 = load in panel to the left of dd'; and n = number of panels in the truss. Equation (5) is satisfied when a wheel passes joint d. For maximum compression in dd', the train load extends from the right end of the

span into panel *de*, train headed to the left. The criterion for load position for maximum stress is

$$\frac{G}{2n} - G_1 = 0 \tag{6}$$

in which G = total load on span; G_1 = load in the panel to the right of dd'; and n = number of panels in the truss. Equation (6) is satisfied when a wheel passes joint e. Corresponding conditions hold for members ee' and ff'.

Illustrative Problem.—Calculate the stress in member dd' of Fig. 73 due to a dead load of 2,000 lb. per ft., a uniform live load of 3,000 lb. per ft., and for E-50 train loading.

Dead panel load = (2,000)(20) = 40,000 lb. $R_1 = (^{11}\!/_2)(40) = 220$. Shear in panel $cd = 220 - (2)(40) = 140$. From Eq. (4)

$$T = 1{,}000[40 - (\tfrac{1}{2})(140)] = 30{,}000 \text{ lb. compression}$$

which is the dead load stress in dd'.

Live panel load, uniform loading = (20)(3,000) = 60,000 lb. For maximum compression in dd', load joint e and all joints to the right. Shear in panel $cd = R_1 = {}^{60}\!/_{12}(1 + \quad . + 8) = 180$. From Eq. (4), with $W_d = 0$,

$$T = 1{,}000[0 - (\tfrac{1}{2})(180)] = 90{,}000 \text{ lb. compression.}$$

For maximum tension, load d and all joints to the left. Shear in panel $cd = R_1 - (2)(60) = 150 - 120 = +30$. Load at d = 60,000 lb. From Eq. (4)

$$T = 60{,}000 - (\tfrac{1}{2})(30{,}000) = 45{,}000 \text{ lb. tension.}$$

Maximum stress, dead load and uniform live load = 30,000 + 112,500 = 142,500 lb. compression. Minimum stress = −30,000 + 45,000 = 15,000 lb. tension.

Maximum compression in dd' for train loading will occur when some wheel is placed at e, train headed to the left. The criterion of Eq. (6) is satisfied when wheel 2 is placed at e. For this load position, shear in panel $cd = R_1 = \frac{1}{240}[20{,}455 + (355)(64) + \frac{1}{2}(2.5)(64)^2] = 200.1$; load at joint $d = {}^{100}\!/_{25} = 5$ Then from Eq. (4)

$$T = 1{,}000[5 - (\tfrac{1}{2})(200.1)] = 95{,}050 \text{ lb. compression.}$$

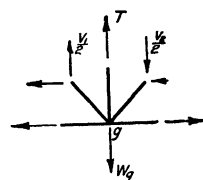
For maximum tension in dd', substitution in Eq. (5) shows that wheel 3 at d, train headed to the right, satisfies the criterion. For this load position, R_2 = 38,940; load at d = 80,190; and load at e = 14,380. Hence shear in cd = −(38,940 − 80,190 − 14,380) = +55,630 lb. From Eq. (4)

$$T = 80{,}190 - (\tfrac{1}{2})(55{,}630) = 52{,}370 \text{ lb. tension.}$$

Maximum and minimum stresses for dead load and train loading are as follows: Maximum compression, 30,000 + 95,050 = 125,050 lb.; maximum tension, − 30,000 + 52,370 = 22,370 lb.

Fig. 76.

The stress in Gg, Fig. 73 may be determined from the conditions existing at joint g. Figure 76 shows the system of concurrent forces acting at joint g. Let V_1 and V_2 = simultaneous shears in panels fg and gh. Assume these shears to be positive. Then $T = W_g + \frac{V_2}{2} - \frac{V_1}{2}$. But $V_2 = V_1 - W_g$. Hence

$$T = \tfrac{1}{2}W \tag{7}$$

that is, the stress in Gg is independent of the shear and depends only on the joint load at g. Maximum and minimum stresses in Gg are determined by the same methods as used for the end verticals in a Pratt truss.

When the top chord of a K-truss is curved, the above general methods must be modified so as to include the effect of the vertical component of top chord stress. The method of moments used in the analysis of the curved chord Pratt truss may readily be applied to the curved chord K-truss.

SKEW BRIDGES

17. General Considerations.—When a bridge crosses a stream or a street at an angle, it is generally impossible to place the ends of the

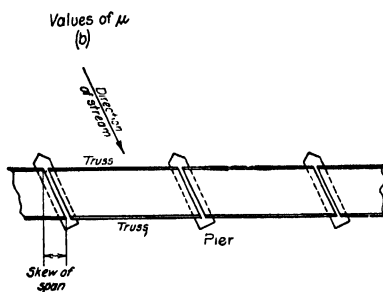
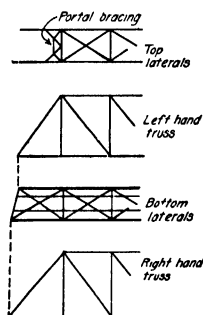



FIG. 77. FIG. 78.

two trusses in any span opposite each other. When the trusses are arranged in the manner described above, the span is called a skew span. Figure 77 shows a typical skew bridge layout. While it is always

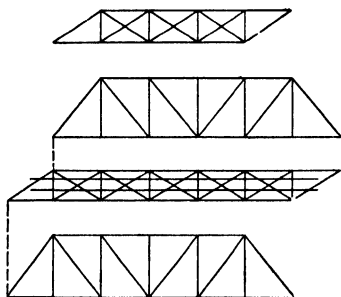
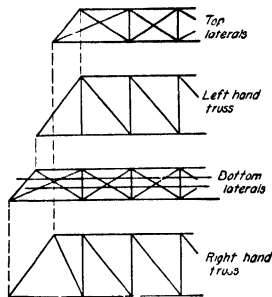



FIG. 79. FIG. 80.

desirable to arrange a span with square ends, it is not advisable to do so if this would cause the piers or abutments to present a partial side view to the direction of the stream flow. In the case of city bridges forming overhead crossings it is desirable to have the abutments parallel to the street lines in order to reduce the right of way area to a minimum.

The conditions encountered in skew spans generally lead to unsymmetrical trusses. It is desirable in skew bridges to place the floorbeams at right angles to the main trusses in order to provide simple and direct connections between floorbeams and trusses. When the skew is small, the inclinations of the end posts may be made unequal, as shown in Fig. 78. This construction is advisable only in case it permits the use of portal bracing in a plane surface. This bracing may be placed at an angle which is the mean of the end post inclinations. In rare cases it may be possible to make the panel length equal to the skew. The arrangement shown in Fig. 79 may then be adopted. Figure 80 shows a typical arrangement resulting in two unsymmetrical trusses. Note that the end suspender is lacking in one of the end panels. Figure 81 shows unsymmetrical trusses with inclined end suspenders, an arrangement which is used frequently in practice. The layouts shown in Figs. 79, 80 and 81 permit the use of portal bracing all parts of which lie in the same plane, except the end connections.

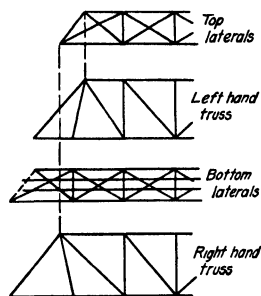



Fig. 81.

Two methods of supporting the end stringers are shown in Fig. 82. Figure 82*a* shows an end floorbeam to which the stringers are riveted at a_1 and a_2. The stringer loads are carried by the floorbeam to the main shoes at a and a'. This arrangement has the following disadvantages: The connections between stringers and floorbeams and between the end floorbeams and the trusses are complicated and not as rigid as right

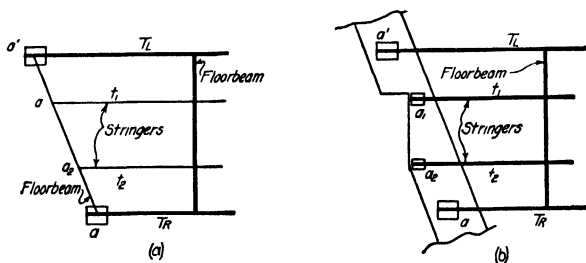



Fig. 82.

angle connections; the stringers t_1 and t_2 being of unequal length have unequal deflections; and cross ties near the abutment end of the span rest partly on the masonry and partly on the stringers. Figure 82*b* shows the arrangement generally adopted. The main trusses are supported by shoes at a and a', and the stringers are supported by independent shoes at a_1 and a_2. This arrangement permits the use of stringers t_1 and t_2 of equal length, and the ties at the end of span rest entirely on the stringers.

18. Determination of Stresses in Skew Bridges.—General methods for the determination of stresses in skew bridges will be illustrated by means of the structure shown in Fig. 83. This structure is so arranged as to allow the use of two equal unsymmetrical spans. It will be assumed that all stringers are of equal length and that the end stringers are supported on independent shoes.

The dead panel loads will not be equal. Assuming a dead load of w lb. per ft. per truss, all of which is applied at the lower chord, the panel load at b, Fig. 83a is $\frac{1}{2}w(30 + 25)$ lb.; at c, d, and e, the panel load is $25w$ lb.; and at f the panel load is $\frac{1}{2}w(25 + 20)$. Stresses in all members may readily be determined by the methods given for trusses with horizontal chords. If a portion of the dead load is assumed as applied at the

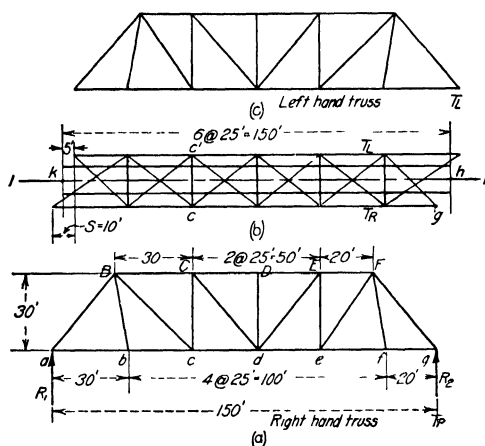



Fig. 83.

upper chord joints, the assumed dead load must be divided into parts which represent the weight of the floor system and the trusses. This division of load is best made from a study of existing structures. The portion of the dead load representing the weight of the trusses may be divided into two equal parts, one of which is assumed to represent the top chord weight and the other represents the lower chord weight. At B and F the end posts may be assumed to have the same weight as the top chord member. Each joint load may then be determined as above.

Panel loads for uniform live load will all be equal for the arrangement shown in Fig. 83, since the effective panel lengths are all equal. Live load positions for maximum stress may be taken the same as for horizontal chord trusses.

Live load stresses due to train loading may be determined by assuming that the position of loads for maximum moment or shear is the same as for a square ended truss with panel lengths equal to the stringer lengths taken

along the axis 1-1 of Fig. 83*b*. For the truss of Fig. 83, we then have six 25-ft. panels. Load positions for this assumed span may be determined by the methods given in Sec. 1. Having determined the position of the live load, the reactions, moments and shears may be determined by either of the two methods explained in detail below. These methods are: (*a*) Method of Panel Concentrations; (*b*) Method of Direct Moments. These methods give identical results, but in most cases, the calculations required by the second method are considerably shorter than those of the first method.

18*a*. Method of Panel Concentrations.—After the load position has been determined by the method suggested above, calculate the panel concentrations due to this loading, assuming that the load on both rails is applied along the axis 1-1 of Fig. 83*b*. Methods for the calculation of these panel concentrations are given in Art. **71**, p. 140. These panel loads may then be divided equally between the two main trusses and the stresses in all members may be calculated by the usual methods.

18*b*. Method of Direct Moments.—The reactions at the end supports may also be determined by considering the equilibrium of the

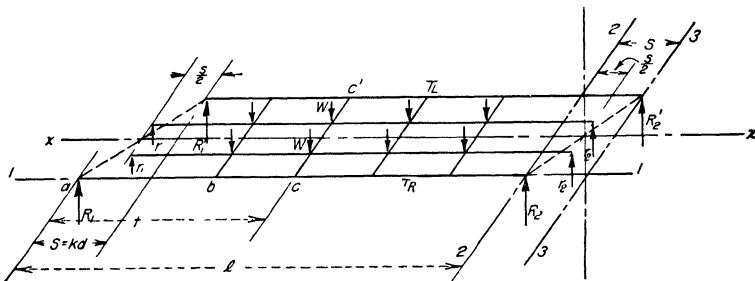



FIG. 84.

structure as a whole. Figure 84 shows an isometric projection of the plane of the lower chord of the truss drawn with respect to the intersecting axes 1-1 and 3-3. Let R_1, R_1', R_2 and R_2' represent the main truss reactions. Assume the stringers in the end panels to be of equal length and let r_1 and r_2 be the stringer reactions at the ends of the truss. Cut a section x-x along the center line of the truss. Since the stringers are assumed to be equal in length, the loads W on any floorbeam are equal and the shear in the portion of the floorbeam between stringers is zero. To determine R_1, consider the portion of the structure below axis x-x. From moments about axis 2-2 of forces acting perpendicular to the plane of the lower chord, we have

$$R_1 l + r_1\left(l - \frac{s}{2}\right) - r_2 \frac{s}{2} - M_2 = 0$$

from which

$$R_1 = \frac{M_2}{l} + \frac{s}{2l}(r_1 + r_2) - r_1 \qquad (1)$$

In Eq. (1) M_2 = moment of applied loads about axis 2-2; and s = skew at end of span. Note that loads to the right and left of axis 2-2 must be included in calculating M_2. To determine R_1', take moments about axis 3-3, considering forces above axis $x - x$, from which

$$R_1' = \frac{M_3}{l} - \frac{s}{2l}(r_1 + r_2) - r_1 \qquad (2)$$

in which M_3 = moment of applied loads about axis 3-3. Similar methods may also be used when the skew is unequal at the two ends of the bridge.

When the load on the bridge is a uniform load of $2w$ lb. per ft. of bridge, and the panel lengths are equal, the above equations reduce to simple forms. Let $l = nd$, where n = number of panels and d = panel length, and let $s = kd$ where k = a fraction. We then have

$$M_2 = \frac{w}{2}\left(l - \frac{s}{2}\right)^2 - \frac{ws^2}{8} = \frac{wd^2}{2}\left(n - \frac{k}{2}\right)^2 - \frac{wk^2d^2}{8} = \frac{wd^2}{2}n\left(n - k\right)$$

and

$$r_1 = r_2 = \frac{wd}{2}$$

Substituting in Eq. (1)

$$R_1 = \frac{wd}{2}\left(1 - \frac{k}{n}\right)(n - 1) \qquad (3)$$

In a similar manner

$$R_1' = \frac{wd}{2}\left(1 + \frac{k}{n}\right)(n - 1) \qquad (4)$$

If desired, similar equations for reactions due to partial loading may also be derived.

The moment at any joint c, Fig. 84, is equal to moments about an axis c-c' through the given center. Considering vertical forces below axis x-x, we have

$$M_c = R_1t + r_1\left(t - \frac{s}{2}\right) - M_1 \qquad (5)$$

in which M_1 = moment about axis c-c' of applied loads between this axis and the left end of the span. The shear in any panel bc, Fig. 84, is

$$V = R_1 + r_1 - \Sigma W$$

in which ΣW = joint load at b due to loads in panel bc plus loads from b to the left end of the span.

Illustrative Problem.—Calculate the moment at joint c of the right-hand truss of Fig. 83 due to a uniform live load of 6,000 lb. per ft. of bridge.

Panel Concentration Method.—Panel loads at each joint of each truss = (½)(6,000)(25) = 75,000 lb. Place loads of 75,000 at joints b to f of Fig. 83a for maximum moment at c. From moments about g,

$$R_1 = \frac{75,000}{150}(20 + 45 + 70 + 95 + 120) = 175,000 \text{ lb.}$$

Then

$$M_c = [(R_1)(55) - (75,000)(25)] = 7,750,000 \text{ ft.-lb.}$$

Direct Moment Method.—From Eq. (3), with $2w = 6,000$; $k = \frac{s}{d} = \frac{10}{25} = 0.4$: $n = 6$, and $d = 25$,

$$R_1 = \frac{(3,000)(25)}{2}\left(1 - \frac{0.4}{6}\right)(6 - 1) = 175,000 \text{ lb.}$$

For moment at c, use Eq. (5), with $R_1 = 175,000$; $t = 55$ ft.; $s = 10$ ft.; $r_1 = \frac{wd}{2} = 37,500$ lb.; and M_1 = moment of two panels of uniform load about c = (3,000)(50)(25) = 3,750,000. Then

$$M_c = (175,000)(55) + (37,500)(55 - 5) - 3,750,000$$
$$= 7,750,000 \text{ ft.-lb.}$$

Illustrative Problem.—Calculate the maximum positive shear in panel cd of the right-hand truss of Fig. 83 due to a uniform live load of 6,000 lb. per ft. of bridge.

Panel Concentration Method.—Using the conventional method of loading, joint loads of (3,000)(25) = 75,000 lb. are to be placed at joints d, e, and f, Fig. 83a. Then,

$$\text{shear in panel } cd = R_1 = \frac{75,000}{150}(20 + 45 + 70) = 67,500 \text{ lb.}$$

Direct Moment Method.—The uniform live load of 3,000 lb. per ft. per truss covers the span from joint d to the right end of the stringers To conform to the conventional method of loading, an additional concentrated load of 37,500 lb. must be placed at each stringer on the floorbeam at d. Taking moments about an axis through g for loads on the lower side of axis 1-1, Fig. 83b, noting that $r_2 = 37,500$ lb. we have

$$R_1 = \frac{1,000}{150}[(3)(75)(32.5) + (37.5)(5) + (37.5)(70)]$$
$$= 67,500 \text{ lb.}$$

which is the required shear in panel cd.

Illustrative Problem.—Calculate the moment at joint c of the right-hand truss of Fig. 83 due to E-50 train loading.

Assuming the train load to be located along the axis 1-1, it will be found from Fig. 141, p. 114 that wheel 7 at c gives maximum moment at c.

Panel Concentration Method.—By the methods given in Art. **71,** p. 140, the panel concentrations at the joints of the right-hand truss of Fig. 83a due to the above position of loads are as follows: b = 87,750 lb.; c = 71,700 lb.; d = 82,550 lb.; e = 80,050 lb.; and f = 61,700 lb. Taking moments about g, R_1 = 186,375 lb. Moment at c = (186,375)(55) − (87,750)(25) = 8,056,875 ft.-lb.

Direct Moment Method.—To determine R_1, use Eq. (1), p. 324. In this equation M_2 = moment at g, Fig. 83b, due to train load with wheel 7 at c. Note that the uniform load extends 5 ft. to the right of g and 23 ft. to the left. Then

$$M_2 = 1,000[20,455 + (355)(23) + (½)(2.5)(23)^2 - (½)(2.5)(5)^2]$$
$$= 29,250,000 \text{ -ft.lb.}$$

The stringer reaction at k is $r_1 = 10{,}000$ lb., and at h, the stringer reaction is $r_2 = 31{,}250$ lb. Then from Eq. (1)

$$R_1 = \frac{29{,}250{,}000}{150} + \frac{10}{300}(10{,}000 + 31{,}250) - 10{,}000$$
$$= 186{,}500 \text{ lb.}$$

The moment at c, as given by Eq. (5) is

$$M_c = (186{,}375)(55) + (10{,}000)(55 - 5) - 2{,}693{,}750$$
$$= 8{,}056{,}875 \text{ ft.-lb.}$$

Note that the results given by the two methods are identical. However, the amount of work required by the second method is generally less than that required by the first method. This is due to the extra work required for the calculation of the floorbeam reactions.

Illustrative Problem.—Calculate the shear in panel bc of the right-hand truss of Fig. 83 due to E-50 train loading.

Considering the loads as applied along the center line of the bridge, wheels 3 and 4, train headed to the left, answer the criterion for shear in panel cd. It was found that wheel 4 gave the maximum shear.

Panel Concentration Method.—The panel loads at the several joints for wheel 4 at d are as follows: $b = 24{,}000$ lb.; $c = 94{,}550$ lb.; $d = 65{,}700$ lb.; $e = 91{,}000$ lb.; and $f = 72{,}750$ lb. On taking moments about the right end of the truss, we find $R_1 = 146{,}742$ lb. Hence, shear in panel $cd = 146{,}742 - 24{,}000 = 122{,}742$ lb.

Direct Moment Method.—For the given loading $M_2 = 1{,}000\,[20{,}455 + (355)(4) + (\frac{1}{2})(2.5)(4)^2 - (\frac{1}{2})(2.5)(5)^2] = 21{,}863{,}750$ ft.-lb. Also, $r_2 = 29{,}500$ lb. Then from Eq. (1),

$$R_1 = \frac{21{,}863{,}750}{150} + \frac{10}{(2)(150)}(29{,}500) = 146{,}720 \text{ lb.}$$

as before, shear in panel $cd = 146{,}742 - \dfrac{600{,}000}{25} = 122{,}742$ lb.

BRIDGES ON CURVES

19. General Considerations.—The spacing of trusses for bridges on curves must be increased to allow for the tilting of the cars due to the

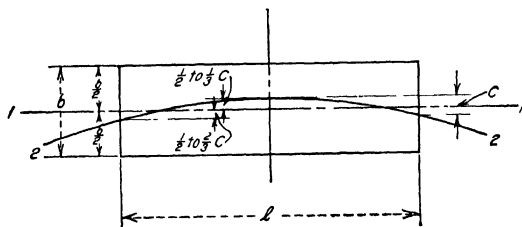



FIG. 85.

superelevation of the outer rail and also to allow for the swing of the portion of the cars overhanging the trucks.

Figure 85 shows a deck structure with the center line of track on the curve 2-2. If c = middle ordinate of the curve of the center line of track, l = length of span, and R = radius of curve 2-2, it can be shown that

$$c = R - \sqrt{R^2 - \left(\frac{l}{2}\right)^2} \tag{1}$$

When the degree of curve is less than ten, we may use in place of Eq. (1) the approximate formula

$$c = \frac{l^2}{8R} \tag{2}$$

If D = degree of curve, Eq. (2) may be written

$$c = \frac{l^2D}{45{,}840} = 0.0000218l^2D \tag{3}$$

The track is sometimes placed so that the mid-ordinate c is divided by the axis 1-1, Fig. 85. If b_1 = distance between girders required for track on tangent

$$b = b_1 + c$$

In other cases the axis 1-1 is made to pass through a point $\frac{1}{3}c$ from the vertex of the curve 2-2.

The distance between trusses in a through bridge on a curve must provide for the usual clearance diagram increased to allow for the tilting of the cars, as shown in Fig. 86, and for the swing of the cars, as shown in Fig. 87. As shown in Fig. 86, a tilting of the car to the left causes point 1 to move to 1′ and point 2 to move to 2′. The vertical and horizontal components of the movement of these points are shown on the figure. These values are given in terms of E, the superelevation of the outer rail; G, the gage of the track; H and h, the height of points 1 and 2 above the top of rail; and B the width of the car.

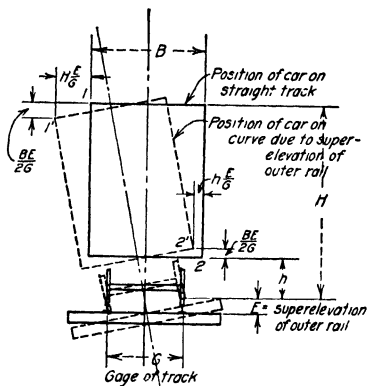



FIG. 86.

Figure 87 shows the conditions governing the distance between trusses due to the swing of a car of length A, width B, distance between trucks T and vertical height as shown in Fig. 86. It can be seen from Fig. 86 that point 1′ will determine the clearance on the inside of the curve and point 2′ will determine the clearance on the outside. The paths swept out by these points are shown in Fig. 87 by curves 1-1 and 2-2.

If R and L represent the distances from the center line of track at the bridge center to the centers of the trusses, it can readily be seen from Fig. 87 that

$$R = \frac{1}{2}t_1 + c + c_1 + c_2 + \frac{B}{2} + \frac{HE}{G} \tag{4}$$

$$L = \frac{1}{2}t_2 + c - c_2 + c_3 + \frac{B}{2} - \frac{hE}{G} \tag{5}$$

In these equations t_1 and t_2 represent the width of the truss at the end post and at the center of the truss; c = minimum allowable clearance between rolling stock and bridge, generally taken as 2 ft.; c_2 and c_3 = the mid-ordinates respectively for the distance center to center of trucks and for the length of the car; c_1 = mid-ordinate for chord l, Fig. 87a determined

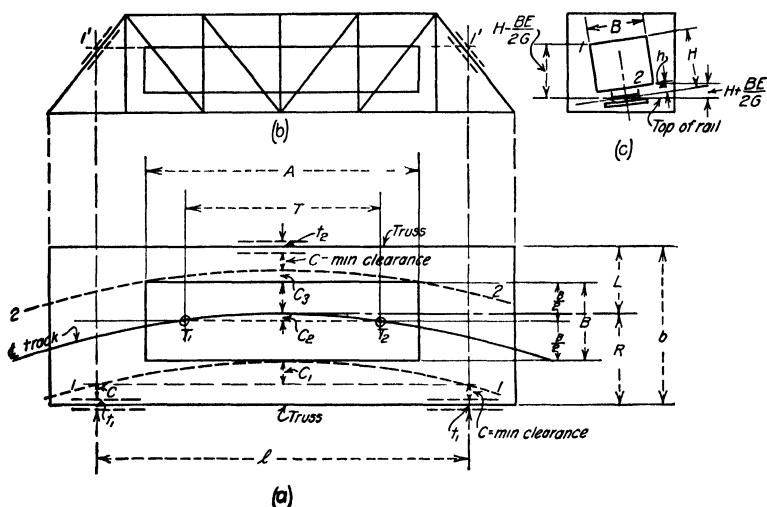



FIG. 87.

by projecting point 1, Fig. 87c, to an intersection at 1′-1′, Fig. 87b with the outside edges of the end posts; and the last terms in each equation represent the tilting of points 1 and 2, Fig. 86, due to elevation of the outer rail.

Two general methods used in locating the stringers in a bridge on a curve are shown in Fig. 88. When the mid-ordinate does not exceed about 6 in., the distance between stringers is increased somewhat and all of the stringers are placed in the same line as shown in Fig. 88a. When the degree of curve is large, the stringers in adjacent panels are offset, as shown in Fig. 88b. The amount of offset in adjacent panels must be sufficient to prevent interference between stringer connections.

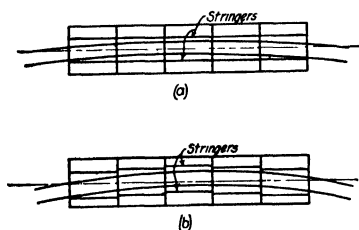
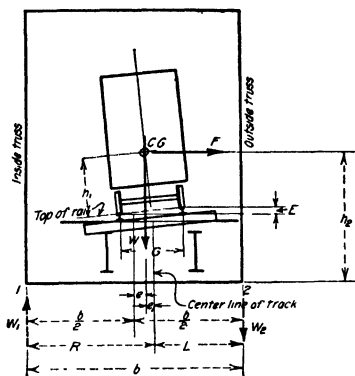



FIG. 88.

20. Stresses in Bridges on Curves.—The forces acting on a bridge on a curve are shown in the cross-section of Fig. 89. These forces are W, the weight of the train, and F, the centrifugal force due to motion of the train. In calculating the stresses in the structure, the effect of forces W and F may be determined separately and the resulting stresses added to give the total stress in any part of the structure.

20a. Effect of Weight of Train.—As shown in Fig. 89, load W is, in general, eccentric with respect to the main trusses. The stresses due to W may be then calculated in two parts, (a) one part due to W considered as a centrally applied load, and (b) the other part due to the eccentricity of application of load W. For any member, the total stress is then the sum of the two partial stresses.

Fig. 89.

To determine the stresses due to any given loading, let W = floorbeam load at any joint and let e = eccentricity of load W with respect to center line of bridge. On taking moments about joint 1 for the conditions shown in Fig. 89, the load on the outside truss, considered as applied at joint 2, is

$$W_2 = \frac{W}{2} + W\frac{e}{b} \tag{6}$$

and for the inside truss

$$W_1 = \frac{W}{2} - W\frac{e}{b} \tag{7}$$

From Fig. 89,

$$e = \frac{b}{2} - (L + e_1) \tag{8}$$

If h_1 = distance from top of rail to the center of gravity of the train load, usually taken as 6 ft., we have $e_1 = h_1\frac{E}{G}$, where E = elevation of outer rail and G = gage of track. Equation (5) gives the value of L, and $b = L + R$. In a similar manner, panel loads may be determined for each joint in the truss. Note that the value of e differs for each panel, depending upon the position of the track at that point. The load W is the total panel load for both rails. In general, the loads $W\frac{e}{b}$ cause an increase in the stresses in the outside truss of Fig. 89 and a decrease in the stresses in the inside truss.

In calculating stresses in a truss for the loads given by Eqs. (6) and (7), the portion of the stress due to $\frac{W}{2}$ is the same as for the given loads centrally applied. The portion of the stress due to loads $W\frac{e}{b}$ is determined by placing these loads at the proper joints and calculating the stresses by the usual methods. These stresses are the same in amount but opposite in kind for the two trusses. The total stress in any member is the sum of these partial stresses.

The calculation of the loads $W\frac{e}{b}$ for train loading is generally rather a tedious process, because of the length of the calculations required for the determination of panel concentrations. Since the determination of the eccentricities at the several panels is based on assumptions which are only approximate, and since the stresses due to the loads $W\frac{e}{b}$ are small compared to those due to loads $\frac{W}{2}$, it is reasonably correct to determine loads $W\frac{e}{b}$ from an equivalent uniform load obtained as explained in Art **68***h*, p. 133. Load positions for maximum combined stress may be taken the same as for the centrally applied loads. Methods of stress calculation are given in the preceding chapters. Loads on stringers and floorbeams may be determined by similar methods. In determining the eccentricity e for a stringer, the average value for the panel in question may be taken. For floorbeams, use an average value of e taken for the half stringer panel each side of the floorbeam in question.

20*b*. **Effect of Centrifugal Force.**—The centrifugal force F due to a body of weight W lb. moving in a curve whose radius is R ft. at a velocity v miles per hour is

$$F = \frac{v^2}{gR} W \tag{9}$$

in which g = acceleration of gravity = 32.2 ft. per sec. If g be expressed in miles per hour, and if D = degree of curve the above equation may be written

$$F = 0.0000117v^2DW \tag{10}$$

The centrifugal force F may be determined from the above equations in terms of panel loads or load per foot.

The centrifugal force F is sometimes taken as a percentage of the live load plus impact. In the 1941 edition of the specifications of the A.R.E.A. we find the following recommendation:

208. Centrifugal Force.—On curves, the centrifugal force in percentage of the live load is $0.00117S^2D$. S = speed in miles per hour, D = degree of curve.

It shall be assumed to act 6 ft. above the rail and shall be taken without impact.

The table on following page gives the permissible speeds and the corresponding centrifugal force percentages for curves with the amounts of superelevation shown. It is based on a maximum speed of 100 miles per hour and a maximum superelevation of 7 in., resulting in a maximum centrifugal force of 17 5 per cent.

If the conditions at the site restrict the permissible speeds to less than those shown in the table, the centrifugal force percentage shall be taken for the greatest speed expected.

In Eq. (10) the centrifugal force F is proportional to the applied loads W. If W is caused by a uniform load, then also will F be uniform. If

D	E	S	C	D	E	S	C
0°-10′		100	1.95	2°-30′	7	77	17 5
0°-20′		100	3 90	3°-00′	7	71	17.5
0°-30′	0 33	100	5 85	3°-30′	7	65	17 5
0°-40′	1 44	100	7 80	4°	7	61	17.5
0°-50′	2 56	100	9 75	5°	7	55	17 5
1°-00′	3 67	100	11 7	6°	7	50	17 5
1°-15′	5 33	100	14 6	8°	7	43	17 5
1°-30′	7	100	17 5	10°	7	39	17 5
1°-45′	7	93	17 5	15°	7	32	17 5
2°-00′	7	87	17.5	20°	7	27	17 5
2°-15′	7	82	17 5				

D = degree of curve
E = superelevation in inches
S = permissible speed in miles per hour
C = centrifugal force in percentage of the live load

$$C = 0.00117S^2D = 1.755(E + 3)$$

$$E = \frac{2}{3} \cdot \frac{S^2D}{1000} - 3 = \frac{C - 5.265}{1.755}$$

$$S^2 = \frac{1500}{D}(E + 3)$$

W is due to wheel concentrations, then F will be a similar set of loads proportionately as great as the vertical loads.

The action of loads F on a truss is exactly the same as a wind force, or other lateral force, acting on the side of a train. Methods of stress analysis are given in Sec. 4. It is there shown that a load F, Fig. 89, causes stress in the lateral system of the loaded chord and also causes stresses in the vertical trusses.

Stresses in the lateral system of the loaded chord are to be determined for a system of horizontal loads of amount kW, $\left(k = \frac{C}{100}\right)$, applied in the plane of the lateral system. For Fig. 89, these loads are applied in the plane 1-2, the plane of the lower chord. Note that load W of Eq. (10) is the total load for two rails.

Stresses in the vertical truss are similar in nature to overturning effect explained in Art. **4,** Sec. 4. At joint 2, Fig. 89, a downward load $F\frac{h_2}{b} = kW\frac{h_2}{b}$ acts on the outside truss and at joint 1, a similar load acts upward. In general, the stress in any member of the vertical truss is $\frac{2kh_2}{b}$ times the stress due to a centrally applied vertical load equal to the applied load on both rails.

Floorbeams and stringers are also affected by the load F. The portion of the floorbeam adjacent to the outside truss will have stresses equal to $\frac{2kh_2}{b}$ times those due to vertical loading. These stresses are due to the overturning effect of load F. Stringers also are stressed by overturning effect. They are also subjected to lateral forces due to the hori-

zontal load on the panel in question. Methods for the determination of these stresses are the same as for the main lateral and vertical trusses.

20c. Maximum Stresses.—The maximum stress in any member of a truss in a bridge on a curve is due to the combined effect of vertical loading, eccentricity of vertical loading, and the centrifugal force F. In determining maximum stress in any member, the stresses to be combined are the simultaneous stresses for a given position of the loads. An exact determination of maximum stress therefore involves considerable work. However, it will generally be found that the position of loads giving maximum stress in the vertical trusses will generally give the combined maximum stress.

STRESSES DUE TO TRACTIVE FORCES

An application of the brakes while a train is crossing a bridge has frequently been assumed to exert on that structure a force equal to 20 per cent of the weight of the moving load on the span. This force is assumed as applied 6 ft. above the top of rail. The 1941 A.R.E.A. Specifications for Steel Railway Bridges states:

214. Longitudinal Force.—The longitudinal force resulting from the starting and stopping of trains shall be the larger of

(*a*) Force due to braking.
Fifteen per cent of the live load without impact

(*b*) Force due to traction.
Twenty-five per cent of the weight on the driving wheels, without impact.

The longitudinal force shall be taken on one track only and shall be assumed to act 6 ft. above the top of the rail.

For bridges where, by reason of continuity of members or frictional resistance, much of the longitudinal force will be carried directly to the abutments (such as ballasted deck bridges of only three or four spans), only one-half of the longitudinal force shall be considered effective.

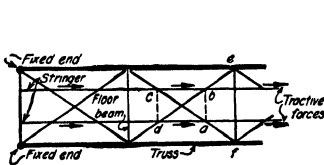
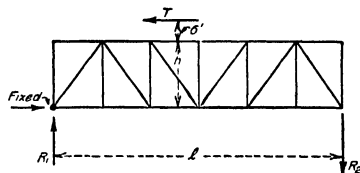



FIG. 90. FIG. 91.

In through bridges the members affected by tractive forces are the floorbeams, stringers, and the lateral system of the loaded chord. This includes the chord members of the loaded chord. The tractive forces are transferred directly to the stringers, and are carried to the floorbeams, which must resist these forces as beams in bending in a plane perpendicular to the web plate. The tractive forces transferred to any floorbeam are due to the loads in a single panel.

When the laterals are connected to the stringers, as shown at *a, b, c,* and *d* of Fig. 90, and cross braces *ab* and *dc* are provided, small trusses *f, a, b, e* are formed which offer some assistance in carrying tractive forces. However, the portion of the tractive forces carried by these trusses is generally small.

The floorbeam reactions at *e* and *f*, Fig. 90, due to tractive forces are transferred to the chords of the main trusses. If the train advances on the bridge from the fixed end, these chord stresses are tensile. They are compressive when the train advances from the free end. The stresses in the chord members increase uniformly from the free end to the fixed end of the span, receiving an increment of stress at each floorbeam.

In a deck structure which is supported at the plane of the top chord, the effect of tractive forces is the same as described above for the through bridge. When the truss is supported at the plane of the lower chord, as shown in Fig. 91, the force T tends to lift one end of the truss and force down on the other, as indicated by the directions shown for the reactions R_1 and R_2. Hence, in addition to top chord stresses similar to those described above for the through bridge, all members of the deck bridge are subjected to stresses due to the reactions R_1 and R_2. However, these stresses are generally small, for in a truss as usually designed, $(h + 6)$ is generally from one-fifth to one-sixth of the span. Hence $R_1 = \frac{1}{5}T$ to $\frac{1}{6}T$. But $T = 20$ per cent of the applied vertical loads. Therefore, in general, R_1 due to tractive forces will not exceed from 4 to $3\frac{1}{3}$ per cent of the reaction due to vertical loading. The effect of tractive stresses on vertical trusses may generally be neglected.

SECTION 4

LATERAL TRUSSES AND PORTAL BRACING

A bridge truss, considered as a framed structure in space, must be designed to provide for the following forces: (*a*) vertical forces due to the weight of the structure; (*b*) lateral forces due to wind pressure on the exposed surface of the truss and on the vertical projection of the live load; lateral forces due to the nosing action of locomotives caused by unbalanced conditions in the mechanism and also to the lurching movement of the trucks under cars against the rails because of the play between wheels

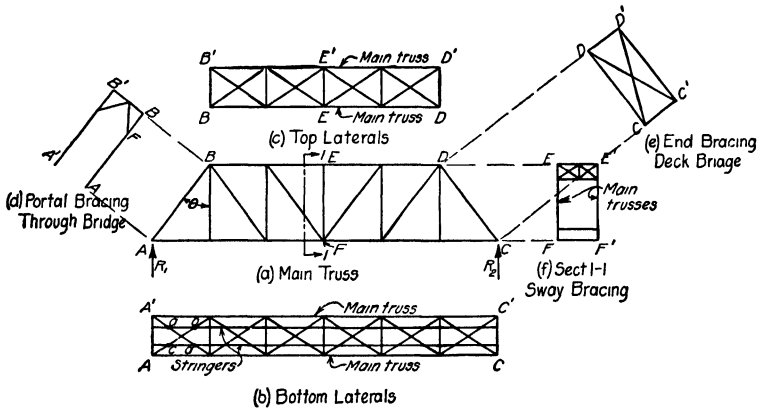



FIG. 1.

and rails; and lateral forces due to centrifugal forces when the track is curved (see p. 330).

Forces of the first class mentioned above are carried by the main trusses shown in Fig. 1, which represents a typical bridge truss. The nature of the vertical forces acting on these trusses and methods for the determination of the resulting stresses for different types of loading have been given in detail in the preceding sections.

Forces of the second class mentioned above are generally assumed to act in a horizontal plane and in directions perpendicular to the plane of the main trusses. To resist these forces, trusses in a horizontal plane, known as *lateral trusses*, are usually provided. The most convenient location for these trusses is in the plane of the upper and lower chord of the main trusses. In this way a single member can be used to form a chord member common to the main and lateral trusses. Figure 1*b* shows a typical bottom lateral system, and Fig. 1*c* shows a top lateral system. Note that *BD* is common to the main and top lateral trusses and *AC* is common to the main and bottom lateral trusses.

In order to form a framework which will be structurally rigid, it is necessary to provide bracing in a plane perpendicular to the planes of the main and lateral trusses. If bracing is placed in the plane of the end posts

of the truss, the resulting space frame will in general be rigid and statically determinate, or may be rendered so by simple and reasonable assumptions. In the case of a deck truss, one or both of the diagonals shown in the end bracing of Fig. 1*e* may be used. For through bridges, a clear space must be provided between the trusses for the passage of the live load. A typical form of bracing used in through bridges is shown in Fig. 1*d*. This is known as *portal bracing*. The framework shown in Fig. 1*d* is incomplete and the desired rigidity is secured by subjecting certain of the members to bending.

Although the truss systems and end or portal bracing are sufficient to form a rigid framework, it is usually considered advisable to provide additional bracing placed in the plane of all vertical members, except those at the ends of the bridge where portal or end bracing is in place. Typical bracing of this sort, which is known as *sway bracing*, is shown in Fig. 1*f*. The object of this bracing is to bind the structure firmly together, and also to brace the top chord compression members so that the column length may be taken as equal to the truss panel length.

In short span deck-girder bridges generally only a single lateral system is provided. This lateral truss is placed in the plane of the top chord. Cross frames placed at frequent intervals are used to support and stiffen the lower chord. End cross frames are provided to carry all lateral forces to the supports on the abutments.

LATERAL TRUSSES

1. Loads on Lateral Truss Systems.—The lateral load to which a bridge truss will be subjected cannot be determined with any great degree of precision. In general, the amount of load for which provision should be made in order to secure a rigid structure is based on past experience as gained from existing structures. The following is taken from the 1941 specifications of the A.R.E.A. for railway bridges:

209. Wind on Loaded Bridge.—The wind force shall be considered as a moving load acting in any horizontal direction. On the train it shall be taken at 300 lb. per lin.-ft. on one track, applied 8 ft. above the top of rail. On the bridge it shall be taken at 30 lb. per sq. ft. of the following surfaces:

(*a*) For girder spans, one and one-half times the vertical projection of the span.

(*b*) For truss spans, the vertical projection of the span plus any portion of the leeward trusses not shielded by the floor system.

(*c*) For viaduct towers and bents, the vertical projections of all columns and tower bracing.

The wind force on girder spans and truss spans, however, shall not be taken at less than 200 lb. per lin.-ft. for the loaded chord or flange, and 150 lb. per lin.-ft. for the unloaded chord or flange.

210. Wind on Unloaded Bridge.—If a wind force on the unloaded bridge of 50 lb. per sq. ft. of surface as defined in Art. **209**, combined with the dead load, produces greater stresses than those produced by the wind forces specified in Art. **209** combined with the stresses from dead load, live load, impact, and centrifugal force, the members wherein such greater stresses occur shall be designed therefor.

212. Nosing of Locomotives.—The lateral force to provide for the effect of the nosing of locomotives (in addition to the other lateral forces specified) shall be a single moving force of 20,000 lb. applied at the top of the rail, in either lateral direction, at any point of the span. The resulting vertical forces shall be disregarded.

213. Bracing between Compression Members.—The lateral bracing of the compression chords or flanges of trusses and deck girders and between the posts of viaduct towers shall be proportioned for a transverse shear in any panel equal to 2½ per cent of the total axial stress in both members in that panel, in addition to the shear from the specified lateral forces.

In the material quoted above, Art. 209 represents the maximum wind pressure under which trains may operate in safety, for experience has shown that pressures greater than 30 lb. per sq. ft. will overturn unloaded cars. Article 210 represents pressures encountered in hurricanes and other violent wind storms. Under such pressures trains cannot operate and such loads would be borne by a structure without live load.

Since the loadings given above are approximate in nature, it follows that in calculating panel loads on lateral trusses refined assumptions regarding the distribution of lateral forces are unwarranted. It is sufficiently accurate to assume that all of the lateral wind force is applied at the joints of the windward truss. Panel loads at each joint of the bottom lateral system may be considered as equal and in amount equal to the load per foot times a panel length. A similar assumption may be made for the top laterals, and joint loads at B and D of Fig. 1 may be taken as equal to the load at an interior joint E.

2. Forms of Lateral Truss Systems.—The form of lateral trussing used in any structure depends on the type of structure and to some extent on its size. In small structures, such as plate girders or half-through highway bridges, in which the length of members in the lateral trusses is small, a single Warren system may be used.

Figure 1 shows types of lateral truss framing of the Pratt type. This system of bracing is in common use for steel spans of any form. In the top lateral system of Fig. 1*c*, the members are generally rather long. It is therefore usually assumed that the diagonals take tension only, as column action in members of this length would require large sections to meet rigidity conditions, while only a small section would be required to meet stress conditions. Two members are generally provided in each panel, as shown in Fig. 1*c*. It is assumed that one member is in action when the direction of lateral force is such as to cause tension in that member. The other member is assumed to be inactive at that time. When the direction of lateral force is reversed, the active and inactive members are also reversed. The cross struts, such as EE', are in action as compression members regardless of the direction of the lateral force.

Similar conditions are sometimes also assumed for the bottom lateral system of Fig. 1*b*. However, it is the usual practice to connect

the lateral diagonals to the stringers at points a, b, c, and d where they cross. In this way, the length of the diagonals is reduced to such an extent that these members may readily be considered as compression members. When this condition holds, it is generally assumed that both diagonals in any panel are in action at the same time, one acting as a tension member, and the other as a compression member.

3. Action of Lateral Forces on the Structure.—Figure 2a is an isometric projection of a through bridge showing the lateral forces in position. Loads W_1 represent the panel loads applied to the top lateral system. Loads W_2 represent the effect of lateral loads due to the train loading. Under the A.R.E.A. specifications, loads W_2 are due to a uniform moving load of 300 lb. per lin. ft. Loads W_3 represent

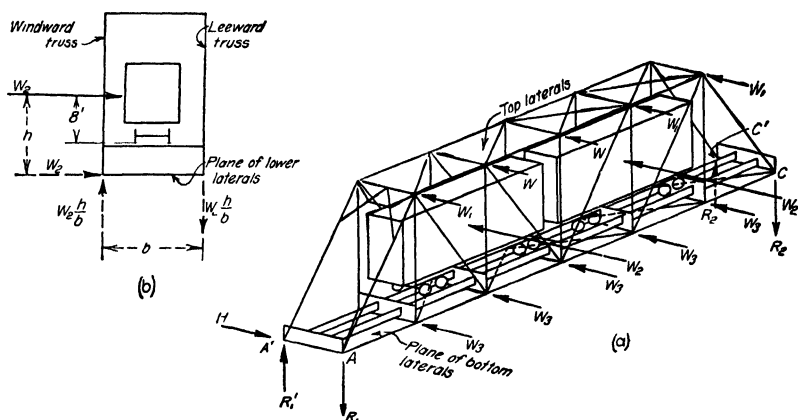



Fig. 2.

the panel loads on the bottom lateral system. Panel loads W_1 and W_3, under the A.R.E.A. specifications, are due respectively to uniform moving loads of 150 and 200 lb. per ft.

Loads W_1 of Fig. 2a cause stresses in all members of the top lateral system. These stresses may be determined by methods given in the preceding sections. The loads W_1 also cause stress in certain members in the main vertical trusses. From Fig. 2a it can be seen that the reactions for loads W_1 are supplied at the points of support for the main trusses, at A, A', C, and C'. Hence the overturning effect of loads W_1 will cause reactions R_1, R_1', R_2, and R_2' which may be determined by taking moments about axes in the plane of the bottom laterals. In determining the values of these reactions, attention must be paid to the nature of the bracing in the plane of the end posts. In Art. **8** it is shown that when the end posts are fixed at their bases, a moment M_o exists at these points acting in the plane of the end posts. From Eq. (4), p. 346, and Eq. (8), p. 347, $M_o = \frac{P}{4}(c - e)$, in which P = total load

carried to supports by the portal bracing; and c and e = respectively the length of end post and the distance from top of portal to foot of portal bracing (see Fig. 8, p. 346). On taking moments about AC or $A'C'$ of Fig. 1, or Fig. 2 as an axis, we have

$$R_1 = R_1' = \frac{Ph_1 - 2M_o(\cos \theta)}{b}$$

$$= \frac{P}{2b}[2h_1 - (c - e) \cos \theta] \tag{1}$$

in which θ is the angle which the end post makes with the vertical and h_1 is the vertical distance between the top and bottom lateral trusses. For the direction of lateral forces shown in Fig. 2, R_1 and R_1' act as shown in Fig. 2a. These reactions cause a stress in the entire lower chord member of the main truss which is constant over the entire chord and equal to

$$S = \frac{P}{2b}[2h_1 - (c - e) \cos \theta] \tan \theta \tag{2}$$

This stress, which is known as the *portal effect*, is tension in $A'C'$ and compression in AC. In Art. **9**, this same result is obtained by another method. When the portal frame is hinged at the base, or when full diagonal bracing may be employed in the plane of the end posts, as in the case of deck bridges, $M_o = 0$, and Eqs. (1) and (2) become

$$R_1 = R_1' = \frac{Ph_1}{b}; \qquad S = \frac{Ph_1}{b} \tan \theta$$

Loads W_3 of Fig. 2a cause stresses in all members of the lower lateral system. Since these loads are carried directly to the supports A, A', C, and C', they cause stress only in the members of the bottom lateral system.

The load W_2, which acts on the side of the train, as shown in Figs. 2a and 2b, tends to cause an overturning of the entire bridge about a horizontal axis due to the fact that W_2 is applied eccentrically with respect to the main and lateral truss systems. In determining the effect of loads W_2 on the bridge, we may make use of a principle of statics which states that any force, as W_2, may be resolved into an equal force W_2 applied in any convenient plane parallel to the line of action of the original force, and a couple equal in magnitude to W_2 times the distance between the original force and the selected plane. As shown in Fig. 2b, the loads W_2 may be assumed as transferred directly by the stringers and floorbeams to the plane of the lower lateral system. Hence, by placing in the plane of the lower lateral system two equal and opposite forces W_2, we may resolve load W_2 into a horizontal force W_2 (shown by the dotted arrow) acting in the plane of the lower laterals and a couple of magnitude W_2h,

where h = distance from W_2 to plane of lower laterals. This couple W_2h is probably resisted jointly by the main and lateral trusses, since it acts on the structure as a whole. However, the main trusses are generally more rigid than the lateral trusses, and it is therefore reasonable to assume that the couple W_2h is resisted entirely by the main trusses. If the main trusses are placed at a distance b apart, as shown in Fig. 2b, the effect of the couple W_2h on the main trusses may be represented by a couple consisting of forces $\frac{W_2h}{b}$ acting in vertical planes at a distance b apart. In Fig. 2b, a downward force $W_2\frac{h}{b}$ in the plane of the leeward truss and an upward force $W_2\frac{h}{b}$ same as acting upward in the plane of the windward truss, therefore, represent the overturning action of load W_2. Similar loads are to be placed at each panel point of the main trusses.

The loads determined above cause stresses in all members of the vertical trusses. On the leeward truss, these loads act downward, as shown in Fig. 2b. Hence the character of stress in any member is similar to that for any downward uniform load. On the windward truss, the loads act upward. Hence the stresses in the members of the windward truss are equal in magnitude to those in the leeward truss, but they are opposite in character.

Similar methods of analysis apply with but slight modification to deck structures.

4. Stresses in a Through Bridge Due to Lateral Forces.—In the preceding article it was shown that lateral forces cause stresses in members of both lateral systems and also, under certain conditions, in the members of the main trusses. In general, the stresses caused in members of the main trusses by lateral loads are small when compared to the stresses in the same member due to dead load, live load, and impact. However, it was noted that the lower chord members of the main trusses received stresses from lateral loads in three ways, as follows: (1) As chord members of the bottom lateral system. This is known as *lateral truss effect;* (2) as bottom chord members of the main truss due to the overturning effect of the wind on the train. This is known as *overturning effect;* and (3) a uniform stress along the entire chord member due to overturning effect of the lateral forces on the top laterals. This is known as *portal effect.* It should also be noted that for any given member, the three stresses mentioned above are alike in character, being tension for the lower chord of the leeward truss and compression for the windward truss. Hence for the leeward truss, all stresses in the lower chord member due to lateral forces are of the same character as the stresses due to vertical loading. The sum of all lateral stresses may often be found to form an appreciable portion of the total stress in these members due to vertical and lateral loading.

The conditions mentioned above are provided for by Art. **217** of the specifications of the A.R.E.A. in the following manner:

217. Members subject to stresses produced by a combination of dead load, live load, impact, and centrifugal force, with other lateral forces and with longitudinal force, or with bending due to such forces, may be proportioned for unit stresses 25 per cent greater than those specified in Art. **301**; but the section of the member shall not be less than that required for the combination of dead load, live load, impact, and centrifugal force.

Therefore, in designing main truss members, it is unnecessary to take into account any additional stresses in these members due to lateral forces unless these lateral stresses exceed 25 per cent of the stresses due to dead load, live load, and impact. It will generally be found that it will be necessary to calculate combined stresses due to vertical and lateral loading only for the chord members of the loaded chord. Lateral stresses in the chord members of the unloaded chord (top chord in a through bridge) and in all web members of the vertical trusses are generally much below 25 per cent of the stresses due to dead and live load. Since the web members of both top and bottom lateral have stresses due only to lateral forces, these stresses must be calculated in every case. The illustrative problem given below shows the general methods of procedure to be followed in calculating stresses due to lateral forces.

It will sometimes be found that under certain live load conditions, the total compression in the chord members in the end panels of single track through bridges due to lateral forces may be so great that the tension due to dead and live load is reduced to a very small value, and may even be reversed to compression. The live loading under which this condition may exist is when a train of unloaded cars weighing about 2,400 lb. per ft. of bridge is standing motionless on the structure. Due to the light live loading and the absence of impact, the resulting live load stresses in the above mentioned chord members may be subject to reversal of stress. For this reason it is generally specified that the tension chord members in the end two panels shall be so constructed as to be capable of resisting compression.

Illustrative Problem.—Assume a structure of the dimensions shown in Fig 3 and assume that this structure is subjected to the loadings given below. Calculate (*a*) stresses in web members of top lateral system, (*b*) stresses in all members of the bottom lateral system, (*c*) stresses in lower chord members due to overturning effect, and (*d*) stresses in lower chord due to portal effect. Calculate also the stresses in lower chord members due to the given dead, live, and impact loadings, and compare these stresses with the total lateral load stresses for the same members.

The assumed loadings[1] are as follows: dead load, 1,200 lb. per ft. per truss; live load, 3,000 lb. per ft. per truss; impact, 59.1 per cent; wind load on top chord, 150 lb. per ft. of bridge; wind load on bottom chord, 200 lb. per ft. of bridge; wind load on train,

[1] Somewhat higher, for impact and wind load on train, than 1941 A R.E.A. specifications for railway bridges. A.R.E.A., however, requires that the effect of nosing of locomotives be included as an additional lateral force (Sec. 4, Art. 1).

700 lb. per ft. of bridge assumed as applied 8 ft. above top of rail. All wind loads to be considered as moving loads. Assume all lateral forces applied to the windward side of the bridge.

Stresses in Top Lateral Web Members.—Panel load = (24)(150) = 3,600 lb. per panel. Assume diagonals take tension only. For the direction of lateral forces shown by the arrow, the form of the top lateral truss is shown by Fig. 3c. Using the conventional method of loading (see p. 93) shear in panel BC = 5,400 lb.; shear in panel CD = 1,800 lb. Sec $\theta = \frac{29.8}{18} = 1.66$. Stress in BC' = (shear in BC) (sec θ) = (5,400)(1.66) = 8,950 lb. tension. Stress in CD' = (2,700)(1.66) = 4,460 lb. tension. Stress in CC' = shear in panel BC = 5,400 lb. compression. Stress in DD' = joint load at D = 3,600 lb. compression. Stress in member BB', which forms a part of the portal bracing, is determined by the methods given in Art. **8**.

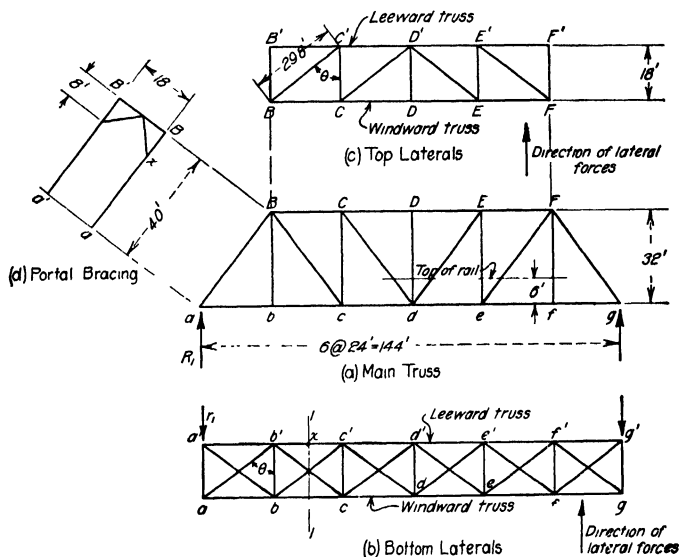



Fig. 3.

As a matter of interest, the chord stress in CD will be computed for lateral forces and for dead, live, and impact loads. For lateral forces, stress in CD = 3,600 $[(3/2)(2) - 1]^{24}/_{18}$ = 9,600 lb. compression. For dead, live, and impact loading, the panel loads on the vertical truss are each [1,200 + 3,000 + (3,000)(59.1)]24 = 143,500 lb.

Stress in CD = 143,500$[(5/2)(3) - (1 + 2)]^{24}/_{32}$ = 485,000 lb. compression. Hence, stress due to lateral loading is only 1.98 per cent of stress due to vertical loading.

Stresses in Members of Bottom Lateral System.—Total lateral load = (200 + 700) = 900 lb. per ft. Panel load = (900)(24) = 21,600 lb. Assume that web members take both tension and compression and that they are both in action, each taking one-half the shear on the section. To determine stress in any chord member, as bc, Fig. 3b cut a section 1-1 through the intersection of the diagonals and take moments about x, a point on $b'c'$ opposite the intersection of the diagonals. Since stresses in diagonals are equal but of opposite character, their moments about x are zero. Then stress in $bc = \frac{M_x}{18}$. The calculations for chord stresses are as follows:

$ab = 21{,}600\ (\frac{5}{2})(\frac{1}{2})^2\frac{24}{18} = 36{,}000$ lb. compression; $a'b' = 36{,}000$ lb. tension; $bc = 21{,}600[(\frac{5}{2})(\frac{3}{2}) - \frac{1}{2}]^2\frac{24}{18} = 93{,}600$ lb. compression; $b'c' = 93{,}600$ lb. tension; $cd = 21{,}600](\frac{5}{2})(\frac{5}{2}) - (\frac{1}{2} + \frac{3}{2})^2\frac{24}{18} = 122{,}500$ lb. compression; $c'd' = 122{,}500$ lb. tension.

Using the conventional method of loading (see p. 93) the shears in the several panels of the bottom lateral system are as follows: $ab = 54{,}000$ lb.; $bc = 36{,}000$ lb.; $cd = 21{,}600$ lb. Sec $\theta = 1.66$. Assuming that each member takes one-half the shear in the panel, the stresses in the diagonal web members for the direction of lateral forces shown by the arrow are as follows:

$ab' = (\frac{1}{2})(54{,}000)(1.66) = 44{,}600$ lb. tension; $ba' = 44{,}600$ lb. compression; $bc' = (\frac{1}{2})(36{,}000)(1.66) = 29{,}800$ lb. tension; $cb' = 29{,}800$ lb. compression; $cd' = (\frac{1}{2})(21{,}600)(1.66) = 17{,}900$ lb. tension; $dc' = 17{,}900$ lb. compression. By the same method as used on p. 319 for the determination of stress in member Gg of the K-truss of Fig. 73, it can be shown that the stresses in bb', cc', and dd' are each compression equal to one-half the joint load at b, c, and d, that is $(\frac{1}{2})(21{,}600) = 10{,}800$ lb. compression.

Stresses in Lower Chord Members Due to Overturning Effect.—The lateral load tending to overturn the truss is 700 lb. per ft. Hence panel load = (700)(24) = 16,800 lb. This load is applied 8 ft above top of rail. From Fig. 3*a*, top of rail is 6 ft. above plane of lower laterals. Hence h of Fig. 2*b* = 8 + 6 = 14 ft. Panel loads on vertical truss $= W_2\frac{h}{b} = (16{,}800)(\frac{14}{18}) = 13{,}080$ lb. The desired chord stresses may be determined from the values given in the table below for dead, live, and impact loading on the vertical truss by multiplying these stresses by the ratio of panel loads, which is $\frac{13{,}080}{143{,}500}$. The resulting stresses are given in the table below in the column headed Overturning Effect.

Stresses in Lower Chord Due to Portal Effect.—Use Eq. (2), p. 338 with $P = (\frac{5}{2})(3{,}600) = 9{,}000$; $b = 18$ ft.; $h_1 = 32$ ft.; $c = 40$ ft. = length of end post aB; $e = 8$ ft. = distance Bx, Fig. 3*d*; $\cos\theta = \frac{32}{40} = 0.8$; $\tan\theta = \frac{24}{32} = 0.75$. Substituting these values in Eq. (2) we have

$$P.E. = \frac{9{,}000}{36}[64 - (32)(0.8)](0.75) = 7{,}200 \text{ lb.}$$

Stresses in Lower Chord Due to Dead, Live and Impact Loading.—As stated above, the panel load for dead, live and impact loading is 143,500 lb. Stresses in chord members are readily determined by the usual methods. These stresses are given in the table below.

Total Stresses in Chord Members Due to Combined Vertical and Lateral Loading.—The following table gives the stress in the lower chord member of the leeward truss as calculated above for lateral and vertical loading:

Member	Lateral truss effect	Overturning effect	Portal effect	Total stress due to lateral loading	Stress due to D L, L.L., and impact on vertical truss	Stress due to lateral loading in per cent of stress due to vertical loading
$a'b'$	+ 36,000	+24,500	+7,200	+ 67,700	+269,000	25.2
$b'c'$	+ 93,600	+24,500	+7,200	+125,300	+269,000	46.6
$c'd'$	+122,500	+39,200	+7,200	+168,900	+430,000	39.3

+ = tension

The percentages given in the last column of this table show that all chord members must be proportioned for the total stress due to lateral and vertical loadings subject to the requirements of the specification quoted on p. 340.

Illustrative Problem.—Assume the same bridge and loadings as in the preceding problem except that the live and impact loading is replaced by a train of empty cars weighing 2,400 lb. per ft. of bridge. Determine the combined stress in the lower chord of the windward truss due to the revised loading.

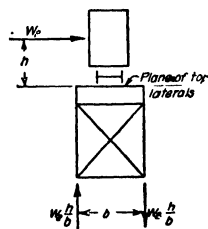
The stresses in the lower chord member of the windward truss due to lateral loading will be equal in amount but opposite in character to those found above. These stresses are as given in the table below. For dead and live load, the panel load for each truss is [1,200 + (½)(2,400)](24) = 57,600 lb. The stresses in the chord members are as given in the table.

Member	Stress due to lateral loading	Stress due to D.L. and Min.L.L.
ab	− 67,700	+108,000
bc	−125,300	+108,000
cd	−168,900	+173,500

On comparing the values given in the above table we note that a reversal of stress occurs in member *bc*. However, a wind pressure of 30 lb. per sq. ft., corresponding to a wind velocity of 100 miles per hr. represents extreme conditions which should be anticipated, but which rarely occur.

5. Stresses in Deck Bridges Due to Lateral Forces.—Stresses in the members of the top and bottom lateral systems of deck bridges are determined by the same general methods as given in the preceding article for through bridges. In determining stresses in the vertical trusses due to overturning effect of loads W_2, the panel loads $W_2\frac{h}{b}$, shown in Fig. 4, are determined for h measured from the plane of the top laterals. Note that the stress in the top chord member of the windward truss is tension due to loads acting as shown in Fig. 4. The stress in this same member due to lateral forces on the train and on the top chord is found to be compression. Since these stresses are of opposite character, it will generally be found that the combined effect of all stresses due to lateral loading will give stresses in the top chord members of a deck truss which are less than 25 per cent of those due to vertical loading.

FIG. 4.

In deck trusses, the end bracing may be provided by means of members subjected only to direct stress. The character of this bracing depends upon the manner in which the end supports are arranged. Figure 5 shows several common arrangements. In Fig. 5*a* the truss is

supported at A and the end bracing is placed in the plane of member Ab. In Fig. 5*b*, the end bracing is placed in the plane of Aa and the structure is supported at a. In Fig. 5*c* the structure is also supported at a. Loads brought to point B from the top lateral system will be transferred to point a along the two paths BAa and Ba. Since path Ba can be made much more rigid than path BAa, it is reasonable to assume that all lateral loads reach the support at a by the path Ba and the main end bracing should be placed in the plane of member aB. Bracing should

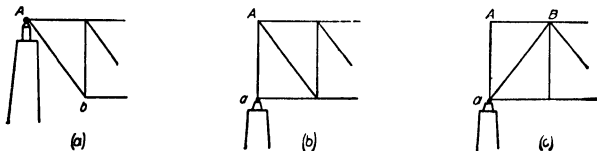



Fig. 5.

also be provided in plane Aa which will transfer to a any load which may be applied at A.

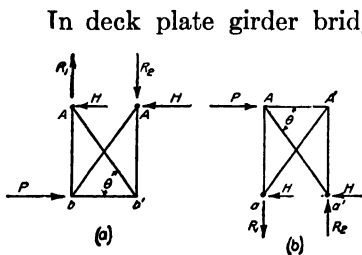
Figure 6*a* shows the type of end bracing which would be placed in the plane of Ab, Fig. 5*a*. In Fig. 6*a*, P = load brought to point b, Fig. 5*a* by the lower lateral system. Two diagonals are shown in position in Fig. 6*a*. If these members are relatively long, it will be best to assume that they take tension only. For the conditions shown, member Ab' is in action and its stress is $P \sec \theta$. Stress in bb' = joint load at b.

In deck plate girder bridges bracing of the type shown in Fig 6*b* is generally used. Since the diagonals are comparatively short, it is reasonable to assume that both members act at the same time, one in tension and the other in compression. Each member is assumed to take one-half the horizontal shear. Hence, stress in $Ab' = \frac{1}{2}P \sec \theta$, tension, stress in $A'b = \frac{1}{2}P \sec \theta$, compression. For the assumed condition, the stress in bb' is equal to $\frac{1}{2}P$ compression. Figure 6*b* shows the type of bracing which would be used in the plane Aa of Fig. 6*b* or in the plane aB of Fig. 6*c*. The methods of analysis are the same as for Fig. 6*a*.

Fig. 6.

6. Lateral Trusses in Bridges with Curved Chords.—The lateral bracing in the plane of the curved chord does not lie in any one plane but in several planes. Any load applied to this lateral truss also causes stresses in the main trusses. An exact determination of these stresses is a difficult matter. However, these stresses in the main trusses are generally small and may be neglected. A correct determination of the stresses in the web members of the lateral system may be made by the

truss as flattened out into one plane. The panel lengths of the truss thus formed are unequal in length, being equal to the lengths of the corresponding top chord members. Panel loads may still be taken as equal in amount, and they may be determined by multiplying the specified load by the horizontal panel length. Stresses may be calculated by the methods given in the preceding articles.

PORTAL AND SWAY BRACING

7. Form of Portal Bracing.—The general nature and object of portal bracing has been discussed on p. 335. Figure 7 shows a few of the forms in common use. The forms shown in Fig. 7*a*, *b*, and *c* are used when rather shallow bracing must be used because of limited vertical clearance. Figure 7*d* shows a form which is used when very shallow bracing must be used. This type of bracing lacks rigidity and is applicable only to very small trusses. The A-frame portal of Fig. 7*e* is a modified form of the one shown in Fig. 7*d*. The portal of Fig. 7*e* forms a very satisfactory portal for spans of from 100 to 300 ft. Figure 7*f* shows a form which may be used in double track two truss spans.

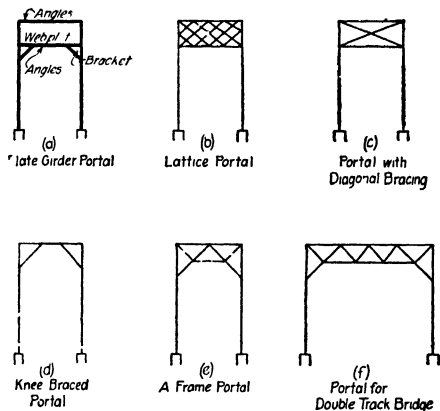



FIG. 7.

8. General Methods for Determining Stresses in Portal Bracing.—Stresses in portal frames may be determined by the same general methods as given in Art. **10**, p. 205 for the determination of stresses in a truss with knee-braces. A few modifications will now be made in the methods there given in order to take into account certain differences in the conditions existing in the two structures.

Figure 8*a* shows a typical portal frame considered as a free body in space with all forces in position. Load P applied at B' represents the load brought to the portal by the top lateral bracing. If it be assumed that the points of support, A and A', are wholly or partially fixed, the forces acting at these points are as shown in Fig. 8*a*. Due to the action of these forces, the distortion of the vertical members of the portal frame will be as shown, and the bending moment diagram for one of the posts is as shown to the left of Fig. 8*a*.

Values of H_1 and H_2, the horizontal forces acting at A and A' respectively, may be determined by the same assumptions as made on p. 206

for the determination of similar values for a roof truss with knee-braces. We then have

$$H_1 = H_2 = \frac{P}{2} \tag{1}$$

To determine the values of the other forces acting at A and A', assume the portal frame of Fig. 8a to be divided into two parts at the point of inflection. The parts above and below the point of inflection,

FIG. 8.

assumed to be located a distance x_o above the base of the portal, are shown in Figs. 8b and 8c respectively. From moments about C, Fig. 8b.

$$V_1 = \frac{P(c - x_o)}{b} \tag{2}$$

and from moments about C',

$$V_2 = \frac{P(c - x_o)}{b} \tag{3}$$

Note that V_1 and V_2 are equal in magnitude but they act in opposite directions. To determine M_o, the moment at the foot of the end posts, take moments about A or A', Fig. 8c. Neglecting the moment of V_1 and V_2 due to the deflection of the end post, we have

$$M_o = \frac{P}{2} x_o \tag{4}$$

In the work to follow, M_o of Eq. (4) will be called the *overturning moment* for the end post.

Since the conditions for the two posts are equal, the moments at the foot of the posts are also equal, and are given by Eq. (4). On solving Eq. (4) for x_o, we have

$$x_o = \frac{2M_o}{P} \tag{5}$$

Values of V_1 and V_2 may also be determined from moments about A or A' of Fig. 8*a*, from which

$$V_1 = V_2 = \frac{Pc - 2M_o}{b} \tag{6}$$

On substituting the value of M_o given by Eq. (4) in Eq. (6), the latter takes the form given by Eqs. (2) or (3).

The position of the points of inflection, C and C' of Fig. 8*a*, depends on conditions existing at the foot of the end post. As shown in Fig. 8*d* the end post bears at points a and b on a pin which is supported by the main shoe. The axis of the pin lies in the plane of the portal. As shown in Fig. 8*d*, the end post is held on the pin by the force S which is the end post stress due to loads on the vertical trusses. In general, the stress S should be calculated for minimum loading conditions on the vertical trusses which occur at the time the lateral forces acting on the top chord are a maximum. This condition will generally be found to occur when the dead load only is acting on the main trusses. Any force $\frac{P}{2}$ acting on the side of the end post tends to overturn the post by causing one of the supports to raise from the pin. This overturning is resisted by the moment of stress S about a or b. If M_R = resisting moment against overturning, we have

$$M_R = \tfrac{1}{2}Sf \tag{7}$$

in which f = distance between points of bearing on the pin.

If M_o, the overturning moment on the end post as given by Eq. (4), is less than M_R, the resisting moment as given by Eq. (7), the post may be considered as fixed at the base. When the post is fixed at the base, it can be shown that the point of inflection is located half-way between points A and E of Fig. 8*a*. Hence, for a post with fixed ends

$$x_o = \tfrac{1}{2}(c - e) \tag{8}$$

On substituting x_o from Eq. (8) in the above formulas, their values for fixed end conditions may be determined.

If in any case M_o is greater than M_R, the base of the post cannot be assumed as fixed. The moment developed at the foot of the post under

such conditions cannot exceed M_R. To locate the point of inflection when the post may not be assumed as fixed, replace M_o of Eq. (5) by M_R and solve for x_o. The value of x_o thus obtained is to be used in the above formulas.

The procedure to be adopted in determining stresses in any portal is as follows: Assume first that the end posts are fixed at the base and assume that the point of inflection is located half way between points A and E of Fig. 8a. Calculate values of M_o from Eq. (4) and M_R from Eq. (7). If M_R is greater than M_o, the assumed position of the point of inflection is correct. Remove the portion of the portal above the points of inflection, as shown in Fig. 8b and proceed with the calculations as outlined in the following articles. If M_R is less than M_o, substitute the calculated value of M_R in Eq. (5) and determine the true position of the inflection points. Proceed as before to analyze the portion of the portal above the points of inflection.

When the connection at the foot of the end posts is arranged so that no resisting moment can be developed, the portal may be assumed as hinged at the base. Under such conditions, the points of inflection may be regarded as located at the base of the portal and the calculation of stress made on that assumption. Such cases will seldom be encountered in practice.

The end posts in a portal frame of the type shown in Fig. 8 are subjected to moment, shear, and direct stress. Let x = distance from base of portal to any point, as shown in Fig. 8a. For the conditions shown, the moment at this point is

$$M_x = M_o - \frac{P}{2}x \tag{9}$$

Equation (9) holds for values of x from zero, at A', to c-e at E'. On substituting the value of M_o given by Eq. (4) we have

$$M_x = \frac{P}{2}(x_o - x) \tag{10}$$

The moment diagram represented by Eqs. (9) and (10) is of the form shown to the left of Fig. 8a. When the base of the portal is fixed, M_E and M_A are equal; when the base is partially fixed, M_E is greater than M_A.

The total direct stress in the end post below point E' is found by combining the stress in the end post due to loading on the main vertical trusses and the value of V_1 as given by Eq. (2). Note that both stresses are compression for the leeward post. For the windward post V_2 is tension. Above point E, the stress in the end post depends upon the form of portal bracing. These stresses will be considered in detail for various types of portals in the articles which follow.

The shear at any point in the end post is generally so small that it never affects the design of the end post member. From A' to E', Fig. 8a, the shear is equal to $\frac{P}{2}$. Above E', the value of the shear depends upon the arrangement of portal bracing adopted.

Maximum combined stresses in the end post, considered as a part of the portal frame, are generally determined with respect to moment and direct stress, the shear being neglected. The maximum fiber stress due to bending and direct stress will be found to occur on the inner edge of the leeward post at point E'. As stated in the discussion given above, the direct stress due to loads on the vertical truss and to V_1, the portal reaction at A', are both compression. From the character of the deformations shown in Fig. 8b, it is evident that the bending stress on the inner edge of the post at E' is also compression. Hence the maximum combined fiber stress occurs at the point mentioned above.

8a. Stresses in the Plate Girder Portal.—Figure 9 shows the portion of a plate girder portal which lies above the points of inflection, located by the methods given in the preceding article. Let c_1 = distance from top of portal to point of inflection.

The reactions at the supports may be determined from the formulas given in the preceding article. From Eq. (1)

$$H_1 = H_2 = \frac{P}{2}$$

From Eqs. (2) and (3), noting that $(c - x_o) = c_1$ we have

$$V_1 = V_2 = \frac{Pc_1}{b}$$

These reactions are shown in position on Fig. 9a. Values of V_1 and V_2 may also be obtained from moments about C and C' of Fig. 9a. Thus, $V_1b - Pc_1 = 0$, from which

$$V_1 = \frac{Pc_1}{b}$$

Bending moments in the posts are a maximum at E' and E. These moments are $M_E = M_{E'} = \frac{P}{2}(c_1 - e)$. At the foot and top of the posts the moments are zero. The variation in moment is as shown in the moment diagram on the left of Fig. 9a. The direct stress in the posts due to load P is equal to $V_1 = P\frac{c_1}{b}$ compression for the leeward post and $V_2 = \frac{Pc_1}{b}$ tension for the windward post. The direct stress diagram to the left of Fig. 9a shows the variation in direct stress from C' to E'. Note that the stress is constant. Above point E' the direct stress is taken up

by the shear at the junction between the end post and the web, becoming zero at the top of the post. The direct stress diagram above E' is not shown; it depends upon the assumptions made regarding the variation in shear across the web.

Maximum fiber stress due to bending and direct stress occurs on the inside edge of the leeward post at E'. The direct stress includes V_1 and the end post stress in the main truss due to dead and live load. Fiber stresses due to direct stress and bending are all compressive at this point, as stated in the preceding article.

The internal stresses in the plate girder connecting the two posts will be determined on the assumption that the web plate takes all of the shear and that the angles forming the flanges carry all of the moment.

FIG. 9.

To determine these internal stresses, cut a section 1-1 distance x from the leeward post and consider the portion to the left of this section, which is shown in Fig. 9b. Let F_1 and F_2 = stresses in top and bottom flange respectively, and let V = shear on section. From a summation of vertical forces

$$V = V_1 = \frac{Pc_1}{b}$$

Since V is independent of x, the shear is constant across the whole beam.

Values of the flange stresses may be derived from moments about section 1-1. To determine stress in the upper flange, take moments about F_2 at section 1-1, from which

$$\frac{Pc_1}{b}x - \frac{P}{2}(c_1 - e) + F_1 e = 0$$

Solving for F_1 we have

$$F_1 = \frac{Pc_1}{be}\left(\frac{b}{2} - x\right) - \frac{P}{2} \tag{11}$$

The variation in stress across the upper flange may be determined by substituting particular values of x in Eq. (11). At B', where $x = 0$,

$$F_1 = +\frac{P}{2e}(c_1 - e) \tag{12}$$

Since e is less than c_1, the upper flange stress at B' is tension. At B, the right end of the upper flange, $x = b$, and we have

$$F_1 = -\frac{P}{2e}(c_1 + e) \tag{13}$$

The minus sign indicates that the flange stress at B is compression. Since x appears in Eq. (11) in the first power, the variation in stress across the flange is represented by a straight line, as shown above Fig. 9a. At the center of the flange, where $x = \frac{b}{2}$, we find that $F_1 = -\frac{P}{2}$, the minus sign again denoting compression.

The stress in the lower flange is given by moments about F_1 at section 1-1, from which

$$F_2 = \frac{Pc_1}{be}\left(\frac{b}{2} - x\right) \tag{14}$$

At E', where $x = 0$

$$F_2 = \frac{Pc_1}{2e} \text{ compression.} \tag{15}$$

and at E, where $x = b$

$$F_2 = \frac{Pc_1}{2e} \text{ tension.} \tag{16}$$

The variation in stress across the flange is shown above Fig. 9a. At the center of the flange, where $x = \frac{b}{2}$, $F_2 = 0$. In designing portals of this type, upper and lower flange sections are generally made of the same size, using the stress given by Eq. (13).

Portals of this type are sometimes provided with brackets attached to the lower flange and the end posts, as shown in Fig. 7a. In determining stresses the presence of these brackets is usually disregarded and the stresses determined as above. If the brackets are very rigid it is reasonable to assume that the section of maximum moment in the end posts occurs at the foot of the brackets instead of at the lower flange.

Illustrative Problem.—Assume that the truss of Fig. 3, p. 341 is supplied with a plate girder portal of the form and dimensions shown in Fig. 10a. Using the loadings given in the problem on p. 340, calculate the flange stresses at the four corners of the plate girder, and calculate also the maximum bending moment and direct stress in the leeward end post.

The method of procedure to be adopted in the solution of problems of this nature has been outlined on p. 348. Assume first that the point of inflection is located halfway between A and E of Fig. 10a. Then $x_o = \frac{1}{2}(40 - 6) = 17$ ft. The panel load at each top chord joint of Fig. 3 is $(150)(24) = 3{,}600$ lb. Hence the load P to be carried by the portal is $(\frac{5}{2})(3{,}600) = 9{,}000$ lb. Substituting these values of x_o and P in Eq. (4), we have $M_o = \frac{P}{2}x_o = (\frac{1}{2})(9{,}000)(17) = 76{,}500$ lb. The resisting moment must be determined for the most unfavorable loading conditions existing on the main truss. Since the top chord lateral loadings are independent of the live load, the end post stress is to be determined for dead load only. Assumed dead load = 1,200 lb. per ft. per truss. Panel load = $(24)(1{,}200) = 28{,}800$ lb. Stress in end post of Fig. 3 = $S = (\frac{5}{2})(28{,}800)(\frac{5}{4}) = 90{,}000$ lb. compression. Assume points of support on pin (f of Fig. 8d) = 18 in. Then from Eq. (7) $M_R = \frac{1}{2}Sf = (\frac{1}{2})(90{,}000)(\frac{18}{12}) = 67{,}500$ ft.-lb. The post is not fixed as assumed, as M_o is greater

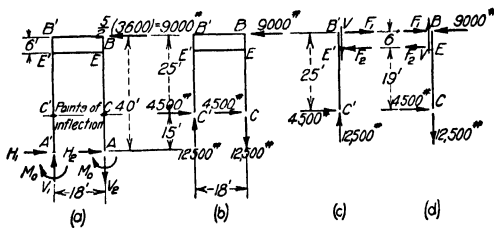



FIG. 10.

than M_R. To determine the true location of the point of inflection, use Eq. (5), replacing M_o with $M_R = 67{,}500$, and with $P = 9{,}000$ Then $x_o = \frac{2M_o}{P} = \frac{(2)(67{,}500)}{(9{,}000)} = 15$ ft. The true point of inflection is therefore 15 ft. above the base of the post, or $(40 - 15) = 25$ ft. below the top. Figure 10b shows the dimensions of the portal to be analyzed.

To determine the flange stress at B, Fig. 10b, use Eq. (13) with $P = 9{,}000$; $c_1 = 25$ ft.; and $e = 6$ ft. Then

$$F_1 = \frac{9{,}000}{(2)(6)}(25 + 6) = 23{,}250 \text{ lb. compression.}$$

The stress at B' is given by Eq. (12), from which

$$F_1 = \frac{9{,}000}{(2)(6)}(25 - 6) = 14{,}250 \text{ lb. tension.}$$

Stresses in the lower flange are given by Eqs. (15) and (16), from which $F_2 = \frac{(9{,}000)(25)}{(2)(12)} = 18{,}750$ lb. compression at E' and tension at E.

The stresses in the flanges may also be determined directly by cutting sections close to the posts and taking moments. Figure 10d shows a section taken close to the windward post. To determine F_1, take moments about E, and we have $-(19)(4{,}500) - (9{,}000)(6) + F_1 6 = 0$ from which $F_1 = 23{,}250$ lb. as before. Values of the remaining forces may be determined in a similar manner.

The maximum bending moment in the end post occurs at point E'. From Fig. 10b,

$$M_{E'} = (4{,}500)(25-6) = 85{,}500 \text{ ft.-lb.}$$

This moment may also be determined from Eq. (10) with $x_o = 15$, and $x = (40 - 6) = 34$, from which $M_{E'} = (4{,}500)(34 - 15) = 85{,}500$ ft.-lb. The maximum direct stress in the end post must include the effect of dead, live, and impact loading on the main truss. From the problem on p. 340, the total panel load on the vertical trusses is 143,500 lb. Hence, stress in end post, Fig. 3, = (143,500)(5/2)(5/4) = 448,500 lb. From Eq. (2) or from moments about C, Fig. 10b, $V_1 = \frac{(9{,}000)(25)}{18} = 12{,}500$ lb. Therefore total direct stress in end post = 448,500 + 12,500 = 461,000 lb. compression.

8b. Stresses in the Lattice Portal.—Stresses in the lattice portal of Fig. 11 are determined by the same methods as used in the preceding article. To determine the stresses in the diagonal web members, cut any section 1-1 through the intersection of the diagonals whose stress is desired. Let D_1 and D_2 indicate the stresses in these members. Assuming the shear on section 1-1 as distributed equally between the members cut by the section, we have

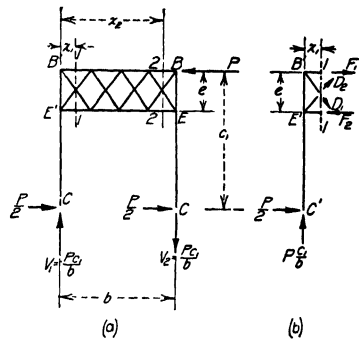
$$D_1 = D_2 = \frac{Pc_1}{2b} \sec \theta \qquad (17)$$

Fig. 11.

Stress D_1 is tension and D_2 is compression. The portal of Fig. 7b is so arranged that four diagonals are cut by any section. If n = number of diagonals cut by any section, as 1-1 Fig. 11, Eq. (17) may be written in the form

$$D = \frac{Pc_1}{nb} \sec \theta \qquad (18)$$

The members sloping in one direction will be in tension, and those sloping in the other direction will be in compression. Since the shear is constant across the entire portal, the stresses in all diagonals will be equal.

To determine the stresses in the flanges of the portal of Fig. 11, cut a vertical section 1-1 through the intersection of the diagonals, as shown in Fig. 11b. Since the stresses in the diagonals cut by section 1-1 are equal but of opposite character, their moments about any point will be zero. For stress F_1, take moments about F_2 at its intersection with section 1-1. The resulting expression is the same as given by Eq. (11). In the same manner, the stress in F_2 is the same as given by Eq. (14). For the form of portal shown in Fig. 11, the least value of x_1 is $\frac{b}{8}$ and the maximum value is $\frac{7}{8}b$, as shown for section 2-2. Maximum stress in the

flange occurs in the top flange at section 2-2. With $x_1 = \frac{7}{8}b$, we have from Eq. (11),

$$F_1 = \frac{P}{8e}(3c_1 + 4e) \tag{19}$$

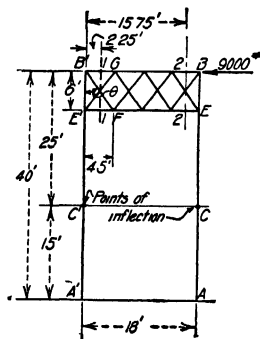
This stress is compression. Maximum stress in the lower flange occurs at either section 1-1 or section 2-2. For section 1-1 with $x_1 = \frac{1}{8}b$, Eq. (14) gives

$$F_2 = \frac{3Pc_1}{8e} \tag{20}$$

Fig. 12.

At section 1-1, F_2 is compression and at section 2-2, F_2 has the value given by Eq. (20) but the stress is tension.

Moment shear and direct stress in the posts are the same as given in the preceding article.

Illustrative Problem.—Assume that the truss of Fig. 3, p. 341 is constructed with a portal of the form and dimensions shown in Fig. 12. Calculate the maximum top and bottom flange stresses and the stresses in the diagonal members for the loadings given in the problem on p. 340.

Since the loadings and over all portal dimensions are the same as for the problem on p. 351 the points of inflection will again be located 15 ft. above the base of the portal, as shown in Fig. 12. Then c_1 = 25 ft. Load P = 9,000 lb.

The stresses in the diagonals may be obtained from Eq (17) or from Eq. (18) with $n = 2$. For the dimensions shown in Fig. 12, $\sec\theta = \frac{B'F}{E'B'} = \frac{7.5}{6} = 1.25$. Then, stress in $B'F = \frac{(9{,}000)(25)}{(2)(18)}(1.25) = 7{,}820$ lb. compression. Stress in $E'G = 7{,}820$ lb. tension. Stresses in other diagonals are the same as for those given.

To determine the maximum lower chord flange stress consider forces acting on section 1-1, Fig. 12, or use Eq. (20). From Eq. (20)

$$F_2 = \frac{(3)(9{,}000)(25)}{(8)(6)} = 14{,}060 \text{ lb. compression.}$$

Maximum stress in top flange may be determined from moments on section 2-2, Fig. 12, or from Eq. (19). Thus

$$F_1 = \frac{(9{,}000)}{(8)(6)}[(3)(25) + (4)(6)] = 18{,}560 \text{lb. compression.}$$

Note that Eqs. (17), (19) and (20) apply only to the form of portal bracing shown in Fig. 11.

8c. Stresses in a Portal with Diagonal Bracing.—Figure 13 shows the portion of the portal above the points of inflection. Two diagonals are usually provided, each capable of taking tension only. For the direction of lateral force shown in Fig. 13, the full line member

$B'E$ is in action and the dotted member BE' is inactive. When P is applied at B', acting to the right, the action of the diagonals is reversed. The reactions at the points of inflection are as shown in Fig. 13a.

To determine the stresses in the portal members, cut a section 1-1 and remove the portion of the structure to the left of this section, which is shown in Fig. 13b. From a summation of vertical forces,

$$S_2 = \frac{Pc_1}{b} \sec \theta = \frac{Pc_1 f}{be} \text{ tension.} \quad (21)$$

From moments about E,

$$S_1 = \frac{P}{2e}(c_1 + e) \text{ compression.} \quad (22)$$

Fig. 13.

and from moments about B',

$$S_3 = \frac{Pc_1}{2e} \text{ compression.} \quad (23)$$

The bending stresses in the posts are the same as for the portal of Art. 8a. For the leeward post, the direct stress due to load P is uniform from C' to B' and equal to $\frac{Pc_1}{b}$ compression. For the windward post, the direct stress from E to C is tension equal to $\frac{Pc_1}{b}$. Above E, the direct stress is zero. The shear in both the posts from the base to the lower edge of the portal is equal to $\frac{P}{2}$. In the leeward post from E' to B', the shear is $\frac{P}{2}$ − stress in $E'E = \frac{P}{2}\left(1 - \frac{c_1}{e}\right)$. Since c_1 is always greater than c, the shear above E' is opposite in character to that below E'. In the windward post from B to E, the shear is P − stress in $B'B = \frac{P}{2}\left(1 - \frac{c_1}{c}\right)$, the same as for the leeward column.

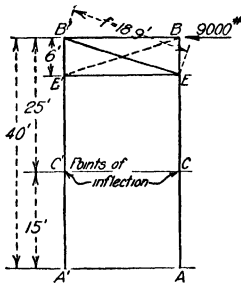
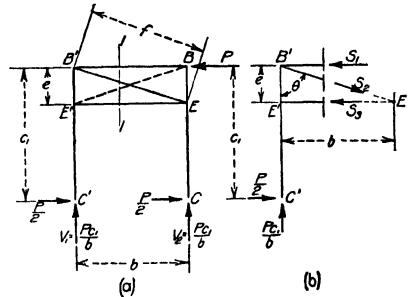



Fig. 14.

Illustrative Problem.—Assume that the truss of Fig. 3 is constructed with a portal of the form and dimensions shown in Fig. 14. Calculate the stresses in BB', $B'E$, and EE' for the loadings given in the problem on p. 310.

Since the loadings and over all portal dimensions are the same as for the problem on p. 351, the points of inflection will again be located 15 ft. above the base of the portal, as shown in Fig. 14. Then c_1 = 25 ft. Load P = 9,000 lb. For the dimensions shown on Fig. 14, f = length of $B'E$ = 18.9 ft.

From Eq (21)

$$\text{Stress in } B'E = \frac{(9{,}000)(25)(18.9)}{(18)(6)} = 39{,}400 \text{ lb. tension.}$$

From Eq. (22)

$$\text{Stress in } BB' = \frac{(9{,}000)}{(2)(6)}(25 + 6) = 23{,}250 \text{ lb. compression.}$$

From Eq. (23)

$$\text{Stress in } EE' = \frac{(9{,}000)}{(2)(6)}(25) = 18{,}750 \text{ lb. compression.}$$

8*d*. Stresses in a Knee-braced Portal.—Figure 15 shows the portion of a knee-braced portal frame which lies above the points of

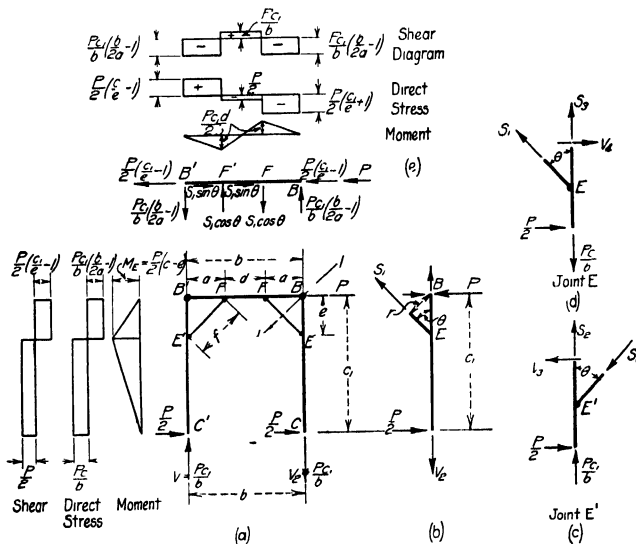



FIG. 15.

inflection. Member BB' and the end posts form continuous members which are subjected to moment, shear, and direct stress. Members EF and $E'F'$, the knee-braces, are subjected only to direct stress. In determining stresses in the members of this portal, it is generally assumed that the end posts are connected to member BB' by means of hinges at B and B', and that EF and $E'F'$ are hinged to the sides of the other members, which are continuous over these points of attachment. The reactions at C and C' are the same as for the portals analyzed in the preceding articles.

To determine the stress in EF, pass a section 1-1 through point B and remove the portion of the frame which lies to the right of this section, as shown in Fig. 15*b*. Since BB' and BC are hinged at B, the moment at this point is known to be zero. Placing equal to zero a moment

equation for point B, Fig. 15b, we find that $S_1 = \frac{Pc_1}{2r}$. But from similar triangles in Figs. 15a and b, $r = \frac{ea}{f}$. Hence

$$\text{Stress in } FE = \frac{Pc_1 f}{2ea} \text{ tension.} \tag{24}$$

The stress in $E'F'$ is equal in magnitude but opposite in kind to that in EF.

Moments, shears and direct stresses in the end posts below E and E' are the same as for the portals considered in the preceding articles. The variation in moment is shown by the diagram to the left of the leeward post. The moment diagram for the windward post is exactly the same. To determine the direct stress in the leeward post above joint E', consider vertical components of forces acting on joint E' as shown in Fig. 15c, from which $S_2 - S_1 \cos\theta + \frac{Pc_1}{b} = 0$. Hence

$$S_2 = S_1 \cos\theta - \frac{Pc_1}{b} = \frac{Pc_1}{b}\left(\frac{b}{2a} - 1\right) \tag{25}$$

S_2 is tension, since a is always less than $\frac{b}{2}$. Below E', the direct stress in the post is $\frac{Pc_1}{b}$ compression. The direct stress diagram to the left of the leeward post in Fig. 15a shows the variation in direct stress for the leeward post. In the same manner, the direct stress in EB, as given by summation of vertical forces in Fig. 15d is $S_3 = -\frac{Pc_1}{b}\left(\frac{b}{2a} - 1\right)$. The minus sign indicates compression. It is to be noted that if a is less than $\frac{b}{4}$, the compression in BE will exceed the compression in $E'C'$.

The shear in the leeward post above joint E' may be determined from a summation of horizontal forces in Fig. 15c, from which

$$V_3 = S_1 \sin\theta - \frac{P}{2} = \frac{P}{2}\left(\frac{c_1}{e} - 1\right) \tag{26}$$

Since $\frac{c_1}{e}$ is greater than unity, the shear above point E' acts as shown in Fig. 15c. Below E', the shear is $\frac{P}{2}$, acting as shown by the horizontal force $\frac{P}{2}$ in Fig. 15c. The complete shear diagram is shown to the left of the leeward post in Fig. 15a. Shears for the windward post have the same values as for the leeward post.

The stresses in the upper member BB' may be determined by removing this member from the frame and indicating all forces acting on the

member. Figure 15e shows all forces acting on member BB'. The horizontal and vertical loads at B' are the shear and direct stress, respectively, in $B'E'$. At B, the vertical force is the direct stress in BE; one horizontal force is the shear in BE and the other is the load P. The forces at F and F' are the vertical and horizontal components of the stress in the knee-braces.

The moment at F of member BB' for the conditions shown on Fig. 15e is

$$M_F = \frac{Pc_1}{b}\left(\frac{b}{2a} - 1\right)a = \frac{Pc_1}{b}\left(\frac{b}{2} - a\right)$$

But $\frac{b}{2} - a = \frac{d}{2}$. Hence

$$M_F = +\frac{Pc_1 d}{2b} \tag{27}$$

It can readily be seen that the moment at F' is equal to that at F, but is negative. Since BB' is assumed as hinged at the ends, moments at B and B' are zero. These moments are plotted to form the moment diagram of Fig. 15e. Note that the moment is zero at the center of BB'.

The direct stress in BB' can readily be determined by considering summations of horizontal forces acting on the several parts of the member. From B' to F',

$$\text{Stress in } B'F' = +\frac{P}{2}\left(\frac{c_1}{e} - 1\right) \tag{28}$$

The plus sign indicates tension.

From F' to F,

$$\text{Stress in } F'F = \frac{P}{2}\left(\frac{c_1}{e} - 1\right) - S_1 \sin\theta = -\frac{P}{2} \tag{29}$$

The minus sign indicates compression.

From B to F,

$$\text{Stress in } BF = -\left[\frac{P}{2}\left(\frac{c_1}{e} - 1\right) + P\right] = -\frac{P}{2}\left(\frac{c_1}{e} + 1\right) \tag{30}$$

The minus sign indicates compression. These stresses are plotted to form the direct stress diagram of Fig. 15e.

The shears in the several parts of BB' may be determined from a summation of vertical forces for the conditions shown on Fig. 15e. For $B'F'$,

$$\text{Shear in } B'F' = -\frac{Pc_1}{b}\left(\frac{b}{2a} - 1\right) \tag{31}$$

This shear is negative, since the vertical force at B' acts downward. For $F'F$,

$$\text{Shear in } F'F = -\frac{Pc_1}{b}\left(\frac{b}{2a} - 1\right) + S_1 \cos\theta$$

$$= +\frac{Pc_1}{b} \tag{32}$$

This shear is positive, as indicated by the sign of Eq. (32). For FB,

$$\text{Shear in } FB = -\frac{Pc}{b}\left(\frac{b}{2a} - 1\right) + S_1 \cos\theta - S_1 \cos\theta$$
$$= -\frac{Pc}{b}\left(\frac{b}{2a} - 1\right) \qquad (33)$$

This shear is also negative. The variation in shear is as shown in the shear diagram of Fig. 15e.

Fig. 16.

Illustrative Problem.—Assume that the truss of Fig. 3 is constructed with a portal of the form and dimensions shown in Fig. 16. Calculate the stress in FE and $F'E'$ and the moment, direct stress and shear in member $B'B$ due to the loadings given in the problem on p. 340.

Since the loadings and over all portal dimensions are the same as for the problem on p. 351, the points of inflection will again be located 15 ft. above the base of the portal, as shown in Fig. 16. Then c_1 = 25 ft. Load P = 9,000 lb.

For the dimensions shown on Fig. 16, the stress in FE, as given by Eq. (24), is

$$FE = \frac{(9{,}000)(25)(8.5)}{(2)(6)(6)} = 26{,}600 \text{ lb. tension.}$$

Stress in $F'E'$ = 26,600 lb. compression.

The moments in $B'B$, as given by Eq. (27), are as follows:

$$M_F = +\frac{(9{,}000)(25)(6)}{(2)(18)} = 37{,}500 \text{ ft.-lb.}$$

At F', M_F = − 37,500 ft.-lb.

The direct stresses in $B'B$, as given by Eqs. (28), (29) and (30), are as follows:

$$B'F' = +\frac{(9{,}000)}{2}\left(\frac{25}{6} - 1\right) = 14{,}250 \text{ lb. tension.}$$
$$FF' = -\frac{P}{2} = 4{,}500 \text{ lb. compression.}$$

and

$$BF = -\frac{(9{,}000)}{2}\left(\frac{25}{6} + 1\right) = 23{,}250 \text{ lb. compression.}$$

The shears in $B'B$, as given by Eqs. (31), (32) and (33), are as follows:

$$B'F' = -\frac{(9{,}000)(25)}{18}\left(\frac{18}{12} - 1\right) = -6{,}250 \text{ lb.}$$
$$F'F = +\frac{(9{,}000)(25)}{18} = +12{,}500 \text{ lb.}$$
$$FB = B'F' = -6{,}250 \text{ lb.}$$

8e. Stresses in an A-frame Portal.—The stresses in the A-frame portal shown in Fig. 7e may be determined as a special case of the knee-braced portal of Fig. 7d. It can be seen from these figures that if the knee-braces of Fig. 7d are so arranged as to meet at a point, the full line portal of Fig. 7e will be obtained. Hence by placing $d = 0$ and $a = \frac{b}{2}$ in the equations of the preceding article, the desired stresses for the A-frame portal may be determined.

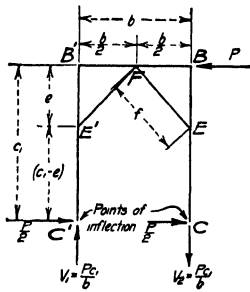



Fig. 17

Figure 17 shows the portion of the portal under consideration which lies above the points of inflection. To determine the stress in FE, place $a = \frac{b}{2}$ in Eq. (24), and we have

$$\text{Stress in } FE = \frac{Pc_1f}{be} \text{ tension} \tag{34}$$

Stress in FE' is equal in amount but is compression.

The direct stresses in $B'E'$ and BE, the upper ends of the end posts, are found to be zero, for on substituting $a = \frac{b}{2}$ in Eq. (25) a zero value results. The shears in these portions of the posts are unchanged, as Eq. (26) for $B'E'$ and the corresponding value for BE do not contain a.

The moment and shear in the upper member $B'FB$ reduce to zero, for on substituting $d = 0$ in Eq. (27) and $a = \frac{b}{2}$ in Eqs. (31) and (33), these equations reduce to zero. Member $B'FB$ is therefore subjected to direct stress only. It will be found that Eqs. (28) and (30) give directly the desired stresses in $B'FB$. We then have

$$\text{Stress in } B'F = +\frac{P}{2}\left(\frac{c_1}{e} - 1\right) \tag{35}$$

and

$$\text{Stress in } BF = -\frac{P}{2}\left(\frac{c_1}{e} + 1\right) \tag{36}$$

The plus sign in Eq. (35) indicates tension and the minus sign in Eq. (36) indicates compression.

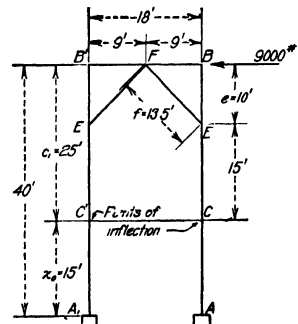



Fig. 18.

Illustrative Problem.—Assume that the truss of Fig. 3 is constructed with a portal of the form and dimensions shown in Fig. 18. Calculate the stresses in BF, $B'F$, FE and FE' due to the loadings given in the problem on p. 340.

Since a change has been made in the depth of the portal bracing, it will be necessary to locate the points of inflection. On p. 352 the dead load stress in the end post is

given as 90,000 lb. Assume in this case that the points of support on the pin are 2 ft. apart. Then from Eq. (7), $M_R = (\frac{1}{2})(90{,}000)(2) = 90{,}000$ ft.-lb. Assuming that the point of inflection is located half-way between A and E of Fig. 18, we have $x_o = 15$ ft. From Eq. (4), $M_o = (\frac{1}{2})(9{,}000)(15) = 67{,}500$ ft.-lb. Hence the portal is fixed at the base, since M_o is less than M_R. The points of inflection are located at C, 15 ft. above the base as shown in Fig. 18, and the effective height of the portal frame is the same as before.

From Eq. (34)

$$FE = \frac{(9{,}000)(25)(13.5)}{(18)(10)} = 16{,}900 \text{ lb. tension}.$$

Stress in FE' = 16,900 lb. compression.

From Eq. (35),

Stress in $B'F = (\frac{1}{2})(9{,}000)(\frac{25}{10} - 1) = 6{,}750$ lb. tension.

From Eq. (36)

Stress in $BF = (\frac{1}{2})(9{,}000)(\frac{25}{10} + 1) = 15{,}750$ lb. compression.

9. Determination of Portal Effect.—In Art. 4 it was shown that the overturning effect of lateral loads on the top lateral system produced a stress in the lower chords of the main truss. This stress was called the portal effect.

Stresses due to portal effect may also be determined from a consideration of the direct stresses acting at the base of the portal frame. The loads V_1 and V_2 which act at the foot of the portal are transferred to the abutments through the main shoes. At the roller end, the vertical component of V_1, indicated by R_1 in Fig. 19, is taken by the supports. The horizontal component of V_1 cannot be taken by the rollers, but must be transferred by the lower chord to the fixed end of the truss. This causes a uniform stress which is

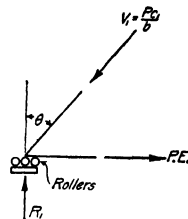



Fig. 19.

$$\text{P.E.} = V_1 \sin\theta = \frac{Pc_1}{b} \sin\theta \tag{37}$$

On the leeward side of the main truss, P.E. is tension and on the windward side P.E. is compression.

Illustrative Problem.—Calculate the portal effect for the truss of Fig. 3, using the method of this article. Assume the portal to be fixed at the base.

For the data given in the problem of Art. 4, $P = 9{,}000$ lb.; $c_1 = 24$ ft.; $b = 18$ ft.; and $\sin\theta = \frac{24}{40} = 0.6$. Then from Eq. (37)

$$\text{P.E.} = \frac{(9{,}000)(24)}{18}(0.6) = 7{,}200 \text{ lb.}$$

This checks the value given in the table on p. 342.

10. Skew Portals.—Figure 20 shows a skew portal and a portion of the end of the main truss and the lateral systems. The load P brought to the portal by the top lateral system may be resolved into the compo-

nents P sec α and P tan α. The former is the load to be carried by the portal, as shown in Fig. 20*b*, and the latter force causes additional stresses in the top chord members.

The reactions at the base of the portal may be assumed to act as shown in Fig. 20*b*. Then $H_1 = H_2 = \frac{1}{2}P \sec \alpha$, and $V_1 = V_2 = (P \sec \alpha)\frac{h_1}{b}$.

The portal effect for this case is P.E. $= V_1 \sin \theta + H_1 \sin \alpha$.

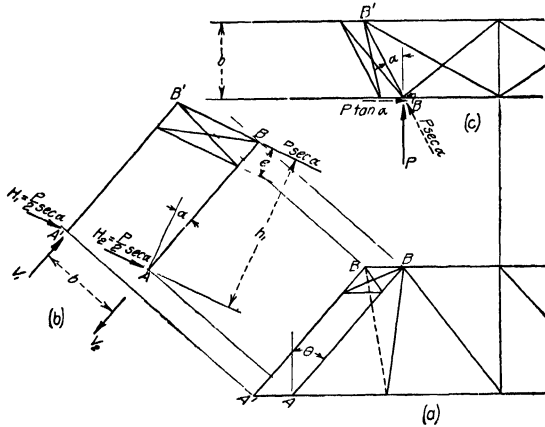



Fig. 20.

11. Stresses in Sway Bracing.—The object of sway bracing is to stiffen the structure against lateral vibrations. It is generally constructed from the minimum sizes of rolled sections with the maximum slenderness ratios for compression members which are allowable under

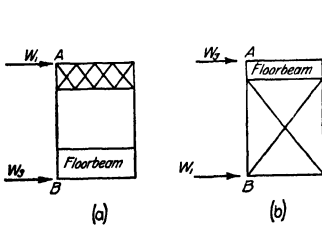
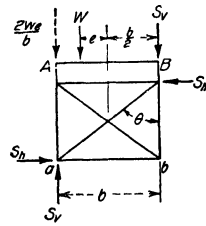



Fig. 21. Fig. 22.

good specifications. Sway bracing proportioned in this manner will be found to answer the purpose for which it is intended.

Stresses in sway bracing are in general statically indeterminate, for as stated on p. 334, sway bracing is added to a structure which is already stable. Hence, in determining stresses in sway bracing, the relative flexibility of all parts of the span must be taken into consideration.

Figures 21*a* and *b* show typical forms of sway bracing for through and deck bridges respectively. If the upper and lower lateral trusses are

subjected to full loading, and if the relative rigidities of the two systems are equal, the horizontal deflection of points A and B will be equal. Hence the sway bracing will be unstressed. When the live load is not on the structure, the lower chord in the through bridge of Fig. 21a is subjected to about one-third of its maximum load while the top chord is fully loaded. If the relative rigidities of the two lateral systems are equal, their horizontal deflections will not be equal. This will cause distortion of the sway bracing and a portion of the load on the top lateral system will be transferred to the lower lateral system, causing stresses in the sway bracing. The amount of load thus transferred depends upon the relative flexibility of the several truss systems under the given loading. It is reasonable to assume that one-half the lateral load may be so transferred, and the sway bracing may be designed accordingly, using the same methods as given in the preceding articles for portal frames. A similar analysis may also be applied to the sway bracing shown in Fig. 21b.

Some specifications require the sway bracing in double track structures to be designed to equalize the eccentric effect of live load on one track so that the deflections of the two main trusses shall be equal. Figure 22 shows a double track deck bridge with a load W representing the load on one track. The floorbeam reaction at A is $\frac{W}{b}\left(\frac{b}{2} + e\right)$ and at B, the load is $\frac{W}{b}\left(\frac{b}{2} - e\right)$. Hence the excess of load at A over that at B is $\frac{2We}{b}$, which is shown in position at A, Fig. 22. Assuming tension diagonals, aB must transfer half this load to B, and its stress will be $\frac{We}{b} \sec \theta$. However, in order that aB may receive this stress, it is necessary that the lateral bracing in the planes ab and AB must be able to transfer to the abutments the forces S_h (horizontal component of stress in aB) and similar forces at other joints, without causing unequal distortions of the lateral systems. This condition is seldom realized and as a result, the amount of load transferred by aB to the other truss is less than one-half, as assumed. The exact amount can be determined by the general methods given in the chapter on Stresses in Redundant Members.

SECTION 5

DEFLECTION OF TRUSSES, REDUNDANT MEMBERS AND SECONDARY STRESSES

DEFLECTION OF TRUSSES

1. General Considerations.—The deflection of a truss is the result of changes in length of members or distances between panel points, whether caused by strains, temperature, errors in the lengths of members during fabrication, or play between the pins and pin-holes. The treatment in all cases is the same after the changes in length have been determined.

The stress in any member of a truss under the influence of external forces is accompanied by a corresponding strain or change in length of the member. The strains which the various members experience when under stress, cooperate in a general distortion of the truss, somewhat after the manner illustrated in Fig. 1. The full lines represent the configuration of the truss when the members are under no strain. When loads are applied gradually at B, C and G, the truss undergoes a gradual distortion, and finally conforms to the shape indicated by the dotted lines, when the loads have reached their full magnitudes, P_1, P_2 and P_3. The dotted outline gives an exaggerated idea of the distortion which may be expected in any practical problem.

The first step in finding the movement or displacement of any point in a truss subjected to any set of applied loads is the determination of the strain or deformation of each member subjected to stress. This is readily accomplished by means of the well established law of mechanics that

$$D = \frac{Sl}{AE}$$

where

D = strain, or change in length of the member.
S = stress in the member.
l = length of member.
A = area of cross-section.
E = modulus of elasticity of material composing the member.

Hence the strain or change in length of a member may be determined if its length and cross-sectional area, the stress which it resists and the modulus of elasticity of its material are known. Experiments show that for any given material, the modulus of elasticity is approximately constant for all unit stresses below a certain limit, called the elastic limit. The elastic limit for structural steel is about 60 per cent of its ultimate strength; hence the permissible unit stress in all current practice is well within the elastic limit. The modulus of elasticity for ordinary structural steel is about 29,000,000 lb. per sq. in.; hence if a member 50 ft. long, having a cross-sectional area of 20 sq. in., is subjected to a tensile stress of 300,000 lb., the strain or elongation of the member will be

$$D = \frac{(300,000)(50)(12)}{(20)(29,000,000)} = 0.31 \text{ in.}$$

Deflections of trusses may be determined by algebraic or by graphical methods. In general, algebraic solutions are more precise than graphical solutions, for algebraic solutions can be carried to any desired number of significant figures in the results while graphical solutions are limited by the size of drawing which can conveniently be made and also by inherent inaccuracies which exist in all graphical work. However, a complete determination of the movement of all joints of a truss by the algebraic methods in common use requires two deflection calculations for each joint of the truss (vertical and horizontal movement of each joint). On the other hand, a graphical solution gives a single displacement diagram from which the resultant motion of all joints of the truss can be determined at the same time.

In practically all of the problems encountered in practice, the graphical solutions yield results which are sufficiently accurate. It is seldom that algebraic methods are required to produce the desired accuracy. Graphical methods are therefore recommended for use because of the time which can be saved by their use, and because a greater amount of information can be obtained from a single solution. Both methods will be treated in the articles which follow.

2. Algebraic Method.

2a. **Deflection Due to Stress in Members.**—The algebraic solution for this case may be developed by equating the external work done by the load supported by the truss to the internal work performed by the members of the truss.

Figure 1 shows a truss supporting loads P_2 and P_3 due to any given loading conditions. The vertical component of the deflection of joint B due to these loads is required. To determine the required deflection apply at joint B an arbitrary load P_1 acting in the direction of the desired deflection. For the given conditions P_1 is a vertical load acting downward, as shown in Fig. 1.

Let Δ_1, Fig. 1, represent the vertical component of the deflection of joint B in the direction of the force P_1 due to the loads P_1, P_2 and P_3 which are assumed to be applied gradually, increasing from zero to their full values. During the movement of the point from B to B', the force P_1 performs work to the amount of $\frac{1}{2}P_1\Delta_1$. Each member under stress, and thereby subject to strain, contributes its share to the deflection of B. Consider any member HK of length l and area A Let S_1, S_2 and S_3 represent the stresses in HK caused by the loads P_1, P_2 and P_3

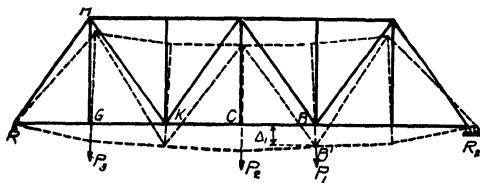



FIG. 1.

respectively, and let $S_1 + S_2 + S_3 = S =$ the total stress. The strain in HK is

$$D = \frac{Sl}{AE}$$

and the work performed upon HK resulting from the load P_1 is

$$\frac{1}{2}S_1D = \frac{1}{2}S_1\frac{Sl}{AE}$$

Let $\Sigma\frac{1}{2}S_1\frac{Sl}{AE}$ represent the sum of the work performed upon each member by the stress resulting from the load P_1. Since the external work done by the loads must equal the internal work performed upon the members, then

$$\frac{1}{2}P_1\Delta_1 = \Sigma\frac{1}{2}S_1\frac{Sl}{AE}$$

whence

$$\Delta_1 = \Sigma\frac{S_1}{P_1}\frac{Sl}{AE}$$

in which S_1 is the stress in any member due to the load P_1 and S is the stress in the same member due to the three loads P_1, P_2 and P_3.

The value of Δ_1 given above represents the deflection due to loads P_1, P_2 and P_3. To determine the deflection due to the applied loads P_2 and P_3, assume that load P_1 approaches a zero value and let Δ represent the deflection of point B due to the loads P_2 and P_3. As P_1 approaches zero, the limit of Δ_1 is Δ, and the limit of values of S become the stresses in the several members due to loads P_2 and P_3.

Since the stress S_1 in any member varies directly as the load P_1, it is obvious that the ratio $\frac{S_1}{P_1}$ is a constant for all values of P_1. Let $\frac{S_1}{P_1} = u$. Then

$$\Delta = \sum \frac{Sl}{AE} u \tag{1}$$

Equation (1) may also be derived by another method. At the point whose deflection is desired, for example point B of Fig. 1, apply a load W acting in direction of the desired deflection. Let P be the stress in any member, as HK, due to load W and let z be the linear distortion of the member due to load W. The average internal work done on member HK is then $\frac{1}{2}Pz$. For all members of the truss, the total average internal work is $\frac{1}{2}\Sigma Pz$. If d = deflection of point B due to the distortion of all members, the average external work is $\frac{1}{2}Wd$. Equating internal and external work, we derive

$$d = \sum \frac{P}{W} z$$

Now W, the load applied at point B, may be given any desired value. Assume that it is a 1-lb. load. Then $\frac{P}{W}$ of the above equation is a ratio which for any member is equal to the stress in that member due to a 1-lb. load applied at the point whose deflection is desired and acting in the direction of the desired deflection. As before, call this ratio u. We may then write

$$d = \Sigma uz$$

Suppose now that any set of loads is applied to the structure, and let e denote the distortion of any member due to these loads Since the deflection of joint B is relatively small compared to the dimensions of the structure, it is reasonable to assume that the deflection at point B due to distortions in any member due to different applied loadings is proportional to the distortions caused by these loadings. If Δ = deflection at B due to any set of applied loads we may write the proportion

$$d : \Delta :: z : e$$

from which

$$\Delta = \frac{de}{z}$$

Substituting the value of d given above in this equation we derive

$$\Delta = \Sigma eu$$

From mechanics, the distortion e of any member of length l, and A, and modulus of elasticity E for a stress S is

$$e = \frac{Sl}{AE}$$

Substituting this value of e in the equation for Δ, we have

$$\Delta = \sum \frac{Sl}{AE} u$$

To calculate values of the ratio u for each member, place a 1-lb. load at the point whose deflection is desired and assume the load to act in the direction of the desired deflection. Compute the stresses in all members due to this 1-lb. loading. The resulting stresses expressed in pounds, when divided by pounds will represent the ratio u for the several members.

For any member, the stress S and the ratio u have positive or negative signs corresponding to tension and compression. The strain in any member causes the point in question to deflect in the direction in which the 1-lb. load is assumed to act if S and u have like signs, and in the opposite direction if they have unlike signs.

In Eq. (1) E is the only quantity which is constant for all members of the truss, provided all members are made of the same material, which is usually the case. Hence the expression $\frac{Sl}{A} u$ may be computed and tabulated for each member, and the deflection found by dividing the algebraic sum $\sum \frac{Sl}{A} u$ by E, as in the problem which follows:

Illustrative Problem.—The truss in Fig 2 supports a load of 240,000 lb. at each bottom chord panel point. In Table 1, the length of each member is given in column 1; the *gross* cross-sectional area in column 2; and the stress, in column 3.

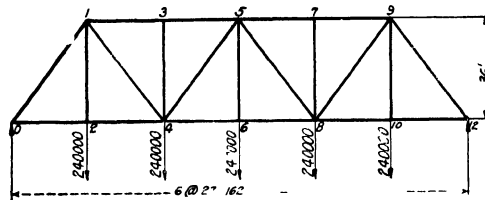



FIG. 2.

Gross areas are used for tension members as well as for compression members due to the fact that in riveted tension members the net areas at sections containing rivets occupy a relatively small portion of the length of the member. Therefore gross areas represent very closely the actual areas which contribute to the strain of the member. The quantities in column 4, when divided by E, represent the strains. To find the deflection at point 6, place 1 lb. at 6 and compute the stresses, as given in Fig. 3*a*. These stresses, when divided by 1 lb., are the ratios u_6 in column 5. In this case S and u_6 have like signs for each member. Hence the quantities in column 6 are all positive, and their algebraic sum is 42,246. Since S was expressed in units of 1,000 lb., the deflection at point 6 is

$$\Delta_6 = \frac{(42{,}246)(1{,}000)}{E} = \frac{42{,}246{,}000}{29{,}000{,}000} = 1.46 \text{ in.}$$

In like manner the deflection at point 4 is found by placing a load of 1 lb. at 4 (Fig. 3*b*) and obtaining the ratios u_4 in column 7. The products of corresponding values in columns 4 and 7 are given in column 8. In this instance, S and u_4 have

TABLE 1

	1	2	3	4	5	6		7	8		9	10
Member	Length in inches (l)	Area in square inches (A)	Stress in thousands of pounds (S)	$\frac{Sl}{A}$	u_6	$\frac{Su_6l}{A}$	Member	u_4	$\frac{Su_4l}{A}$	Member	u_2	$\frac{Su_2l}{A}$
0–1	540	84.0	−750	−4,820	− 5/8	+3,013	0–1	− 5/6	+4,017	0–1	− 25/24	+5,021
1–3	324	69 0	−720	−3,380	− 3/4	+2,540	1–3	−1	+3 380	1–3	− 1/2	+1,690
3–5	324	69 0	−720	−3,380	− 3/4	+2,540	3–5	−1	+3,380	3–5	− 1/2	+1,690
0–2	324	47 4	+450	+3,080	+ 3/8	+1,155	0–2	+ 1/2	+1,540	0–2	+ 5/8	+1,925
2–4	324	47.4	+450	+3,080	+ 3/8	+1,155	2–4	+ 1/2	+1,540	2–4	+ 5/8	+1,925
4–6	324	83.4	+810	+3 150	+ 9/8	+3,540	4–6	+ 3/4	+2,360	4–6	+ 3/8	+1,180
1–4	540	47.4	+450	+5,125	+ 5/8	+3,200	1–4	+ 5/6	+4,270	1–4	− 5/24	−1,070
4–5	540	30 2	−150	−2,680	− 5/8	+1,675	4–5	+ 5/12	−1,120	4–5	+ 5/24	− 560
1–2	432	22.5	+240	+4,610	0	0	1–2	0	0	1–2	+ 1	+4,610
3–4	432	22.5	0	0	0	0	3–4	0	0	3–4	0	0
5–6	432	22.5	+240	+4,610	+ 1	+4,610	5–6	0	0	5–6	0	0
7–8	432	22.5	0	0	0	0	7–8	0	0	7–8	0	0
9–10	432	22 5	+240	+4,610	0	0	9–10	0	0	9–10	0	0
5–8	540	30.2	−150	−2,680	− 5/8	+1,675	5–8	− 5/12	+1,120	5–8	− 5/24	+ 560
8–9	540	47 4	+450	+5,125	+ 5/8	+3,200	8–9	+ 5/12	+2,140	8–9	+ 5/24	+1,070
6–8	324	83.4	+810	+3,150	+ 9/8	+3,540	6–8	+ 3/4	+2,360	6–8	+ 3/8	+1,180
8–10	324	47 4	+450	+3,080	+ 3/8	+1,155	8–10	+ 1/4	+ 770	8–10	+ 1/8	+ 385
10–12	324	47.4	+450	+3,080	+ 3/8	+1,155	10–12	+ 1/4	+ 770	10–12	+ 1/8	+ 385
5–7	324	69 0	−720	−3,380	− 3/4	+2,540	5–7	− 1/2	+1,690	5–7	− 1/4	+ 845
7–9	324	69.0	−720	−3,380	− 3/4	+2,540	7–9	− 1/2	+1,690	7–9	− 1/4	+ 845
9–12	540	84 0	−750	−4,820	− 5/8	+3,013	9–12	− 5/12	+2,009	9–12	− 5/24	+1,004

$\Sigma \frac{Su_6l}{A} = 42,246$

	$\frac{Su_4l}{A}$	$\frac{Su_2l}{A}$
	+33,036	+24,315
	− 1,120	− 1,630
Total	$\Sigma \frac{Su_4l}{A} = 31,916$	$\Sigma \frac{Su_2l}{A} = 22,685$

unlike signs for the member 4-5. Hence the quantity representing the member 4-5 in column 8 has the negative sign, and the strain in that member tends to raise the point 4. The algebraic sum of the quantities in column 8 is 31,916, and the deflection of point 4 is

$$\Delta_4 = \frac{31,916,000}{29,000,000} = 1.1 \text{ in.}$$

Similarly the deflection of point 2 is

$$\Delta_2 = \frac{22,685,000}{29,000,000} = 0.78 \text{ in.}$$

Since the design and loading of the truss in this problem are both symmetrical about the center line, it is obvious that the vertical deflections at the points 8 and 10 are respectively the same as at points 4 and 2.

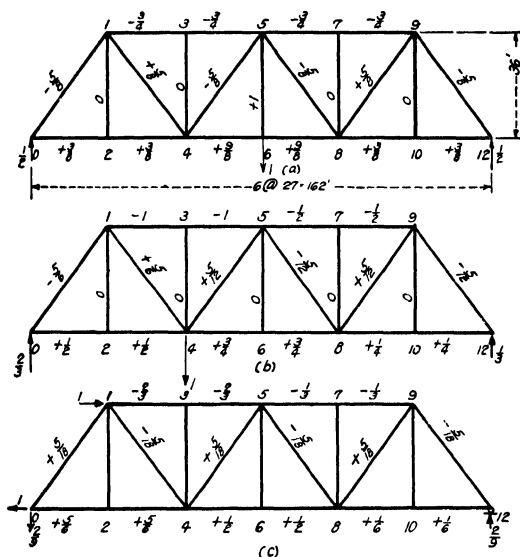



FIG. 3.

The horizontal displacement of any point may be obtained in a similar manner. Suppose that the horizontal displacement of the point 1 in Fig. 2 is required, when the truss is held fast at the left support and rests on rollers at the right support, as shown in Fig. 1. The loads are assumed as in Fig. 2, and the values of $\frac{Sl}{A}$ of column 4, Table 1, are therefore applicable. Place 1 lb. at point 1, acting horizontally either to the right or left; let us say to the right (as in Fig. 3c). Compute the reactions and tabulate the resulting ratios in a column marked u_1, and make the extensions $\frac{Su_1l}{AE}$.

The horizontal displacement of the point 1 will be

$$\Delta_1 = \Sigma \frac{Su_1l}{AE} = \frac{19,890,000}{29,000,000} = 0.685 \text{ in.}$$

Since the 1-lb. load was taken as acting to the right, a positive value for the summation indicates a displacement to the right, that is, in the direction of the 1-lb. load.

2b. Deflection Due to Temperature Changes.—Suppose that we wish to determine vertical displacement of the point 4, Fig. 2, when the temperature of the top chord is 10 deg. above normal and the temperature of the bottom chord is 15 deg. below normal. The coefficient of thermal expansion is 0.0000065. Consider the member 1-3. Its

TABLE 2

Member	Length in inches (l)	Temperature change (t)	lt	u_4	ltu_4
1-3	324	+10	+3,240	−1 0	−3,240
3-5	324	+10	+3,240	−1 0	−3,240
5-7	324	+10	+3,240	−0 5	−1,620
7-9	324	+10	+3,240	−0 5	−1,620
0-2	324	−15	−4,860	+0 5	−2,430
2-4	324	−15	−4,860	+0 5	−2,430
4-6	324	−15	−4,860	+0 75	−3,645
6-8	324	−15	−4,860	+0 75	−3,645
8-10	324	−15	−4,860	+0 25	−1,215
10-12	324	−15	−4,860	+0 25	−1,215

$\Sigma ltu_4 = -24{,}300$

change in length, due to 10 deg. rise in temperature is

$$(324)(10)(0.0000065) = +0.021 \text{ in.}$$

This change in length is treated precisely as if it were a strain resulting from the stress in the member due to loads. Since the coefficient of thermal expansion is the same for all members, the computation may be arranged as in Table 2. The vertical displacement of point 4 is

$$\Delta_4 = (\Sigma ltu_4)(0.0000065) = -0.16 \text{ in.}$$

The negative sign indicates that point 4 is raised by the effect of the temperature.

The problem may be solved graphically by drawing a Williot diagram and using the quantities lt in Table 2 to represent the changes in length.

2c. Deflection Due to Non-elastic Distortion of Members.—Changes in length of truss members are sometimes the result of play of the pins in pin holes and errors in fabricating the members. These changes in length of members produce deflection of the truss which is equal to the deflection caused by similar changes due to strain. Since these changes in length are not produced by the elastic deformation of the member, they are known as non-elastic deformations.

In pin-connected structures it is the usual practice to make the inside diameter of the pin hole about $\frac{1}{50}$ in. greater than the outside

diameter of the pin. This is done in order to facilitate driving of the pins. Hence members pin connected at both ends, when under stress, are shortened or elongated by $\frac{1}{50}$ in., depending upon the character of stress in the member. Members with a pin at one end are shortened or elongated by $\frac{1}{100}$ in.

In fabricating truss members in the bridge shop it sometimes happens that slight errors are made in locating the position of pin holes or in milling the ends of members. These errors result in members whose length differs from that called for on the plans of the structure. When placed in the structure, these errors in length contribute to the deflection of the structure precisely the same as changes in length due to strain.

To estimate the deflection of a structure due to non-elastic distortion, replace the term $\frac{Sl}{AE}$ of Eq. (1) by the estimated or known change in length of the member. Increases in length are to be given a positive sign and decreases in length are to be given a negative sign. Values of u are to be calculated by the methods given in Art. **2a**.

Illustrative Problem.—Assume that the truss shown in Fig. 2 is pin-connected throughout and determine the virtual deflection of joint 6 based on the following assumptions: The play in pin holes is $\frac{1}{50}$ in.; the top chord member is riveted from joints 1 to 9, and to secure proper camber (see Art. **5**) the length of each top chord panel length is increased by $\frac{5}{16}$ in ; web members are joined to the top chord by pin connections.

Table 3 gives the required calculations in tabular form. Since the truss is symmetrical about the point whose deflection is desired, the table may be made up for one-

TABLE 3

Member	Non-elastic distortion (D)	u_6	Du_6
0-1	−0.02	−0.625	+0.0125
1-3	+0 3025	−0.75	−0.226875
3-5	+0 3125	−0.75	−0.234375
0-2	+0 02	+0 375	+0 0075
2-4	+0 02	+0 375	+0.0075
4-6	+0 02	+1 125	+0.0225
1-4	+0 02	+0 625	+0 0125
4-5	−0 02	−0 625	+0 0125
1-2	−0 02	0	0
3-4	−0 02	0	0
5-6	+0 02	+1 0	+0.01[1]
	..... .	. . . .	−0 37625

[1] $\frac{1}{2}Du_6$ used for center vertical

Total deflection = −(2)(0.37625) 0.7525 in. upward

half of the truss and the summations multiplied by two. The deformations for the several members are given in the table. In determining the character of these deformations, it is assumed that the play in the pin holes is taken up by a stress condition corresponding to dead load and full live load. Note that top chord member 1-3 is subjected to pin play for a pin at one end in addition to the intentional elongation of $\frac{5}{16}$ in. Also, member 5-6 is common to both halves of the truss. Hence its distortion is not to be multiplied by two. Values of u are taken from column 5 of Table 1.

3. Graphical Method.—The graphical method for the determination of the deflections of a truss presented in the articles which follow is due to the French engineer Williot and to Professor Otto Mohr. As originally presented by Williot, the method was applicable only to symmetrical structures symmetrically loaded and to cantilever structures. Professor Mohr extended the method to include structures loaded in any manner.

3*a*. Williot—Mohr Diagrams.—The graphical method of finding the deflections of a truss is accomplished by drawing a Williot diagram, after the strain in each member has been determined. In order to compare the results of the algebraic method with the graphic, the latter will be explained in connection with the problem illustrated in Fig. 2. Since the modulus of elasticity is constant for all members, the quantities in column 4 of Table 1 will be taken to represent the strains; and the factors 1,000 and E = 29,000,000 lb. per sq. in. will be introduced at the end of the problem, as in the algebraic solution. Three solutions will be given, based on three different assumptions.

First Solution.—We shall assume that the point 6 is fixed in position and the member 5-6 fixed in direction, and determine the relative displacements of the other points with reference to point 6, when the various members are subject to the strains indicated in column 4, Table 1. First let us consider the triangular unit 4-5-6 of the truss in Fig. 2. For convenience this unit is shown in Fig. 4 and designated as abc. From column 4 the strains are represented by the following quantities: bc = +4,610; ac = +3,150; and ab = −2,680. According to the hypothesis, point c is to remain fixed in position and the member bc is to remain fixed in direction.

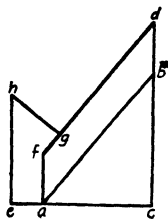



Fig. 4.

The strain in bc is an increase in length, represented by the quantity 4,610; consequently from b we lay off to any convenient scale (not necessarily the scale used in laying off bc) bd = 4,610, above the point b. The point which was originally at b has now moved to d. The location of d was a simple matter, since according to the hypothesis, the member bc (when elongating) had no option except to extend vertically upward.

The location of the new position of the point at a is more complicated. The member ac lengthens; and since c is fixed, the extension is to the left of and away from c. Hence we lay off ae = 3,150. The movement of

a is also influenced by the member ab. Hence, af is laid off equal and parallel to bd, and fd is drawn to represent the original length of ab. The member ab is shortened by the strain 2,680. Since point d has been located, therefore, the point f moves towards d, the amount fg = 2,680. Since the two members ac and ab are connected, the two points e and g, which represent the ends of these members, when under strain, must coincide. Therefore an arc having the radius ce is described about c as a center, and an arc having the radius dg is described about d as a center. The intersection of the two arcs at h marks the new position of the point, which was originally at a. Since, in any practical problem, the arcs are very small in proportion to their radii, it will be sufficiently accurate to draw the straight lines eh and gh perpendicular to ac and ab, respectively.

Fig. 5.

Reference to Fig. 5 shows at once that the displacement of the points a and b may be determined, without including in the diagram the members

Fig. 6.

themselves. Let the point c, which is assumed fixed in position be the origin or fixed point of reference. The member cb lengthens, and the point b moves upward and away from c; therefore lay off cb = 4,610, upward from c. The member ca also lengthens, and the point a moves

horizontally to the left from c; therefore lay off $ca' = 3{,}150$, to the left from c. The member ab shortens, and the point a moves (also) diagonally upward and to the right towards b; therefore lay off $ba'' = 2{,}680$, parallel to ab upward and to the right from b. The perpendiculars through a' and a'' intersect at a. Since c is the origin or reference point, the movements or displacements of the points a and b from their *original* position are indicated by their positions relative to the origin c.

The procedure is similar when the frame consists of a series of triangular units, as illustrated by the truss in Fig. 6*a*. It is assumed that the truss and loads are the same as shown in Fig. 2, and specified in Table 1. The quantities which represent the strains in the various members are taken from column 4 of the table; they are indicated in the figure for convenience. A positive sign indicates elongation, and a negative sign indicates a shortening. Each strain is laid off in the Williot diagram, Fig. 6*b*, parallel to the original direction of the member in which the strain occurs. Joint 6 is assumed fixed in position, and member 5-6 is assumed fixed in direction; hence point 6, Fig. 6*b*, is the origin or reference point in the Williot diagram.

Joint 5 moves upward from joint 6, hence point 5 is laid off to a convenient scale above point 6. Joint 4 moves to the left away from joint 6, and upward to the right toward joint 5; hence the strain in 4-6 is laid off to the left from point 6, and the strain in 4-5 is laid off upward and to the right from point 5. The perpendicular lines drawn from the extremities of these two strains intersect at point 4. The position of point 4, relative to the reference point 6, indicates the movement of point 4 from its original position.

Joint 3 moves to the right toward joint 5, and moves neither upward nor downward with reference to joint 4, since there is no strain in the member 3-4; hence the strain in 3-5 is laid off to the right from point 5, and no strain is laid off from point 4. The perpendicular lines drawn from the extremity of the strain in 3-5, and from point 4, intersect at point 3. The position of point 3, relative to point 6, indicates the movement of joint 3 from its original position.

The remaining joints 1, 2 and 0 are similarly treated in consecutive order, and point 0 is finally located in the diagram.

The scale used in constructing the Williot diagram is too large to be of service in locating the movement or displacement of each joint with reference to the truss itself. If a smaller and more convenient scale is taken, the displacements may be indicated by the dotted outline as shown in Fig. 6*a*; although, even there, the actual distortion is greatly exaggerated in comparison to the dimensions of the truss. On account of symmetry in loading and design, the distortion of the whole truss may be determined by drawing the Williot diagram for either half. The dotted outline gives a true conception of the distortion, when joint 6 is fixed in posi-

tion, the member 5-6 fixed in direction, and the members are subjected to the strains as indicated. It is evident, however, that joints 0 and 12, at the points of support instead of joint 6, remain fixed in elevation. The correction for this error in assumption is made by moving the dotted outline vertically downward until the joints a and b are at the same elevation as 0 and 12. The corresponding correction is made in the Williot diagram by considering A as the origin, or reference point, instead of point 6. The vertical displacement or deflections of joints 2, 4 and 6

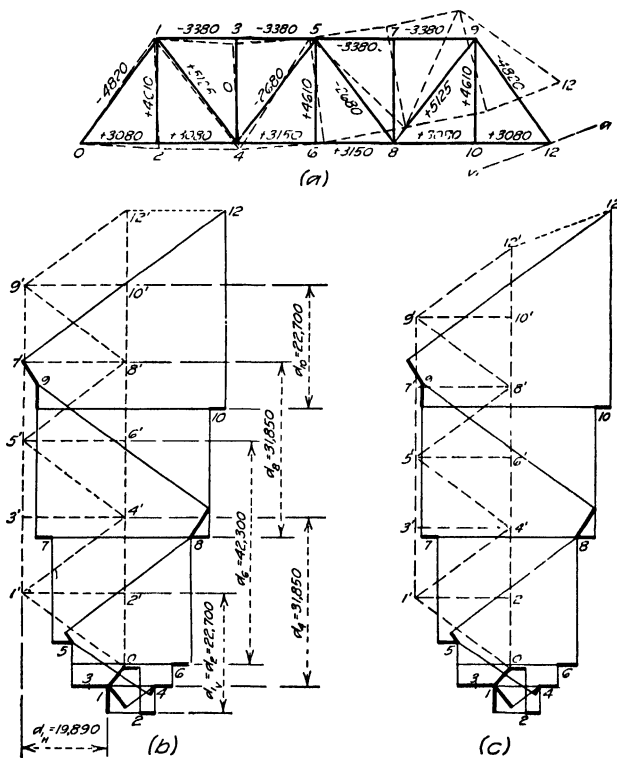



Fig. 7.

are obtained by scaling d_2, d_4 and d_6, multiplying each by 1,000 and dividing by 29,000,000, as in the algebraic solution. The horizontal displacement of any point is obtained by scaling its horizontal distance from the line $A6$. Thus the horizontal, as well as vertical displacements of all joints in the truss, may be found by drawing one Williot diagram.

Second Solution.—If the loading or design in the first solution had not been symmetrical, it is evident that the assumption that the member 5-6 remained fixed in position would not have been true. Hence for unsymmetrical designs or for symmetrical with unsymmetrical loading, the Williot diagram does not represent true conditions. In order that the

Williot diagram may represent true conditions a suitable correction diagram must be applied. The method by which this correction diagram may be made will be illustrated by using the same strains as before and assuming that joint 0, Fig. 7*a* remains fixed in position and that member 0-1 is fixed in direction.

A Williot diagram constructed for the conditions assumed above, is shown in Fig. 7. Point 1 is located first; then points 2, 4, 3, 5, and so on to point 12. The distorted outline may now be drawn, as shown by the dotted lines of Fig. 7*a*. This outline has precisely the same configuration as the distorted outline of Fig. 6*a*. Note, however, that point 12

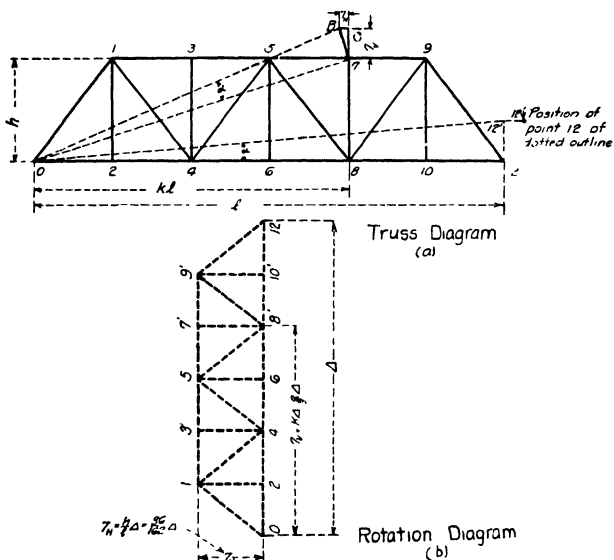



FIG. 8.

of the dotted outline occupies a position to the right of and above the position of point 12 of the original outline. Evidently the position of point 12 given by the Williot diagram is not the correct position of that point, for the presence of rollers at point 12 limits the motion of that point to a movement parallel to the plane of the roller bed, which in this case is assumed as horizontal. The apparent upward motion of point 12 is due to the fact that member 0-1 is not fixed in direction during the distortion of the truss but is subjected to an angular rotation about point 0 as a center.

In order that the distorted outline of Fig. 7*a* may truly represent the deflections of the several joints, it is necessary that the dotted figure be rotated about point 0 until joint 12 of the dotted outline falls on a line line drawn parallel to the plane of the roller bed through point 12 of the original outline. For the assumed conditions, point 12 of the dotted

outline must lie on a horizontal line through point 12 of the original outline. Hence to represent the true deflection of the several joints, the distorted outline of Fig. 7a must be rotated about point 0 until joint 12 of the dotted outline is on a level with joint 12 of the original outline. An equivalent effect may be obtained by rotating the original outline about point 0 until joint 12 is on the same level with joint 12 of the dotted outline. The true deflection of any point may then be determined by measuring the distance from any point in the rotated original outline to the position of the same joint in the dotted outline. This is best accomplished by locating on the Williot diagram the rotated position of the joint of the original outline.

The method of locating on the Williot diagram the rotated positions of the joints of the original outline will now be explained by means of Fig. 8. Let point 12′ represent the point to which joint 12 of the original outline must be rotated in order to bring it to the level of point 12 of the dotted outline, and let α be the angular rotation of point 12 about the center 0. Since the truss is a rigid frame, all joints of the truss are also rotated through the angle α. Consider any top chord joint, as 7. Let $\Delta = 12 - 12' =$ vertical movement of joint 12; $h =$ height of truss; $kl =$ horizontal distance from point 0 to joint 7, where k is a fraction less than unity; and let 7_V and 7_H respectively represent the vertical and horizontal components of motion of joint 7 due to the rotation of the truss about point 0. From the similar triangles 078 and $7BC$, it can readily be shown that

$$\left.\begin{aligned} 7_V &= k\Delta \\ 7_H &= \frac{h}{l}\Delta \end{aligned}\right\} \quad \text{and} \qquad (A)$$

In a similar manner it can be shown that for any lower chord joint, as 8,

$$\left.\begin{aligned} 8_V &= k\Delta \\ 8_H &= 0 \end{aligned}\right\} \quad \text{and} \qquad (B)$$

The rotation diagram of Fig. 8b may be constructed by methods similar to those used for the Williot diagram. Let 0 be the reference point to which all movements are to be referred. Then 0-12 $= \Delta$ represents the movement of joint 12 with respect to point 0. The position of joint 7 in the rotation diagram may be located by means of Eq. (A). The horizontal distance from 0 to joint 7 is two-thirds of the span length. Hence the vertical movement of joint 7 is $\frac{2}{3}\Delta$ as indicated. Since the height of the truss is 36 ft. and the span length is 162 ft. (see Fig. 2), the horizontal movement of joint 7 is $\frac{36}{162}\Delta$. Point 7′ of Fig. 8b therefore represents the rotated position of joint 7. In a similar manner

the rotated positions of the other top chord joints were found and are shown on the diagram by 1′, 3′, 5′ and 9′. To locate the rotated position of any lower chord joint note from Eq. (*B*) that these points are subjected only to a vertical displacement. For point 8, for which $k = 2/3$, the vertical movement is $\frac{2}{3}\Delta$. Point 8′ vertically above point 0 locates the rotated position of point 8.

From Eqs. (*A*) and (*B*) and Fig. 8*a*, it can readily be seen that the rotated position of any point in the rotation diagram with respect to the reference point 0 is in direct proportion to the position of that point in the truss diagram with respect to point 0, the fixed point on the truss. Therefore if the several points in the rotation diagram are connected by straight lines, the resulting figure will be exactly similar in form to the original truss. However, each line in the rotation diagram is perpendicular to the corresponding line in the truss diagram. Therefore, to construct the rotation diagram, lay off from the reference point 0 a distance 0-12′ equal to 12-12′, the vertical displacement of joint 12 of the original outline. On this line 0-12′ construct a figure similar to the truss outline. Since the horizontal components of the rotation of any joint are to the left with respect to the given joint, construct the figure of the truss to the left of the line 0-12′.

To insert the rotation diagram in the Williot diagram of Fig. 7*b*, draw a vertical line through 0 and a horizontal line through 12. Produce these lines to an intersection at 12′. With 0-12′ as a base construct the figure of the truss, as shown by the dotted lines. Label each joint of this figure of truss to correspond with the outline of the truss, using primes, as indicated. The dotted line figure is known as a Mohr correction diagram, and the entire construction of Fig. 7 is known as a Williot-Mohr diagram.

The displacement of any joint may be determined from the Williot-Mohr diagram by measuring the distance from the position of that joint in the dotted line figure to the position of the same joint in the Williot diagram. It will generally be found convenient to measure the deflection of any joint in terms of the vertical and horizontal components of the deflection of that point. Thus for joint 1, the vertical deflection is given by the vertical distance from point 1′ to point 1 of Fig. 7*a*, which is 22,700 units, and the horizontal deflection is the horizontal distance from 1′ to 1, or 19,890 units. To determine the direction of these deflections, note that point 1 is below and to the right of point 1′. Therefore, point 1 deflects downward and to the right with respect to its original position. These deflections check the computed values given in Art. **2**. In a similar manner the deflection of any other point may be determined. Note that point 12 lies directly to the right of point 12′, indicating that the right end of the truss moves in a horizontal direction. This conforms to the assumed initial conditions of a horizontal roller bed.

When the plane of the roller bed is inclined, as shown by the line *a-a* of Fig. 7*a*, the Mohr correction diagram is constructed by drawing through point 12 of the Williot diagram a line parallel to *a-a*, the plane of the roller bed, as shown in Fig. 7*a*. Through 0, draw a vertical, intersecting the line through point 12 at 12′. As before construct the Mohr diagram with 0-12′ as a base. The completed construction is shown in Fig. 7*c*.

Third Solution.—In the first solution, a translation of the distorted outline (Fig. 6*a*) was necessary, because of the erroneous assumption of a

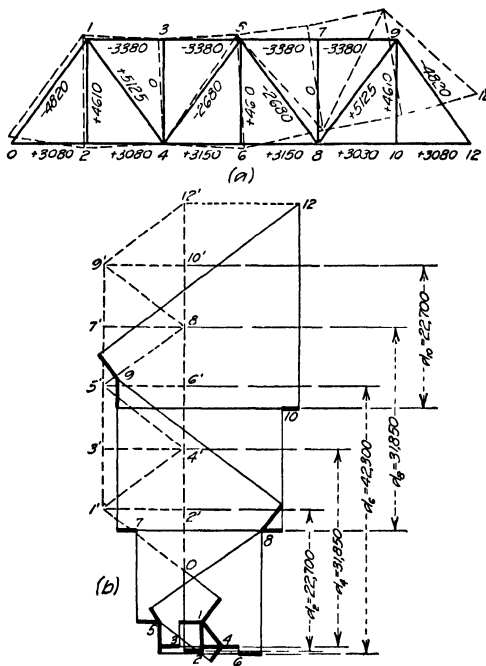



Fig. 9.

joint fixed in position; while in the second solution, a rotation of the distorted outline (Fig. 7*a*) was necessary, because of the erroneous assumption that a member was fixed in direction. The Williot distortion diagram, in Fig. 9*b*, is drawn on the assumption that the joint 4 is fixed in position and the member 4-1 is fixed in direction. Point 4 is the origin, and the other points may be located in the order 1, 2, 0, 3, 5, 6 and so on to 12. The distorted outline of the truss, shown in Fig. 9*a* has precisely the same configuration as in Figs. 6*a* and 7*a*. The Mohr rotation diagram will be drawn on the basis of a fixed support at 0, and a roller support at 12. It is apparent that the distorted outline must be translated vertically downward, then horizontally to the right until joint 0 of the

distorted outline coincides with joint 0 of the original outline. The distorted outline must then be rotated about joint 0, until joint 12 of the distorted outline is level with joint 12 of the original outline. Since joint 0 is now to be considered fixed in position, instead of joint 4, the origin is moved from point 4 to point 0 in Fig. 9b. Draw the vertical through point 0, and the horizontal through point 12, intersecting at 12′, and on 0-12′, as the bottom chord, construct the truss to scale.

The total vertical displacement or deflection of any joint may be considered as the result of three operations—distortion, vertical translation and rotation. As the distortion takes place, joint 2 is lowered the vertical distance from point 4 to point 2 in the distortion diagram. The vertical translation lowers the joint the vertical distance from point 0 to point 4. Finally the rotation lowers the joint the vertical distance from point 2′ to point 0. Hence, the total vertical distance through which the joint 2 moves, is represented by the vertical distance from point 2′ in the translation rotation diagram to point 2 in the distortion diagram, which is $d_2 = 22{,}700$. The vertical displacements or deflections here found agree with those of the previous solutions. The horizontal displacements agree only with those of the second solution, where joint 0 was also subject to horizontal restraint.

3b. Choice of Initial Conditions.—In the preceding article the results given by the three solutions are identical. However, the several Williot-Mohr diagrams differ in form and in size. A study of Figs. 6, 7 and 9 will show that the simplest and most compact diagram is given by Fig. 6. In constructing this diagram the center vertical, which was known to move parallel to itself, was chosen as the reference member. Hence rotation of the truss was not involved and a Mohr correction diagram was not required.

The diagrams of Figs. 7 and 9 were drawn for reference members which are subjected to rotation during the deflection of the truss. Hence Mohr correction diagrams are required. A study of the conditions involved will show that the reference member for Fig. 7 is subjected to a greater rotation than the reference member for Fig. 9. Hence a larger Williot-Mohr diagram is required for Fig. 7 than for Fig. 9, when the same scale is used for both diagrams.

For ease in carrying out the necessary graphical work required in the construction of Williot-Mohr diagrams, it is desirable to have the diagrams as compact as possible. This may be done either by using a small scale in laying out the diagrams, or by a careful selection of the reference member. In general, accurate work cannot be secured by the use of small scales. It is therefore best to use as large a scale as possible, securing the desired reduction in size of the diagram by proper selection of the reference member. To secure the desired reduction in size of diagrams, select a reference member whose motion is parallel to itself, or one which

is subjected to as small a rotation as possible. The first condition is answered by a member at the center of a symmetrical structure symmetrically loaded. The second condition is generally answered by a member near the center of a structure under unsymmetrical loading.

4. Maxwell's Theorem of Reciprocal Deflections.—The proof of Maxwell's theorem as applied to beams, given in the volume on "Structural Members and Connections" is equally applicable to trusses. It is also easily proven by the algebraic method of Art. **2.** Let Δ_a represent

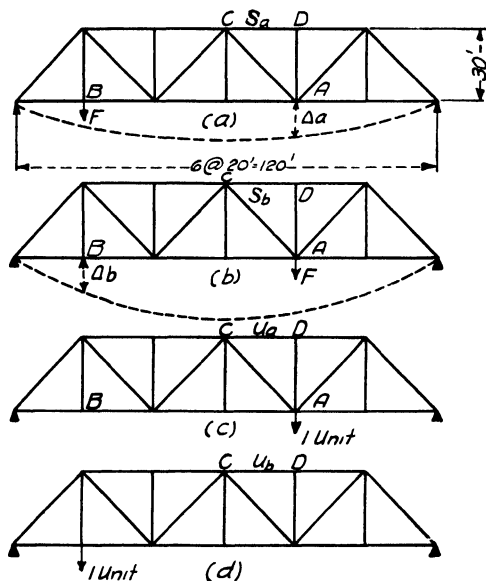



FIG. 10.

the deflection at A, caused by the load F at B (Fig. 10a); and let Δ_b represent the deflection at B, caused by the load F at A (Fig. 10c); then

$$\Delta_a = \sum \frac{S_a u_a l}{AE}$$

and

$$\Delta_b = \sum \frac{S_b u_b l}{AE}$$

where
S_a = stress in any member of Fig. 10a.
S_b = stress in any member of Fig. 10b.
u_a = stress in any member of Fig. 10c.
u_b = stress in any member of Fig. 10d.

Choose any member as CD, then

$$S_a = \frac{2P}{9},\ S_b = \frac{8P}{9},\ u_a = \frac{8}{9},\ \text{and}\ u_b = \frac{2}{9}$$

Hence

$$S_a u_a = S_b u_b$$

If any other member is taken at random it will be found that

$$S_a u_a = S_b u_b$$

Therefore

$$\Delta_a = \Delta_b$$

or the deflection at A, caused by a load at B, equals the deflection at B caused by the same load at A.

5. Camber.—It is desirable that the truss in Fig. 1, when loaded, shall have the configuration represented by the full lines, rather than by the dotted lines. In order to accomplish this end, it is necessary to *camber* the truss, by increasing the length of each compression member and decreasing the length of each tension member by the amount of strain which it experiences under load. Thus, in Fig. 2, the strain in the member 1-3 from Table 1 is

$$\frac{-(3{,}380)(1{,}000)}{29{,}000{,}000} = -0.116 \text{ in.}$$

This member is shortened about ⅛ in. when the truss is loaded, hence its original length is made 27 ft. 0⅛ in. If all members are treated accordingly, and the truss erected on false work, in such a way that the members are carrying little or no stress, the truss will have the configuration of Fig. 11; which is known as a camber diagram. The camber diagram is constructed from a Williot diagram, drawn by using the strains as given in column 4 of Table 1, with opposite signs. Trusses are usually cambered for dead load plus live load, impact not included; sometimes the dead load plus two-thirds the live load is taken.

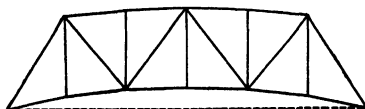



Fig. 11.

The following approximate method is sometimes used. If the average unit stress in the members is 14,000 lb. per sq. in. based on the gross section, the strain in every 10 ft. of length is a little short of 1/16 in. Hence a truss may be cambered by a rule-of-thumb method, if ⅛ in. for every 10 ft. in length is added to the top chord; the length of all other members remaining the same as if no allowance were made. Thus each top chord member of the truss in Fig. 2 would be made 27 ft. 0 5/16 in. long.

STRESSES IN REDUNDANT MEMBERS

6. Necessary Members for a Stable Frame.—When it becomes necessary to transfer any load or loads across the space between any two points by means of a frame work, the most desirable form for that

frame is one in which as few members as possible are used. Also, it is desirable, for purposes of design, that the stresses in the members of the frame may be readily determined by the principles of statics. The members of the framework should, if possible, be subjected only to direct stresses in tension or compression.

Suppose it is desired to transfer a load P to points A and B of Fig. 12. A triangular frame ABC, shown in Fig. 12*a* will accomplish the desired object. The members may be hinged at the joints, and the stresses in all members will be either tension or compression. These stresses may be determined by the principles of statics. Note that only three members are necessary. A rectangular framework, as shown in Fig. 12*b*, will also transfer load P to points A and B. However, unless rigid joints are provided at the four corners, this frame is unstable, as it tends to fold up. When rigid corners are provided, the frame is stable, but its stability depends upon bending moments and shears set up in the several members in addition to direct stresses. These direct stresses, moments and shears are not readily determined by the methods of statics. They can be determined only by the methods given in Sec. 6 for rigid frames. A framework such as Fig. 12*a*, in which the stresses may be determined by statics, is said to be *statically determinate.* The framework of Fig. 12*b* is said to be *statically indeterminate,* since the stresses cannot be determined by simple statics. From the above discussion, it seems evident that the triangle is the simplest truss or frame element.

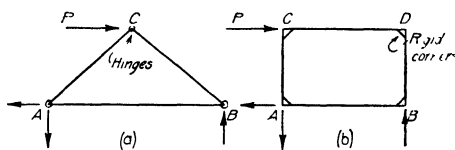



Fig. 12.

Suppose that in forming a given structure it was found necessary to make use of m points, or joints. From the above discussion, it is evident that the simplest statically determinate framework by which these points can be connected will be formed by a system of triangles connecting the several points. Let n = number of members required to connect these points. The relation between members required and points connected may be determined by noting that after the first triangle has been formed there are left n-3 members and m-3 points from which the rest of the structure is to be formed. For each additional triangle which is formed, one point and two members are required. We therefore have the relation

$$n-3 = 2(m-3)$$

from which

$$n = 2m-3 \tag{1}$$

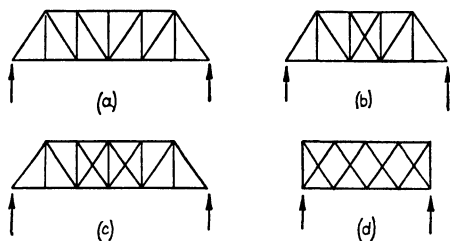
On substituting values of n and m for any given structure, it is possible to determine whether the minimum number of members has been pro-

vided, or whether there are more members than are really necessary. In general this fact can readily be seen from an inspection of the structure. Members which are provided in excess of the minimum number required, as determined by Eq. (1), are known as *redundant members*. A structure may be *singly*, *doubly*, or *multiply* redundant, depending on the number of excess members in place.

Fig. 13.

Illustrative Problem.—Figure 13 shows a few typical trusses. Apply Eq. (1) and determine whether redundant members are present

In addition to the notation used in Eq. (1), let N = number of members in position. Then $N - n$ = number of redundant members present. The results for the several trusses are as follows:

Fig. 13a, $m = 12$; $N = 21$; $n = 21$; $(N - n) = 0$
No redundant members.
Fig. 13b, $m = 10$; $N = 18$; $n = 17$, $(N - n) = 1$
One redundant member.
Fig. 13c, $m = 12$; $N = 23$; $n = 21$; $(N - n) = 2$
Two redundant members.
Fig. 13d, $m = 10$; $N = 18$; $n = 17$; $(N - n) = 1$
One redundant member.

7. Stresses in a Structure with One Redundant Member.—Figure 14 shows a structure which contains one redundant member, for on substituting in Eq. (1) with $m = 4$, we find $n = 5$. The truss as shown

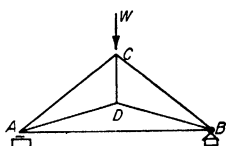



Fig. 14.

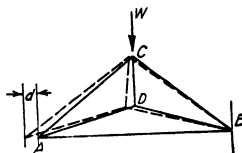



Fig. 15.

contains six members. Hence, as stated above, one redundant member is present.

In attempting to determine the stresses in this structure by the method of successive joints, it will be found that three unknowns exist at every joint. Since the conditions of static equilibrium offer only two simultaneous condition equations, *i.e.*, $\Sigma H = 0$, $\Sigma V = 0$, a solution for stresses by the methods of simple statics cannot be made. A third independent equation must therefore be found by means of which the stress in some one of the members may be determined. In this manner the unknowns at the joint containing the known stress will be reduced to two and all stresses may then readily be determined. This third

equation may be derived subject to the conditions of elastic equilibrium which must exist in the structure, that is, *the distortion of the redundant member is equal to the deflection in the direction of the center line of the redundant member of the statically determinate structure composed of the necessary members.* Any member may be taken as the redundant member. It is only necessary to make certain that a stable structure is left after the removal of the redundant member.

Consider member AB of Fig. 14 as the redundant member. Figure 15 shows the distortion of the loaded structure plotted with respect to the redundant member, assuming B as a fixed point. The dotted lines show the movement of joint A due to the distortion of the structure. Separating the structure into two parts, one part consisting of the redundant member, and the other part consisting of the members necessary to form a stable structure, we have the arrangements shown in Fig. 16.

Fig. 16.

Let S_r denote the stress in the redundant member, assumed to be *tension.* To maintain static equilibrium, forces S_r must be placed as shown in Figs. 16*a* and *b*. From Fig. 16*b*, the distortion of the redundant member is an elongation of

$$d_r = +\frac{S_r l_r}{A_r E} \tag{2}$$

in which l_r and A_r = the length and area respectively of the redundant member, and E = modulus of elasticity of the material composing the member. The plus sign indicates an increase in length of the redundant member, or a movement to the left.

The movement of point A of Fig. 16*a* along the line AB may be calculated from the formula

$$d_n = \sum \frac{Sl}{AE} u \tag{3}$$

in which S = stress in any member of Fig. 16 due to applied loads; l = length of any member; A = its gross area; E = modulus of elasticity of material composing the member; and u = a ratio which is equal to the stress in any member due to a 1-lb. load applied as a tension along the center line of the redundant member. The derivation of Eq. (3) is given on p. 366.

The stresses S are as yet unknown. However, they may be expressed in terms which are readily evaluated by considering the redundant member removed from the structure and its effect on the stresses represented

by forces S_r placed as shown in Fig. 16*a*. The stress S in any member may then be considered as composed of two parts: (*a*) A stress, which will be denoted by S', which is the stress in any member of the structure composed of the n necessary members (redundant member removed) due to the applied loads; (*b*) a stress in each member due to loads S_r applied along the center line of the redundant member, as shown in Fig. 16*a*. The first part S', may readily be determined by the methods of statics. The second part depends upon the value of S_r, as yet unknown. Let u = stress in any member due to a 1-lb. tension acting along the line of the redundant member, as shown in Fig. 16*a*. It is evident that the stress in any member due to S_r is given by the expression $S_r u$. Hence we may write

$$S = S' + S_r u \tag{4}$$

As stated above, the deflection of point A, Fig. 16*a* is desired along the line of member AB. Since S_r has been assumed as tension, A moves to the left. Hence the values of u to be used in Eq. (3) are to be calculated for a 1-lb. load acting to the left. In making up Eq. (4) values of u are required for a 1-lb. load acting to the right. We may then use these values of u in determining the movement of A to the left by changing the sign of Eq. (3). On substituting values of S from Eq. (4) in Eq. (3), and changing signs, we have

$$d_n = -\sum \frac{(S' + S_r u)}{AE} ul \tag{5}$$

Since the elongation of the redundant member and the movement of point A are equal, we have from Eqs. (2) and (5)

$$-\sum \frac{(S' + S_r u)}{AE} ul = + \frac{S_r l_r}{A_r E}$$

from which

$$S_r = -\sum \frac{\sum \frac{S'l}{AE} u}{\sum \frac{u^2 l}{AE} + \frac{l_r}{A_r E}}$$

Since the value of u for the redundant member is unity, the term $\frac{l_r}{A_r E}$ may be included in the summation $\sum \frac{ul^2}{AE}$ and we may write

$$S_r = -\frac{\sum_1^n \frac{S'l}{AE} u}{\sum_1^{n+1} \frac{u^2 l}{AE}} \tag{6}$$

in which $\sum_1^n$ indicates that the summation is to be made for the n necessary members, and $\sum_1^{n+1}$ indicates that the summation is to include the redundant member as well as the n necessary members. Equation (6) is the general equation for stress in a single redundant member. The minus sign in front of Eq. (6) is important, for it indicates the character of the

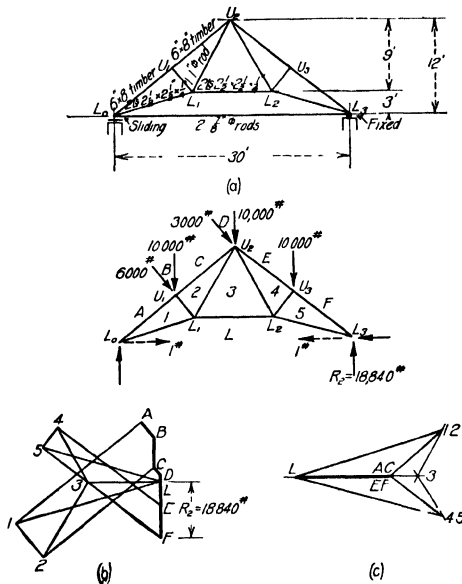



FIG. 17.

stress in the redundant member. A plus sign indicates tension, and a minus sign indicates compression.

If the redundant member as fabricated is not correct in length, it will, when placed in position, cause stresses in all members which are equal to those caused by a stress deformation which is equal to the error in the length of the redundant member. Let e = error in length of the redundant member. If the member is too long, e is to be given a plus sign; if too short, a minus sign is to be used. Since, as noted above, the deflection of the necessary members is equal to the deformation of the redundant member, we may substitute e for $\sum\frac{S'ul}{AE}$ in Eq. (6) and we have

$$S_r = -\frac{e}{\sum\frac{u^2l}{AE}} \tag{7}$$

Equation (7) may be used to determine the stress in any redundant member due to changes in its length from any cause, as imperfections in fabrication, temperature changes, etc. The resulting stresses in other members may then be determined from Eq. (4).

8. Solution of Problems, Structures with One Redundant Member.—The general method of the preceding article will now be applied to the solution of a few typical problems in the determination of stresses in structures with one redundant member.

8*a*. Stresses in a Roof Truss.—Let it be required to calculate all stresses in the truss of Fig. 17. The loads to be carried are shown in position. Sizes of members and dimensions of the structure are as shown on the figure. Note that certain of the members are timber, while others are of steel. The moduli of elasticity will be assumed as 1,000,000 lb. per sq. in. for timber and 30,000,000 lb. per sq. in. for steel.

All necessary data are given in the following table:

Member	Length l (ft.)	Area A (sq. in.)	u	Material	$\frac{ul}{AE}$ times 1,000,000	S' (lb.)	$\frac{S'ul}{AE}$	$\frac{u^2l}{AE}$	S_ru	S
1	2	3	4	5	6	7	8	9	10	11
L_0U_1	9.60	48	+0.805	Wood	+0.161	−52,500	− 8,450	0.130	+21,700	−30,800
U_1U_2	9.60	48	+0.805	Wood	+0.161	−46,200	− 7,450	0.130	+21,700	−24,500
U_2U_3	9.60	48	+0.805	Wood	+0.161	−43,500	− 7,000	0.130	+21,700	−21,800
U_3L_3	9.60	48	+0.805	Wood	+0.161	−49,800	− 8,020	0.130	+21,700	−28,100
L_0L_1	10.3	2.38	−1.71	Steel	−0.244	+48,900	−11,900	0.417	−46,200	+ 2,700
L_1L_2	10.2	2.38	−1.34	Steel	−0.192	+23,600	− 4,530	0.257	−36,200	−12,600
L_2L_3	10.3	2.38	−1.71	Steel	−0.244	+40,800	− 9,950	0.417	−46,200	− 5,400
U_1L_1	3.82	16	0	Wood	0	−13,800	0	0	0	−13,800
U_3L_2	3.82	16	0	Wood	0	− 8,000	0	0	0	−13,800
U_2L_1	10.3	1.0	−0.580	Steel	−0.199	+28,800	− 5,470	0.115	−15,700	+13,100
U_2L_2	10.3	1.0	−0.580	Steel	−0.199	+21,300	− 4,240	0.115	−15,700	+ 5,600
L_0L_3	30.0	1.53	+1.0	Steel	+0.653		. . . .	0.653	+27,000	+27,000
	. . . .	. . . . .	. . . .	. . . . . .	. . . . . .		−67,280	2.494		

\+ = tension − = compression

$$S_r = -\frac{-67,280}{2.494} = +27,000 \text{ lb. tension.}$$

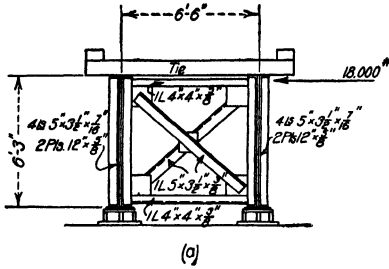
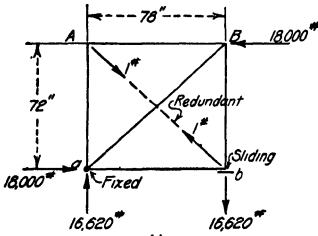
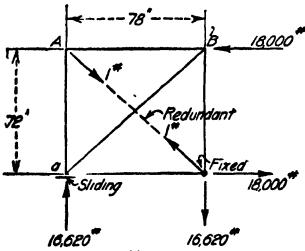
It will be assumed that member L_0L_3 is redundant. Values of u, given in column 4, were determined by means of the stress diagram of Fig. 17*c*, which is drawn for 1-lb. loads acting along the line of member L_0L_3 as shown in Fig. 17. The stresses S' given in column 7 were determined from the stress diagram of Fig. 17*b*, which is drawn for the applied loads acting on the structure with the redundant member removed.

In calculating values of $\frac{ul}{AE}$ in column 6, we may use E for timber = 1, and E for steel = 30. The resulting values of $\frac{ul}{AE}$ will then be one million times their true values. However, the terms $\frac{ul}{AE}$, multiplied by

certain constants, appear in both numerator and denominator of Eq. (6). Hence the multiplier of one million will divide out on substitution in Eq. (6) and it may be neglected in forming the table. This will simplify the work of calculation as it allows the decimal point to be readily located. Values of $\frac{S'ul}{AE}$ and $\frac{u^2l}{AE}$ are given in columns 8 and 9. In calculating values of $\frac{S'ul}{AE}$, careful attention must be paid to the algebraic sign of the result. Note that the sign of $\frac{u^2l}{AE}$ is always positive because u is squared. Summations of values of $\frac{S'ul}{AE}$ and $\frac{u^2l}{AE}$ are given at the foot of columns 8 and 9.

Fig. 18.

The necessary substitutions in Eq. (6) are shown below the table. As indicated by the positive sign of the result, the stress in the redundant member is tension. Stresses in other members of the truss may be determined from Eq. (4). The necessary calculations are given in columns 10 and 11.

In some cases, the tie rod L_0L_3 of Fig. 17 is omitted and both ends of the truss are fixed at the supporting walls, which are assumed as rigid. Under these assumed conditions there will be four unknown reactions, two vertical and two horizontal. One of the horizontal reactions may be assumed as redundant, and a solution for its value may be made by the use of Eq. (6). Since the supporting walls are considered as rigid, this is equivalent to a redundant member whose deformation is zero. This condition may be accounted for by assuming that the redundant member has an infinite area. To modify the above table to account for this condition, place A for member L_0L_3 equal to infinity. Then $\frac{ul}{AE}$ and $\frac{u^2l}{AE}$ for member L_0L_3 will be zero instead of 0.653. The summation $\frac{S'ul}{AE}$ will

remain as given in the table, while $\sum \frac{u^2 l}{AE}$ becomes $(2.494 - 0.653) = 1.841$. Then from Eq. (6)

$$H_1 = -\frac{-67{,}280}{1.841} = 36{,}500 \text{ lb.}$$

acting to the right. The stresses in members may be determined by the method used in the table.

8b. Stresses in the End Cross-frame of a Deck Plate Girder.—Figure 18*a* shows the general dimensions and make-up of sections for the end cross-frame of a deck plate girder bridge. Let it be required to determine all stresses in this frame due to an 18,000-lb. horizontal load applied as shown in Figs. 18*b* and *c*. In determining the reactions at the supports, two assumptions may be made regarding the conditions of these supports. It may be assumed, as shown in Fig. 18*b* that the left end is fixed and the right end free to slide, or as shown in Fig. 18*c*, the opposite conditions may be assumed. The problem will be solved for both sets of assumed conditions. It will be assumed in each solution that member *Ab* is redundant.

The following table gives in convenient form all data and stresses for the conditions shown in Fig. 18*b*. Stresses S' are calculated with the redundant member removed, and stresses u are calculated for a 1-lb. load applied as a tension acting along the line of the redundant member. These forces are shown in position in Fig. 18*b*. Since E, the modulus of elasticity is constant, it may be omitted from the calculations.

TABLE A

Member	Area (sq in.)	Length (in.)	S' (lb.)	u	$\frac{ul}{A}$	$\frac{S'ul}{A}$	$\frac{u^2 l}{A}$	$S_r u$	S
Aa	29.12	72	0	−0.6782	− 1.677	0	+ 1 137	−5,900	− 5,900
Bb	29.12	72	+16,620	−0 6782	− 1.677	−27,870	+ 1.137	−5,900	+10,720
AB	2.86	78	0	−0.7348	−20.04	0	+ 14.72	−6,380	− 6,380
ab	2 86	78	0	−0.7348	−20.04	0	+ 14.72	−6,380	− 6,380
aB	3 05	106 2	−24,500	+1 0	+34.80	−852,600	+ 34.80	+8,690	−15,810
Ab	3.05	106.2	....	+1 0	+34.80	....	+34.80	+8,690	+ 8,690
	.....	.....	....	....	....	−880,470	+101.32		

$$S_r = -\frac{-880{,}470}{+101.32} = +8{,}690 \text{ lb. tension}$$

When the conditions at the supports are as shown in Fig. 18*c*, the necessary calculations are as given in the following table:

TABLE B

Member	Area (sq in.)	Length (in.)	S' (lb.)	u	$\frac{ul}{A}$	$\frac{S'ul}{A}$	$\frac{u^2l}{A}$	S_ru	S
Aa	29 12	72	0	−0 6782	− 1 677	0	+ 1 137	− 8 310	− 8 310
Bb	29 12	72	+16,620	−0 6782	− 1 677	−27,870	+ 1 137	− 8 310	+ 8 310
AB	2 86	78	0	−0 7318	−20 04	0	+ 14 72	− 9,000	− 9 000
ab	2 86	78	+18,000	−0 7318	−20 04	−360 700	+ 14 72	− 9,000	+ 9 000
aB	3 05	106 2	−24 500	+1 0	+34 80	−852,600	+ 34 80	+12,250	−12 250
Ab	3 05	106 2		+1 0	+34 80		+ 34 80	+12 250	+12,250
						−1 241 170	+101 32		

$$S_r = -\frac{-1,241,170}{101.32} = +12,250 \text{ lb. tension}$$

On examining the above tables it will be noted from Table A (conditions of supports as shown in Fig. 18*b*) that the compression diagonal takes $\frac{15,180}{24,500} = 64.5$ per cent of the horizontal shear. Also the verticals are unequally stressed. For the condition of supports shown in Fig. 18*c*, Table B shows that the diagonals each take one-half of the horizontal shear, their stresses being equal in amount but opposite in kind. The vertical and horizontal members are also equally stressed.

The reasons for the unequal distribution of the stresses in the two cases may be determined by a study of the conditions existing in Figs. 18*b* and *c*. In Fig. 18*b*, the fixed point is at *a*. The horizontal load of 18,000 lb. is applied at *B* and is resisted at *a*. In traveling from *B* to *a*, the load is carried over two paths: (*a*) Part of the load is transferred directly from *B* to *a* by the member *aB*; (*b*) the balance of the load is transferred in the order named by members *BA*, *Ab*, and *ba*. Note that the path named in (*b*) is longer, and hence more flexible than the one named in (*a*). For the conditions shown in Fig. 18*c*, point *b* is fixed. In traveling from *B* to *b*, the load is divided over two paths: (*a*) *BA*, *Ab*, (*b*) *Ba*, *ab*. Note that these paths are of equal length, and that they are composed of identical members. Hence the two paths are of equal flexibility. The explanation of the unequal distribution of the load to the members of the frame, in the two cases, therefore, lies in the fact that in Fig. 18*b*, the paths over which the load was forced to travel from *B* to the fixed point were of unequal flexibility; the greater part of the load was transferred by the more rigid path. In Fig. 18*c* the paths from *B* to the fixed point were of equal flexibility and the load divided itself equally over these paths. From the above discussion we may state the following important principle: When a load may be transferred from one point to another in a structure over two or more paths which are of unequal flexibility, the load will be divided over these paths in proportion to the relative flexibility of the several paths. This principle is very useful in deciding whether redundant members should be used in any structure.

The problem solved above gives two solutions for stresses which are based on different conditions at the supports. In practice, the ends of the girders are usually fastened to the shoes by means of bolted connections. If these bolts become loose, it is very probable that the structure will move slightly in a horizontal direction until one or the other of the connections at *a* or *b* comes to a bearing on the bolts at that point. It is then possible that either of the conditions shown in Figs. 18*b* or 18*c* may exist. The stresses for the conditions shown in Fig. 18*b* give the greater stress in the compression diagonal. Hence it would seem best to assume the conditions of the supports as shown in Fig. 18*b*. This requires that, for sections as assumed above, about two-thirds of the shear should be taken by each diagonal instead of one-half, as usually assumed in practice. However, it is the general practice to use sections much in excess of those required for stress in order to obtain a rigid frame-work. Therefore, the sections arrived at by the usual methods are in general large enough to provide for any stress which the member may be called upon to carry.

8c. Pratt Truss with an Odd Number of Panels, Two Riveted Diagonals in Center Panel.—Figure 19 shows a Pratt truss

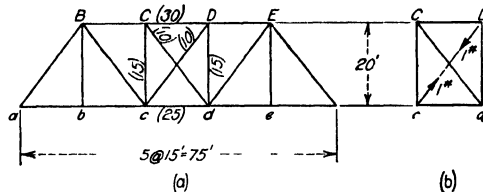



FIG. 19.

with an odd number of panels. The center panel contains two riveted diagonals, each capable of taking both tension and compression. In practice, such members are usually designed on the assumption that each diagonal takes one-half the shear in the panel. The true distribution of shear will now be determined.

Required the stresses in the center panel diagonals of Fig. 19 due to dead panel loads of 20,000 lb. and live panel loads of 60,000 lb. Stresses in the diagonals will be determined for dead load and also for live loads on joints *d* and *e*, the live load position for maximum stress.

Member *Dc* will be considered as the redundant member. Areas of members in the center panel are as given by the figures in parenthesis on Fig. 19*a*. All necessary calculations are given in the following table. Since values of *u* occur only for the members of the center panel, as shown by the loading conditions of Fig. 19*b*, it is unnecessary to include the members of other panels in the table. Stresses in all truss members outside this panel are unaffected by the presence of the redundant member.

CALCULATION OF STRESSES IN CENTER PANEL, FIG. 19
(Member *Dc* redundant)

Member	l (in.)	A (sq. in.)	u	$\frac{ul}{A}$	$\frac{u^2l}{A}$	Dead load				Live load			
						S'	$\frac{S'ul}{A}$	$S_r u$	S	S'	$\frac{S'ul}{A}$	$S_r u$	S
CD	180	30	−0.6	−3.60	2.16	−45,000	+162,000	−228	−45,228	−81,000	+ 291,600	+13,158	−67,840
cd	180	25	−0.6	−4.32	2.59	+45,000	−194,400	−228	+44,772	+54,000	− 233,300	+13,158	+67,160
Cc	240	15	−0.8	−12.8	10.24	0	0	−304	− 304	−36,000	+ 460,800	+17,570	−18,430
Dd	240	15	−0.8	−12.8	10.24	0	0	−304	− 304	0	....	+17,570	+17,570
Cd	300	10	+1.0	+30.0	30.0	0	0	+380	+ 380	+45,000	+1,350,000	−21,930	+23,070
Dc	300	10	+1.0	+30.0	30.0	0	0	+380	+ 380	0	..... .	−21,930	−21,930
	. .	.	.....		85.23		− 32,400			. . . .	+1,869,100		

For dead load,

$$S_r = -\frac{-32{,}400}{85.23} = +380 \text{ lb. tension.}$$

For live load,

$$S_r = -\frac{+1{,}869{,}100}{85.23} = -21{,}930 \text{ lb. compression.}$$

From the above table it will be noted that both diagonals are in tension under symmetrical loading, such as the dead load. For this loading condition, the shear in the center panel is zero. Hence the stresses in the diagonals are due to the deformation of the members in the panel and not to the shear. For the final dead load stresses given in the above table, it will be noted that the top chord member *CD* is shortened, the bottom chord member *cd* is elongated, and the verticals *Cc* and *Dd* are practically unstrained. The result of these deformations on the shape of the panel is such as to cause an elongation of the diagonals. Hence the stress in these members is a small tension, as indicated in the table.

For unsymmetrical loading, such as live load in position for maximum shear in the center panel, the diagonals are unequally stressed, as shown by the final stresses given in the above table. However, the difference in the stresses in the center diagonals is so small that the usual assumption of equal division of shear between the two members is sufficiently accurate for all practical cases.

In some cases the two diagonals in the center panel of Fig. 19 are rods or bars which are capable of carrying tension only. The resulting stresses in these rods will then be the stresses due to loading, calculated by the methods given above, plus the initial tension placed in these rods during the erection of the structure. If the compression due to loading is less than the initial tension, both rods act as tension members. If the compression due to loading exceeds the initial tension, one of the rods becomes inactive and all of the shear is carried by the other rod. It is generally impossible to estimate the amount of this initial tension. Hence the true condition of stress is hard to estimate.

8*d*. Stresses in Double Intersection Trusses with a Single Redundant Member.—In Art. **14**, p. 283, the stresses in double intersection trusses were determined by an approximate method based on the assumption that the main trusses could be resolved into two systems which could act independent of each other. An exact determination of stresses in such structures may be made by means of the theory of redundant members.

As an example of the application of the more exact method, the stress in member *Eg* of Fig. 49, p. 285 will now be determined. Figure 20*a* shows the dimensions of the truss. Areas of the members, deter-

mined for dead and live load stress are shown in parentheses on Fig. 20*a*. These areas are given in square inches. The stress in *Eg* will be determined for live panel loads of 30,000 lb. at *g* and all joints to the right. These loads are shown in position on Fig. 20*c*.

Since any member may be chosen as the redundant member (subject to the condition that a stable structure is maintained) we will consider member *Eg* as redundant. This will simplify the calculations somewhat. Figure 20*b* shows a 1-lb. load acting as a tension in the redundant member. The resulting stresses are the required values of u in Eq. (6). In calculating these stresses, it will be found best to determine first the

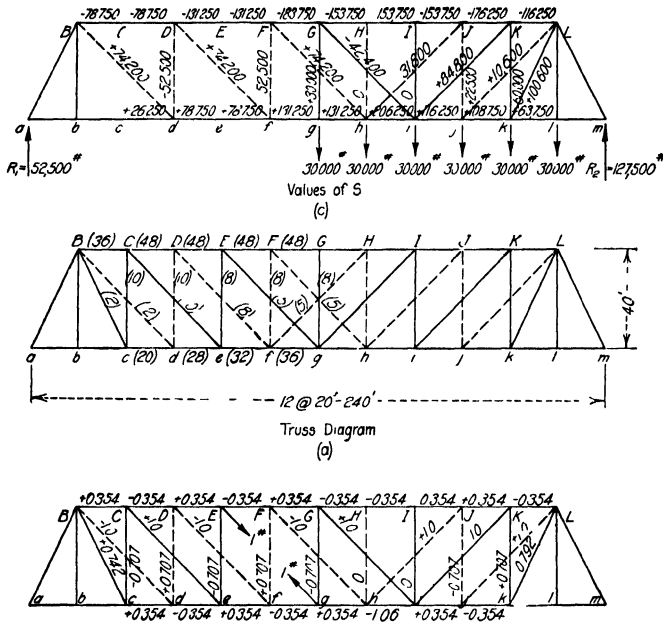



Fig. 20.

stresses in the web members. Starting at joints E and g, the action of the 1-lb. loads may be traced through the full line web members to joints B and L. Then trace the stresses from these points back to the truss center over the dotted line web members. After all web stresses have been determined, determine chord stresses, using the method of successive joints. All stresses are as shown on Fig. 20*b*. In calculating the stresses S' due to the applied loads, we must first assume the form of the truss under the applied loading. This is necessary because counters are present near the center of the truss. It will be assumed that the members shown in Fig. 20*c* are in action for the given loading position. If total dead and live stress is required, the combined loads must be placed in position and the probable form of truss assumed. The redundant member has been

omitted in Fig. 20c. When *Eg* is omitted there will be no stress in the full line web members to the left of *g*. These members have therefore been omitted in Fig. 20c. The stresses as calculated are shown on the members.

All data necessary for the determination of the stress in the redundant member are given in the following table. Since the maximum stress in *Eg* occurs for the given loading, and stresses in other members are not the maximum for these members, values of *S*, the final stresses in the members, are given only for a few of the web members in order to check up on the form of truss as assumed. It was found that the form had been correctly assumed.

Member	l (in.)	A (sq in.)	u	S'	$\frac{S'ul}{A}$	$\frac{u^2l}{A}$	S_ru	S
BC	240	36	+0 354	− 78,750	− 186 000	0 835		
CD	240	42	−0 354	− 78,750	+ 167,000	0 750		
DE	240	48	+0 354	−131,250	− 232 000	0 626		
EF	240	48	−0 354	−131,250	+ 232 000	0 626		
FG	240	48	+0 354	−183,500	− 324,000	0.626		
GH	240	48	−0 354	−153,750	+ 272,000	0 626		
HI	240	48	−0 354	−153,750	+ 272 000	0 626		
IJ	240	48	−0 354	−153,750	+ 272 000	0 626		
JK	240	42	+0 354	−176,250	− 374 000	0 750		
KL	240	36	−0 354	−116 250	+ 274,000	0 835		
cd	240	20	+0 354	0	0	1 502		
de	240	28	−0 354	+ 78,750	− 238,000	1 072		
ef	240	32	+0 354	+ 78,750	+ 209 000	0 940		
fg	240	36	−0 354	+131,250	− 310 000	0 835		
gh	240	36	+0 354	+131,250	+ 319 000	0 940		
hi	240	32	−1 060	+206 250	− 1,640 000	8 430		
ij	240	28	+0 354	+116 250	+ 352 000	1 072		
jk	240	20	−0 354	+108,750	− 462 000	1 502		
Bc	537	12	+0 792	0	0	28 0		
Cc	480	10	−0 707	0	0	24 0		
Ce	680	10	+1 00	0	0	68 0		
Ee	480	8	−0 707	0	0	29 9		
Eg	680	5	+1 00	Redundant		136 0	+43,200	+43,200
Gg	480	8	−0 707	+ 30,000	− 1 270 000	29 9	−30 600	− 600
Gi	680	5	+1 00	− 42 400	− 5 760 000	136 0	+13 200	+ 800
Ii	480	8	0	0	0	0	0	0
iK	680	10	−1 00	+ 84,400	− 5,730,000	68 0	−43,200	+41,200
Kk	480	10	+0 707	− 60,000	− 2 040 000	24 0		
kL	537	12	−0 792	+100,600	− 3,560 000	28 0		
Bd	680	12	−1 00	+ 74,200	− 4,200 000	56 6		
Dd	480	10	+0 707	− 52,500	− 1,780 000	24 0		
Df	680	8	−1 00	+ 74 200	− 6,310,000	85 0		
Ff	480	8	+0 707	− 52 500	− 2,230,000	29 9		
Fh	680	5	−1 00	+ 74 200	−10,100,000	136 0	−43,200	+31 000
Hh	480	8	0	0	0	0	0	0
hJ	680	8	+1.00	− 31,800	− 2,700,000	85 0	+43,200	+11 400
Jj	480	10	−0 707	+ 22,500	− 765,000	24 0	−30,600	− 8,100
jL	680	12	+1.00	+ 10,600	+ 600,000	56 6	+43,200	+53 800
	. .	..	....	....	−47,212,000	1,092 119		

$$S_r = -\frac{-47,212,000}{1,092.1} = +43,200 \text{ lb. tension}$$

The stress in Eg, as calculated above by the exact method, is found to be 43,200 lb. tension. Using the approximate method of Art. **14**, p. 283, the stress in $Eg = \frac{30}{12}(\frac{1}{2} + 2 + 4 + 6)(1.41) = 44{,}200$ lb. tension. The agreement between exact and approximate method is so close in this case that the use of the exact method is entirely unwarranted. In general it will be found that results given by the approximate method are sufficiently accurate for all practical purposes. It is therefore recommended that the approximate method of Art. **14** be used instead of the exact method of the present article.

In certain cases of trusses with an uneven number of panels, or when the arrangement of members is such that the division of the main truss into independent systems is ambiguous, the approximate methods of Art. **14** do not yield reliable results. Under such conditions the exact method must be used.

9. Stresses in Structures with Two or More Redundant Members.—The stresses in structures containing two or more redundant members may be determined by an extension of the methods given in Art. **7.** It will be assumed, as in Art. **7,** that the structure is composed of members subjected to direct stress only.

Let S = stress due to the given loads in any of the n necessary members composing the frame; S_1, S_2, S_3 etc. = the simultaneous stresses in the several redundant members due to the given loads, the subscript in each case denoting the redundant member under consideration; u_1 = stress in redundant member No. 1 due to the load of 1 lb. acting as a tension along the line of that redundant member, assuming that all other redundant members are removed from the structure while values of u_1 are being calculated; u_2, u_3, etc. = corresponding values for redundant member No. 2, No. 3, etc.; S' = stress in any one of the n necessary members due to the given loads, all redundant members considered as removed from the structure; and l, A, and E with proper subscripts = length, gross area, and modulus of elasticity respectively for each member.

As in Art. **7,** the deflection of the necessary members along the line of any one of the redundant members must be equal to the deformation of that redundant member due to its stress. We may then write for the several redundant members, the following condition equations:

$$\sum_0^n \frac{Sl}{AE}u_1 + \frac{S_1l_1}{A_1E} = 0$$

$$\sum_0^n \frac{Sl}{AE}u_2 + \frac{S_2l_2}{A_2E} = 0$$

$$\sum_0^n \frac{Sl}{AE}u_3 + \frac{S_2l_2}{A_2E} = 0$$

The stress S in any member, in terms of S' and the stresses in the several redundant members is given by the expression

$$S = S' + S_1u_1 + S_2u_2 + S_3u_3 + \text{etc.} \tag{8}$$

Substituting the value of S from Eq. (8) in the above condition equations, noting that the value of u_1, u_2, u_3 etc. for the corresponding redundant member is unity, we have the following general equations:

$$\left.\begin{aligned}
S_1\sum_0^{n+1}\frac{u_1^2l}{AE} + S_2\sum\frac{u_1u_2l}{AE} + S_3\sum\frac{u_1u_3l}{AE} + \dots \text{ etc.} &= -\sum_0^{n}\frac{S'u_1l}{AE} \\
S_1\sum\frac{u_2u_1l}{AE} + S_2\sum_0^{n+2}\frac{u_2^2l}{AE} + S_3\sum\frac{u_2u_3l}{AE} + \dots \text{ etc.} &= -\sum_0^{n}\frac{S'u_2l}{AE} \\
S_1\sum\frac{u_3u_1}{AE} + S_2\sum\frac{u_3u_2l}{AE} + S_3\sum_0^{n+3}\frac{u_3^2l}{AE} + \dots \text{ etc} &= -\sum_0^{n}\frac{S'u_3l}{AE}
\end{aligned}\right\} \tag{9}$$

In these equations, $\sum_0^n$ indicates that only the necessary members are included in the summation; $\sum_0^{n+1}$ indicates that the necessary members and redundant member No. 1 are included; $\sum^{n+2}$, $\sum^{n+3}$ indicate corresponding values for redundant members No. 2 and No. 3; and terms similar to $\sum\frac{u_2u_1l}{AE}$ indicate that this summation includes those members which have stresses due to 1-lb. loads acting as tensions in redundant members No. 1 and No. 2, or whatever members may be indicated by the subscripts.

Equations (9) offer a set of simultaneous equations in terms of the unknown stresses in the redundant members and certain other quantities whose values may be determined as soon as the form and dimensions of the truss, the sectional area of its members, and the amount and character of the applied loads are known. In any given case these terms may be evaluated and the resulting set of simultaneous equations may be solved for the required stresses in the redundant members. Note that there will always be available as many independent equations as there are redundant members.

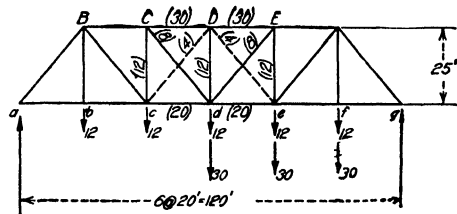



Fig. 21.

9a. Stresses in a Pratt Truss with Rigid Diagonals in Two Panels.—The truss of Fig. 21 has rigid diagonals in the panels each side of the center of the truss. Applying Eq. (1) it will be found that there are two redundant members in place.

Let it be required to determine the stresses in all members of the two center panels due to dead panel loads of 12,000 lb. at all joints, and live panel loads of 30,000 lb. at joints d, e, and f, Fig. 21. Members Dc and De will be assumed as redundant, and their stresses will be denoted by S_1 and S_2 respectively.

Equations (9) of Art. **9** will then take the form

$$S_1 \sum_{o}^{n+1} \frac{u_1^2 l}{AE} + S_2 \sum \frac{u_1 u_2 l}{AE} = -\sum_{o}^{n} \frac{S' u_1 l}{AE}$$

$$S_1 \sum \frac{u_2 u_1 l}{AE} + S_2 \sum_{o}^{n+2} \frac{u_2^2 l}{AE} = -\sum_{o}^{n} \frac{S' u_2 l}{AE}$$

Values of u_1 are due to 1-lb. loads acting in the line of redundant member Dc, and u_2 are corresponding values for De. All necessary data are given in the following table. Dead and live load values of S' are kept separate.

Member	l (in.)	A (sq. in.)	u_1	$\frac{u_1^2 l}{A}$	u_2	$\frac{u_2^2 l}{A}$	$\frac{u_1 u_2 l}{A}$	Dead load: S'	$\frac{S' u_1 l}{A}$	$\frac{S' u_2 l}{A}$
CD	240	30	−0.625	3.13	....	.	. .	−43,200	+216,000	. .
DE	240	30	....	...	−0.625	3.13	. .	−43,200	. .....	+216,000
cd	240	20	−0.625	4.69	. . .	.	..	+38,400	−288,000	
de	240	20	.. .		−0.625	4.69	.....	+38,400	.	−288,000
Cc	300	12	−0.783	15.35	.		.	−6,000	+118,000	.
Dd	300	12	−0.783	15.35	−0.783	15.35	+15.35	0	0	0
Ee	300	12	......	. ..	−0.783	15.35	... ..	−6,000	..	+118,000
Cd	384	8	+1.0	48.0	.	. .		+7,680	+369,000	
dE	384	8			+1.0	48.0		+7,680		+369,000
$S_1 = cD$	384	4	+1.0	96.0		.		.	.	
$S_2 = De$	384	4	.		+1.0	96.0				
			..	182.52		182.52	+15.35	.	+415,000	+415,000

Dead load: $\frac{S' u_1 u_2 l}{A}$	$S_1 u_1 + S_2 u_2$	S	Live load: S'	$\frac{S' u_1 l}{A}$	$\frac{S' u_2 l}{A}$	$\frac{S' u_1 u_2 l}{A}$	$S_1 u_1 + S_2 u_2$	S	Member
. .	+1,310	−41,890	−72,000	+ 360,000			+ 8,400	−63,600	CD
.	+1,310	−41,890	−72,000	. . ...	+360,000	.	− 700	−72,700	DE
. .	+1,310	+39,710	+48,000	− 360,000	.	.	+ 8,400	+56,400	cd
...	+1,310	+39,710	+48,000	. .	−360,000		− 700	+47,300	de
..	+1,640	− 4,360	−30,000	+ 588,000	. .	.	+10,500	−19,500	Cc
0	+1,640	+ 3,280	0	0	0	0	+10,500	+ 9,620	Dd
	+1,640						− 880		
...	+1,640	− 4,360	0	....	0	..	− 880	− 880	Ee
. .	−2,100	+ 5,580	+38,400	+1,845,000	. ..	...	−13,440	+25,060	Cd
...	−2,100	+ 5,580	0	.. ... .	0	.	+ 1,130	+ 1,130	dE
...	−2,100	− 2,100	... .		. .	.	−13,440	−13,440	cD
.	−2,100	− 2,100	....	... .	... .	...	+ 1,130	+ 1,130	De
.		.	. ....	+2,433,000	0				

On substituting the values of the several summations in the above equations, we have the following simultaneous equations:

Dead Load Stresses

$$182.52S_1 + 15.35S_2 = -415{,}000$$
$$15.35S_1 + 182.52S_2 = -415{,}000$$

On solving these equations, we have

$$S_1 = S_2 = -2{,}100 \text{ lb. compression.}$$

From Eq. (8) values of S have been calculated, as given in the table.

Live Load Stresses

$$182.52S_1 + 15.35S_2 = -2{,}433{,}000$$
$$15.35S_1 + 182.52S_2 = 0$$

from which

$$S_1 = -13{,}440 \text{ lb. compression.}$$
$$S_2 = +\ 1{,}130 \text{ lb. tension.}$$

Resulting values of S are given in the table.

On examining the values of final stresses S, it will be noted that for dead load, the stress in the main diagonal *Cd* is 5,580 lb. tension and the stress in the secondary diagonal *Dc* is 2,100 lb. compression. Hence the main diagonal carries about 73 per cent of the shear on a vertical section through panel *cd*. For live load, stress in Cd = 25,060 lb. tension, stress in Dc = 13,440 lb. compression. In this case, the main diagonal carries 65 per cent of the shear.

In practice, it is usually assumed that when two rigid diagonals are provided in any panel, each member takes half the shear on the section. This assumption might be realized if both diagonals were of equal area. In Fig. 20, one diagonal has an area double that of the other. Hence the more rigid member receives the greater portion of the shear.

10. Design of Structures with Redundant Members.—The design of structures with redundant members is in general a difficult matter. From the general equations given in the preceding articles, it will be noted that the areas of the members must be known before the stresses in the members can be determined. Where approximate methods are available, a solution for stresses may be made by these methods and the areas of members determined. The exact methods may then be applied to the resulting structure. In such cases it may be found that the true stresses differ from those determined by approximate methods to such an extent that revision of the sections become necessary. The process must then be repeated until finally an agreement between required and furnished area is obtained at least for a majority of members. It can readily be seen that this is likely to result in long drawn out and

tedious calculations. In some cases a study of existing structures of the same general type will give some idea of the relative sizes of the members. If such data are not available, it may be assumed that all members are of the same area. Stresses calculated on this assumption may be used in determining areas of members. The process may then be carried out as outlined above. It will be found that the use of influence lines will greatly facilitate the placing of loads for maximum stresses in the members.

In general, it is impossible to obtain an exact agreement between required and furnished areas for every member in the structure. For example, in a structure containing one redundant, the distortion of this member is fixed by the deformation along the line of this member, of the necessary members forming the truss. Hence its unit stress is determined by the unit stresses in all other members of the truss. However, this condition assumes importance only when the redundant systems composing the structure are of very unequal flexibility. This emphasizes the importance of so arranging the members that the redundant systems are as nearly as possible of equal flexibility.

SECONDARY STRESSES

11. General Nature of the Problem.—The methods of stress analysis given in the preceding pages are based on the assumption that the members are connected at the joints by frictionless pins which permit the members to turn at the joints. It is further assumed that the gravity axes of members coincide with the center lines of the truss, that all center lines meet at the pin center at each joint, and that all loads are applied at the joints of the truss. Based on these assumptions it may be assumed that all members are straight between joints for all loading conditions, and that the members are subjected only to direct stresses of tension or compression. Stresses calculated under these ideal conditions are known as the *axial*, *direct*, or *primary* stresses in the members. In the work to follow the latter term will be used.

In practice these ideal conditions are not realized. Where pin joints are provided, these are not frictionless as assumed, but offer considerable resistance to turning of members at the joints. Also, the members are not always straight, and gravity axes and center lines of members may not coincide. All of these departures from the assumed ideal conditions given above cause the truss members to be subjected to bending moments, shears, and axial stresses in addition to the primary stresses mentioned above. These additional stresses are known as *secondary stresses*.

The most important causes of secondary stresses in trusses are the bending of members due to rigidity of joints, eccentric joint details, and beam action where members act as beams to carry the weight of the

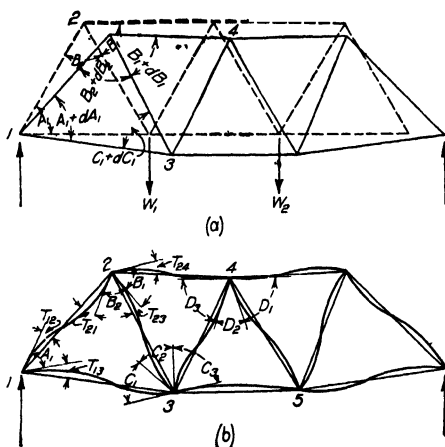
members to the joints or where applied loads between joints must be carried by the truss members.

A general idea of the nature of the bending stresses induced in the members because of rigidity of joints may be obtained from Fig. 22. In Fig. 22*a*, the dotted lines show the outline of an unloaded truss. After the structure is loaded, all members are deformed and the structure is deflected to the position shown by the full lines, assuming the joints to be frictionless pins. This change in form of the truss is due to changes in the angles of the several triangles caused by the deformation of the truss members under direct stress. Thus at joint 2, angle B_1 of the dotted line figure becomes angle $B_1 + dB_1$ of the full line figure. The change in angle is represented by dB_1. Similar conditions exist at other joints. Methods for calculating these angular changes are given in Art. **12**.

Fig. 22.

When the members at any joint are prevented from turning to accommodate themselves to the angular changes because of friction on the pin or because of rigid riveted connections, the angles between the members cannot change as shown in Fig. 22*a*. Since the joints of the truss must deflect to the positions shown in Fig. 22*a*, the members will be bent out of line, as shown by the heavy full lines of Fig. 22*b*. At joint 2 of Fig. 22*b*, the angles B_1 and B_2 are the same as shown for the dotted line truss of Fig. 22*a*. At the ends of any member, as 1-2, Fig. 22*b*, the tangents to the elastic curve of the member make angles shown by τ_{12} and τ_{21}. The determination of these angles and the resulting moments and fiber stresses will be given in the articles which follow.

Secondary stresses due to eccentric joints and to beam action of truss members are due largely to continuity of members because of rigid joints. The moments and fiber stresses due to these effects are often large, as shown by the calculations and discussion given in the following articles. In general there are also certain axial secondary stresses concurrent with those due to bending. However, these axial secondary stresses are generally so small that they may be neglected.

The complete analysis of stresses in structures, as carried out in practice, does not in general include the determination of secondary stresses. Experience gained in practice, and checked by theory, shows that secondary stresses may be kept low if rules, such as those given in

Art. **24,** are followed in designing the truss members. Since secondary stresses depend in general upon deformations of truss members, the primary stresses must first be calculated, the structure designed, and then the resulting secondary stresses for the structure as designed may be determined.

The method of secondary stress calculation which will be presented in the following pages is based on certain fundamental principles first published by Professor Winkler in his "Bruckenbau." The method of Winkler has been used in Modern Framed Structures Part II by Professor Turneaure, who has simplified the notation and devised schemes for carrying out the calculations which has made this very useful method available for practical designing. The tabular forms used in the discussion which follows are modeled after the work of Professor Turneaure.

12. Determination of Changes in Angles of Any Triangle Due to Distortion of Members under Stress.—Let ACB of Fig. 23 show any triangle of a truss. Let a, b, and c represent the lengths of the sides of this triangle and assume that S_a, S_b, and S_c represent the stresses in the members. Assume these stresses to be *tension*. Required an expression for the change in angle CAB due to the above stresses. Let α represent angle CAB, and let $d\alpha$ = change in angle α.

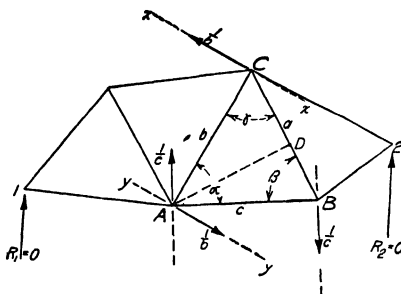



Fig. 23.

Since the deformations of members are all very small, compared to the dimensions of the structure, $d\alpha$ may be taken as equal to the sum of the angular rotations of members AC and AB. The angular rotation α_b of member AC is equal to the deflection of C along a line x-x perpendicular to AC plus the deflection of A along line y-y perpendicular to AC, divided by b, the length of AC. That is

$$\alpha_b = \frac{\sum \frac{Sl}{AE} u}{b}$$

The derivation of this equation, which is Eq. (1), p. 367, is given in Art. **2**a, p. 366, where the notation is also explained. In the case under consideration values of u are to be determined for 1-lb. loads acting along lines x-x and y-y. It can readily be seen from the above equation that the result will be the same if loads of $\frac{1}{b}$ lb., acting as shown in Fig. 23, are substituted for the 1-lb. load. In the same way, loads of $\frac{1}{c}$ acting

perpendicular to AB will give the angular rotation of AB. Adding the angular rotations thus obtained we have

$$d\alpha = \sum \frac{Sl}{AE} u \tag{1}$$

in which values of u are to be determined for the loading conditions shown on Fig. 23. By the methods of statics it can readily be shown that the several values of u are as follows:

$$u_a = +\frac{1}{c} \csc \beta$$

$$u_b = -\frac{1}{b} \cot \gamma$$

and

$$u_c = -\frac{1}{c} \cot \beta$$

Plus signs indicate tension, minus signs indicate compression. In Eq. (1), $\frac{S}{A}$ = unit stress in any member. Let s_a, s_b, and s_c = *unit stresses* in the several members, assumed to be tension in each case. On substituting in Eq. (1) values of u as given above, and lengths of members as shown in Fig. 23, we have

$$d\alpha = \frac{1}{E}(s_a a u_a + s_b b u_b + s_c c u_c)$$

$$d\alpha = \frac{1}{E}\left(s_a \frac{a}{c} \csc \beta - s_b \cot \gamma - s_c \cot \beta\right)$$

If AD be drawn perpendicular to CB, it can readily be shown that $AD = \frac{a}{\cot \gamma + \cot \beta}$. Now $\frac{a}{c} \csc \beta = \frac{a}{c \sin \beta} = \frac{a}{AD}$. Hence $\frac{a}{c} \csc \beta = \cot \gamma + \cot \beta$. On placing this term in the above equation, we have finally

$$d\alpha = \frac{1}{E}[(s_a - s_b) \cot \gamma + (s_a - s_c) \cot \beta] \tag{2}$$

In the same manner, we derive for angles β and γ, the values

$$d\beta = \frac{1}{E}[(s_b - s_a) \cot \gamma + (s_b - s_c) \cot \alpha] \tag{3}$$

and

$$d\gamma = \frac{1}{E}[(s_c - s_b) \cot \alpha + (s_c - s_a) \cot \beta] \tag{4}$$

These angular changes are positive, or increasing, when the sign of the result is positive and decreasing when the sign is negative.

As a check on the calculated values of angular changes in any triangle, we have $d\alpha + d\beta + d\gamma = 0$.

Illustrative Problem.—The triangle of Fig. 24 forms a part of a truss system. Unit stresses and lengths of members are as shown on the figure. Calculate the angular changes due to the given unit stresses.

For the given dimensions, $\cot\alpha = 15/20 = 3/4$, $\cot\beta = 0$, and $\cot\gamma = 20/15 = 4/3$. $E = 30{,}000{,}000$ lb. per sq. in.

From Eqs. (2), (3), and (4),

$$d\alpha = \frac{1}{30{,}000{,}000}\left[(16{,}000 + 8{,}000)\left(\frac{4}{3}\right) + (16{,}000 - 10{,}000)(0)\right]$$
$$= +0.0010667 \text{ radians.}$$
$$d\beta = \frac{1}{30{,}000{,}000}\left[(-8{,}000 - 16{,}000)\left(\frac{4}{3}\right) + (-8{,}000 - 10{,}000)\left(\frac{3}{4}\right)\right]$$
$$= -0.0015167 \text{ radians.}$$

and

$$d\gamma = \frac{1}{30{,}000{,}000}\left[(10{,}000 + 8{,}000)\left(\frac{3}{4}\right) + (10{,}000 - 16{,}000)(0)\right]$$
$$= +0.0004500 \text{ radians}$$

As a check note that the sum of the angular changes is zero Since there are 57.3 deg in one radian, expressed in circular measure, $d\beta = (0.0015167)(57.3)(60) = 5.214$ minutes.

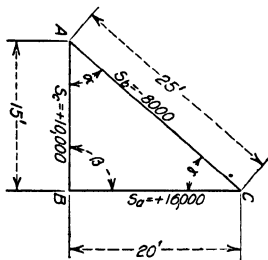
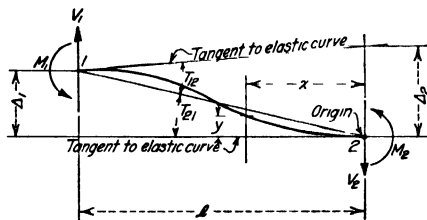



FIG. 24. FIG. 25.

13. Moments at the Ends of a Beam in Terms of the Deflection Angles.

—In Fig. 22, the members of the truss are shown as bent out of line due to the deflection of the truss. Figure 25 shows member 1-2 removed from the structure. The tangents to the elastic curve at the ends of the member make angles with the original axis 1-2 which are indicated by τ_{12} and τ_{21}. In order that the member may assume the form shown in Fig. 25, moments M_1 and M_2 and shears V_1 and V_2 must act at the ends of the member. Since the deformations are relatively small the effect of the axial stress in the member may be neglected in determining the end moments.

From Fig. 25, the deflection angles at the ends of the member have the values $\tau_{21} = \frac{\Delta_1}{l}$ and $\tau_{12} = \frac{\Delta_2}{l}$. Values of Δ_1 and Δ_2 may be determined

by substitution in equation $\frac{d^2y}{dx^2} = \frac{M}{EI}$, which is the general equation of the elastic curve for a bent beam or member.

With respect to an origin at 2, the moment at x, Fig. 25, due to forces to the right of that point is

$$M_x = M_2 - V_2 x$$

From moments about 1, $V_2 = \frac{M_1 + M_2}{l}$. Then

$$M_x = M_2 - (M_1 + M_2)\frac{x}{l}$$

On substituting this value of M_x in the general equation of the elastic curve, we have

$$EI\frac{d^2y}{dx^2} = M_2 - (M_1 + M_2)\frac{x}{l}$$

Integrating twice,

$$EI\frac{dy}{dx} = M_2 x - (M_1 + M_2)\frac{x^2}{2l} + C_1$$

and

$$EIy = \frac{M_2 x^2}{2} - (M_1 + M_2)\frac{x^3}{6l} + C_1 x + C_2$$

To determine the constants of integration, note that when $x = 0$, $\frac{dy}{dx} = 0$ and $y = 0$. Hence $C_1 = 0$ and $C_2 = 0$. From Fig. 25, note that $y = \Delta_1$, when $x = l$. We then have finally,

$$\tau_{21} = \frac{\Delta_1}{l} = \frac{l}{6EI}(2M_2 - M_1) \qquad (5)$$

By a similar process it can be shown that

$$\tau_{12} = \frac{\Delta_2}{l} = \frac{l}{6EI}(2M_1 - M_2) \qquad (6)$$

On solving Eqs. (5) and (6) for M_1 and M_2, we have

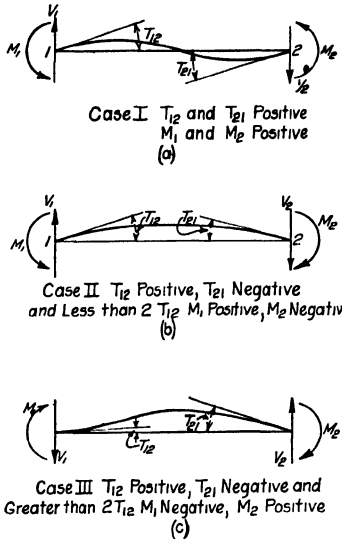
$$M_1 = \frac{2EI}{l}(2\tau_{12} + \tau_{21}) \qquad (7)$$

and

$$M_2 = \frac{2EI}{l}(2\tau_{21} + \tau_{12}) \qquad (8)$$

In Eqs. (5) to (8), counterclockwise moments and deflection angles will be considered as positive.

The values of end moments M_1 and M_2 given by Eqs. (7) and (8) depend upon values of τ_{12} and τ_{21}, the end deflection angles. These

angles may be either positive or negative. The character of the end moments and the shape of the elastic curve of the member therefore depend upon the relative magnitude and character of the end deflection angles. Figure 26 shows three typical cases which may be encountered. It is to be noted that for each case shown in Fig. 26 there is a parallel case in which the signs of the end deflection angles are opposite from those given. Cases I and III represent *double curvature* and Case II represents *single curvature*. A special case of Case II occurs when τ_{21} is negative and equal to $2\tau_{12}$. Then from Eq. (7), $M_1 = 0$, or, point 1 is hinged. Hence, when a moment acts only at one end of a member, the other being hinged, the end deflection angle at the free end is one-half the angle at the restrained end. Cases II and III show that the character of the moment at the end of a member cannot be determined from the sign of the deflection angle at that end of the member. The combined effect of end deflection angles must be considered, as indicated by Eqs. (7) and (8).

Case I T_{12} and T_{21} Positive M_1 and M_2 Positive (a)

Case II T_{12} Positive, T_{21} Negative and Less than 2 T_{12} M_1 Positive, M_2 Negativ (b)

Case III T_{12} Positive, T_{21} Negative and Greater than $2T_{12}$ M_1 Negative, M_2 Positive (c)

FIG. 26.

Illustrative Problem.—Assume that the angular changes calculated in the problem on p. 406 are taken by member AC of Fig. 24 and calculate the bending moments at the ends of the member. The moment of inertia of the member is 4,500 in.[4] (Note that these assumed conditions do not exist, for the angular changes of Fig. 25 affect both members entering the joints in proportion to their relative rigidities.)

Let τ_{AC} = deflection angle at A, Fig. 24, and let τ_{CA} = deflection angle at C. Then from the results given on p. 401, subject to the assumptions made above, we have $\tau_{AC} = d\alpha = +0.0010667$ and $\tau_{CA} = d\gamma = +0.0004500$. From Eqs. (7) and (8) with E = 30,000,000 lb. per sq. in.; I = 4,500 in ^{4}; and l = 25 ft. = 300 in., we have

$$M_A = M_1 = \frac{(2)(30,000,000)(4,500)}{300}[(2)(+0.0010667) - 0.0004500]$$
$$= +\,2,325,000 \text{ in.-lb.}$$
$$M_B = M_2 = \frac{(2)(30,000,000)(4,500)}{(300)}[(2)(+0.00045) + 0.0010667]$$
$$= +\,1,770,000 \text{ in.-lb}$$

Note that the curvature of the member and the character of end moments are covered by Case I of Fig. 26a since τ_{AC} and τ_{CA} have a positive sign.

14. Equilibrium of a Joint.—Since any structure as a whole is in equilibrium, every part of that structure must also be in equilibrium. In Fig. 27 joint 2 of Fig. 22 is removed from the structure with all forces

in position. Assuming that the direct stresses meet at point 2, their moment is zero about that point and the joint is held in equilibrium by the moments induced by the bending of the members and by other moments due to the shear in the members acting at distances d_1, d_2, etc. from the intersection of the members. These moments and shears are equal and opposite to those at the ends of the members shown in Fig. 25.

In general, the moments due to shears are very small compared to those due to bending. The moments due to shear may then be neglected without appreciable error. On this assumption, the conditions of equilibrium for joint 2 are that

$$\Sigma M_2 = M_{24} + M_{23} + M_{21} = 0 \qquad (9)$$

FIG. 27.

In this equation M_{24} = moment at joint 2 in member 2-4. Using the same scheme of notation, M_{42} = moment at 4 in member 2-4. This notation for moments will be adopted in the work which follows.

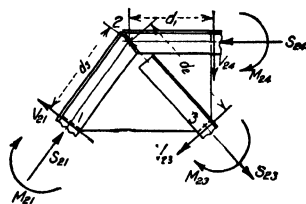
The conditions of equilibrium expressed by Eq. (9) may also be written in terms of end deflection angles by substituting in place of the several moments their values as given by Eqs. (7) and (8). To secure uniform notation for angles and moments, let τ_{24} = deflection angle at 2 for member 2-4, and use similar notation for other angles. We then have from Eq. (9) and Eqs. (7) and (8)

$$\Sigma M_2 = \frac{2EI_{24}}{l_{24}}(2\tau_{21} + \tau_{42}) + \frac{2EI_{23}}{l_{23}}(2\tau_{23} + \tau_{32}) + \frac{2EI_{21}}{l_{21}}(2\tau_{21} + \tau_{12}) = 0$$

Each term in this equation contains a quantity $\frac{I}{l}$. Let $\frac{I}{l} = K$, with a subscript which will indicate the member in question. After dividing by $2E$, the above equation becomes

$$K_{24}(2\tau_{24} + \tau_{42}) + K_{23}(2\tau_{23} + \tau_{32}) + K_{21}(2\tau_{21} + \tau_{12}) = 0 \qquad (10)$$

Equation (10) is the fundamental condition equation which expresses the requirements for equilibrium of any joint in terms of the deflection angles at the ends of the members entering that joint.

Condition equations similar to Eq. (10) may be written for each joint in the structure, giving as many equations as there are joints. However, the values of τ in Eq. (10) are all at present unknown. Since there are two values of τ for each member, it can be shown by Eq. (1), p. 384 that for a truss with m joints, there will be $4m$-6 unknown values of τ. As there are only m condition equations (one for each of the m joints) the values of τ cannot be determined from the available condition

equations of the form of Eq. (10). In the following article, methods will be given for reducing the number of unknowns so that the equations can be solved.

15. Values of Deflection Angles in Terms of Angular Changes. Reference Deflection Angles.—The number of unknown deflection angles at any joint can be reduced to one by means of certain geometrical relations which exist between the several angles at any joint. Figure 28 shows the conditions at joint 2 of Fig. 22. In Fig. 28, B_1 and B_2 are the original angles between members 2-4 and 2-3, and 2-3 and 2-1. Angular changes in these angles are represented by dB_1 and dB_2. The end deflection angles for the several members are represented by τ_{24}, τ_{23}, and τ_{21}.

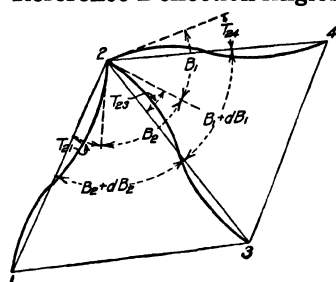



FIG. 28.

From Fig. 28 it can be seen that

$$\tau_{23} + B_1 = B_1 + dB_1 + \tau_{24}$$

or

$$\tau_{23} = \tau_{24} + dB_1 \tag{11}$$

Again

$$\tau_{21} + B_2 + B_1 = B_2 + dB_2 + B_1 + dB_1 + \tau_{24}$$

or

$$\tau_{21} = \tau_{24} + dB_1 + dB_2 \tag{12}$$

In Eqs. (11) and (12), the value of a given deflection angle is expressed in terms of one of the deflection angles, which will be called the *reference deflection angle*, and the sum of the angular changes in the angles between the reference deflection angle and the given deflection angle. The values of the angular changes dB_1, and dB_2 of Eqs. (11) and (12) can be calculated from Eqs. (2), (3), or (4) as soon as the form of truss and loading conditions are known. Therefore, all deflection angles at any joint may be determined in terms of a single unknown reference angle and known angular changes. In this manner the total number of unknowns for the entire structure can be reduced until they are equal to the number of available condition equations.

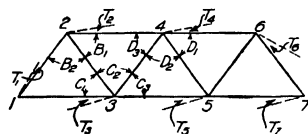



FIG. 29.

The reference angle for each joint may be taken as any convenient deflection angle. However, in order to simplify the arranging of calculations in tabular form, it is best to have some standard method of choosing the reference angles. The following rule will be adopted in

the work which follows: The reference angle for any joint will be taken as the deflection angle of the first member encountered in passing around that joint in a *clockwise direction,* beginning on the outside of the truss. The reference angle will be denoted by τ with a subscript corresponding to the joint notation. Thus in Fig. 29, τ_1, τ_2, τ_3, etc. indicate the reference angles as selected by the above rule.

The above notation for reference angles will now be introduced into Eqs. (11) and (12). We will first adopt some convenient form of notation for the summations of angular changes which appear in these equations. Let $\sum_{23}^{24} dB$ = sum of angular changes in angles between members 2-4 and 2-3 entering joint 2, which in this case is angle B_1 of Fig. 29; also, $\sum_{21}^{24} dB$ = sum of angular changes for all angles between members 24 and 21 of Fig. 29, that is, angles B_1 and B_2. Equations (11) and (12) then take the form

$$\left.\begin{aligned} \tau_{23} &= \tau_2 + \sum_{23}^{24} dB \\ \tau_{21} &= \tau_2 + \sum_{21}^{24} dB \end{aligned}\right\} \tag{13}$$

16. Equilibrium Equation for Any Joint in Terms of Reference Angles.—Values of deflection angles expressed in the form of Eq. (13) may be substituted in Eq. (10), giving an equilibrium equation for the joint in terms of the reference angles. On examining Eq. (10), which is written for joint 2, Fig. 29, it will be found that deflection angles for joints 1, 3, and 4 are involved. Expressed in the form of Eq. (13) these angles are as follows for the conditions shown in Fig. 29:

$$\text{Joint 1; } \tau_{12} = \tau_1$$

$$\text{Joint 3; } \tau_{32} = \tau_3 + \sum_{32}^{31} dC$$

$$\text{Joint 4; } \tau_{42} = \tau_4 + \sum_{42}^{46} dD$$

On substituting these values in Eq. (10) we have

$$\begin{aligned} &K_{24}\left(2\tau_2 + \tau_4 + \sum_{42}^{46} dD\right) \\ &+K_{23}\left(2\tau_2 + 2\sum_{23}^{24} dB + \tau_3 + \sum_{32}^{31} dC\right) \\ &+K_{21}\left(2\tau_2 + 2\sum_{21}^{24} dB + \tau_1\right) = 0 \end{aligned}$$

This equation may be written in the form

$$2\tau_2(K_{21} + K_{23} + K_{24}) + 2\left(K_{21}\sum_{21}^{24} dB + K_{23}\sum_{23}^{24} dB\right) + (K_{21}\tau_1) + \left(K_{23}\tau_3 + K_{23}\sum_{32}^{31} dC\right) + \left(K_{24}\tau_4 + K_{24}\sum_{42}^{46} dD\right) = 0 \qquad (14)$$

Equation (14) is the general form of the independent equilibrium equation for joint 2 of Fig. 29. This equation contains as unknowns the reference τ for joint 2 and the reference τ for the joint at the opposite end of each member entering joint 2. The equation also contains certain known quantities which depend for their values upon K and the end deflection angles for the several members.

For convenience in tabulation, Eq. (14) may be written in the more general form

$$2[(\Sigma K)\tau_n + \Sigma(K\Sigma d\ \angle)] + \left[K_{mn}\tau_m + K_{mn}\sum_{mn}^{\tau_m} d\ \angle\right] + \text{(similar terms for other members entering joint } n) = 0 \qquad (15)$$

In this equation the notation is as follows: Let n = any joint (for example, 2 of Fig. 29); let mn represent any member entering joint 2 (say 42 of Fig. 29); and let m represent the joint at the far end of member mn (say joint 4 of Fig. 29). A little study of Eq. (14) with this notation in mind will make clear the meaning of Eq. (15). Further explanation of the use of Eq. (15) is given in the problem of Art. **19**.

Equations similar to Eq. (15) may be written for each joint in the structure. The resulting equations form a set of linear simultaneous equations containing as unknowns the values of τ for the several joints of the truss Since the number of unknowns is equal to the number of independent equations, the values of τ are readily determined.

17. Moments and Fiber Stresses in Terms of τ.—Having solved the linear simultaneous equations of the form of Eq. (14) and determined all values of τ, the deflection angles at the ends of each member may be determined by means of equations of the form of Eq. (13). On substituting values of these deflection angles in Eqs. (7) and (8), moments at the ends of the several members may be calculated, and from these the extreme fiber stresses are readily determined.

For any member nm, the moment at n is

$$M_{nm} = 2EK_{nm}(2\tau_{nm} + \tau_{mn}) \qquad \textbf{(16)}$$

in which $K_{nm} = \frac{I}{l}$ = moment of inertia of cross-section divided by length of member.

If f_{nm} = fiber stress at n in member nm, we have

$$f_{nm} = \frac{Mc}{I} = \frac{2Ec}{l}(2\tau_{nm} + \tau_{mn}) \tag{17}$$

in which c = distance from neutral axis to extreme fiber.

18. Arrangement of the Calculations.—The determination of secondary stresses by means of the method outlined above is greatly facilitated by the use of convenient and systematic methods of tabulating the calculations. For convenience the work will be divided into the following stages: (*a*) Determination of primary unit stresses due to the applied loading, (*b*) determination of angular changes by means of Eqs. (2), (3) and (4) of Art. **12**; (*c*) tabulation of values of angular changes and cal-

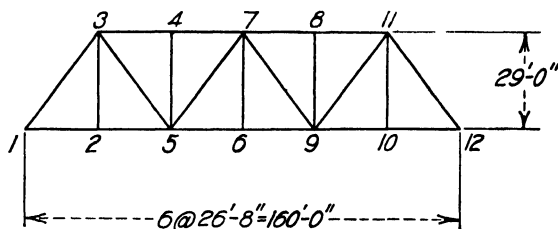



Fig. 30.

culation of ΣdB, ΣdC, etc. of Eq. (14) which will hereafter be referred to as $\Sigma d\angle$; (*d*) formulation of the linear equations similar to Eq. (15) for each joint; (*e*) solution of these equations; and (*f*) calculation of values of τ, and the moments and secondary fiber stresses in the several members.

The general process of secondary stress calculation as given in the preceding articles will be illustrated in the following article by means of a problem.

19. Calculation of the Secondary Stresses in a Riveted Warren Truss. The secondary stresses in the members of the truss of Fig. 30 will be calculated assuming that the truss is loaded with 1,000-lb. loads at each lower chord joint. Table A gives the properties of the members of the truss. In column 5, c = distance from extreme fiber to neutral axis. Where the member is unsymmetrical about the neutral axis, T denotes c for top fiber, B denotes c for bottom fiber. In numbering the joints of the truss, it will be found convenient to follow the order in which the joints would be taken in determining stresses by the method of successive joints.

TABLE A

Properties of Sections

Member	Area (sq. in.)	Length (in.)	I (in.⁴)	c (in.)	$K = \frac{I}{l}$	$2\frac{c}{l}$	Make-up of section
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1–3	60.92	472.75	3,612.5	T 8 275 B 12 850	7.6415	T 0.03008 B 0 054365	1 Cov Pl 24" × 5/8" 4 ∟s 3½" × 3½" × 5/8" 2 Web Pls 20" × 3/4"
3–4 4–7	56.94	320	3,408 9	T 8 378 B 12 747	10 652	T 0.052360 B 0.079670	1 Cov. Pl 24" × 9/16" 4 ∟s 3½" × 3½" × 5/8" 2 Web Pls 20" × 11/16"
1–2 2–5	36.64	320	2,103 4	10.75	6.5732	0.067190	4 ∟s 3½" × 3½" × ½" 2 Web Pls. 21" × 9/16"
5–6	63.20	320	3,236 3	10 75	10 113	0 067190	4 ∟s 3½" × 3½" × ½" 4 Web Pls. 21" × 9/16"
2–3 4–5 6 7	19 32	348	119.50	6 1875	0 34340	0 035562	4 ∟s 6" × 4" × 3/8" 1 Web Pl. 13" × 3/8"
3–5	33.72	472.75	235 34	6 3125	0 49780	0 026701	4 ∟s 6" × 4" × 11/16" 1 Web Pl 13" × 5/8"
5-7	29 42	472 75	805 40	7.50	1.7036	0 031730	2 [s – 15" @ 50 lb

19a. Calculation of Primary Unit Stresses.—Since the truss and its loading are symmetrical about the center of span, stresses need be determined only for one-half the structure. Methods of calculation are given under Stresses in Trusses with Horizontal Chords. The unit stresses in the members of the left half of the truss are shown on Fig. 31. All values are given in pounds per square inch. To assist in the calculation of angular changes, the several unit stresses have been labeled according to the notation used in Eqs. (2), (3) and (4). This procedure is of considerable assistance especially for one unfamiliar with the methods of calculation.

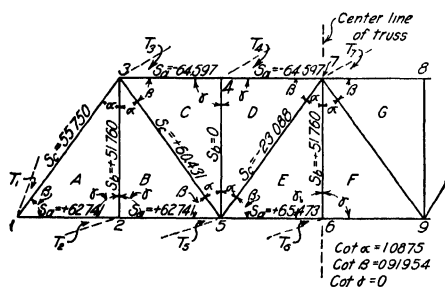



FIG. 31.

19b. Calculation of Angular Changes.—For convenience in making the calculations, the angles in Fig. 31 have been labeled to correspond to the angle notation used in the derivation of Eqs. (2), (3), and

(4). As stated in the preceding article, this has also been done for the unit stresses in the members. Each triangle has been designated by a letter, as shown on Fig. 31.

Since the angles in all triangles are similar, values of cot α, cot β, and cot γ may be determined and placed in a convenient position on Fig. 31. The calculations for angular changes are as given below. Note that the sum of the angular changes for any triangle total zero.

In carrying out the calculations for angular changes as given by Eqs. (2), (3), and (4), the value E will be omitted for the present. In this way considerable difficulty will be avoided in locating the position of the decimal point. Later in the work it will be found that the determination of moments and fiber stresses requires a multiplication by E. It will simplify the calculations to omit E in both places.

TABLE B.—ANGULAR CHANGES, TRUSS OF FIG. 30

Triangle	Angular change	Value
ΔA	$\angle 132 = d\alpha = (+62.741 + 55.750) \cot \beta$	$= +108.95$
	$\angle 213 = d\beta = (+51.760 + 55.750) \cot \alpha$	$= +116.92$
	$\angle 321 = d\gamma = (-55.750 - 51.760) \cot \alpha + (-55.750 - 62.741) \cot \beta$	$= -225.87$
ΔB	$\angle 235 = d\alpha = (+62.741 - 60.431) \cot \beta$	$= +\ 2.1241$
	$\angle 352 = d\beta = (+51.760 - 60.431) \cot \alpha$	$= -\ 9.4298$
	$\angle 523 = d\gamma = (+60.431 - 51.760) \cot \alpha + (+60.431 - 62.741) \cot \beta$	$= +\ 7.3057$
ΔC	$\angle 354 = d\alpha = (-64.597 - 60.431) \cot \beta$	$= -114.97$
	$\angle 435 = d\beta = (0 - 60.431) \cot \alpha$	$= -\ 65.72$
	$\angle 543 = d\gamma = (+60.431 - 0) \cot \alpha + (+60.431 + 64.597) \cot \beta$	$= +180.69$
ΔD	$\angle 457 = d\alpha = (-64.597 + 23.088) \cot \beta$	$= -\ 38.169$
	$\angle 574 = d\beta = (0 + 23.088) \cot \alpha$	$= +\ 25.108$
	$\angle 745 = d\gamma = (-23.088 - 0) \cot \alpha + (-23.088 + 64.597) \cot \beta$	$= +\ 13.061$
ΔE	$\angle 576 = d\alpha = (+65.473 + 23.088) \cot \beta$	$= +\ 81.437$
	$\angle 657 = d\beta = (+51.760 + 23.088) \cot \alpha$	$= +\ 81.398$
	$\angle 765 = d\gamma = (-23.088 - 51.760) \cot \alpha + (-23.088 - 65.473) \cot \beta$	$= -\ 162.835$

ΔF same as ΔE

ΔG same as ΔD

19c. Calculation of Values of $\Sigma d\angle$ and $K\Sigma d\angle$.—Since the structure and its loading are symmetrical about the center of the truss, the tabulations of Table C need be carried only to the truss center. If the truss or its loading is unsymmetrical, the tabulation must include all joints of the truss.

Column 1 of Table C contains the joints taken in order. Column 2 contains the angles at the several joints, beginning with the angle nearest the reference τ for that joint. Thus for joint 3, Fig. 31 shows that the angles in order, beginning at the reference angle τ_3 are 435, 532, and 231, which is the order given in Table C. Column 3 gives the angular changes $d\angle$ which have been calculated in Table B. Column 4 gives

the summations of values of $d\angle$ given in column 3. Thus for joint 3, the angular change for $\angle 435$ is -65.72 and for $\angle 532$ the angular change is $+2.124$. The sum of these changes, which is -63.60, is recorded opposite $\angle 532$. Careful attention must be paid to the sign of the result in every case. In column 5 the members meeting at any joint are given. These members are listed in the order in which they are encountered in passing around the joint to the right beginning at the reference angle. Note that the joint number is stated first, and

TABLE C.—VALUES OF $\Sigma d\angle$ AND $K\Sigma d\angle$

Joint (1)	$\angle$ (2)	$d\angle$ (3)	$\Sigma d\angle$ (4)	Member (5)	K (6)	$K\Sigma d\angle$ (7)
				1–3	7 6415	
1	312	+116 92	+116.92	1–2	6.5732	+ 768 53
					14 2147	768 53
				2–1	6 5732	
	123	−225.87	−225.87	2–3	0.3434	− 77.563
2	325	+ 7 3057	−218.57	2–5	6.5732	−1,436 704
					13 4898	−1,514 267
				3–4	10.6520	
	435	− 65 72	− 65 72	3–5	0 4978	− 32 716
	532	+ 2 124	− 63 60	3–2	0 3434	− 21 840
3	231	+108 95	+ 45 35	3–1	7.6415	+346 542
					19 1347	+ 291 986
				4–7	10.6520	
	745	+ 13 061	+ 13.061	4–5	0 3434	+ 4 485
4	543	+180 69	+193.75	4–3	10 6520	+2,063 825
					21 6474	+2,068 310
				5–2	6.5732	
5	253	− 9 4298	− 9 4298	5–3	0.4978	− 4 6492
	354	−114 97	−124 40	5–4	0 3434	− 42 719
	457	− 38 169	−162 57	5–7	1.7036	− 276 954
	756	+ 81.398	− 81.172	5–6	10 1130	− 820 892
					19 2310	−1,145 211
				6–5	10 1130	
6	567	−162 835	−162 835	6–7	0 3434	− 55 915
	769	−162.835	−325 670	6–9	10.1130	−3,293.501
					20.5694	−3,359.416
				7–8	10.6520	
7	879	+ 25 108	+ 25.108	7–9	1.7036	+ 42.774
	976	+ 81.437	+106.545	7–6	0 3434	+ 36.587
	675	+ 81.437	+187.982	7–5	1.7036	+ 320 246
	574	+ 25.108	+213.090	7–4	10.6520	+2,269.835
					25.0546	+2,669.442

note also that the angle between any two members is listed on the same line as the member more remote from the reference angle. Thus at joint 3, ∠532, which lies between members 35 and 32, is listed on the same line as member 32, which is further from the reference angle τ_3 than is member 35. Column 6 gives the values of K for the several members as recorded in Table A. Finally column 7 gives the products of K column 6 and $\Sigma d \angle$ column 4. The totals of values of K and $K\Sigma d \angle$ for each joint are also given in columns 6 and 7.

19*d*. Formulation of Equations.—In general, independent equations of the form of Eq. (15) are to be formulated for each joint of the structure. However, if the truss and its loading are symmetrical about the center line of the span, there will be some member whose position with respect to the axis of symmetry of the truss is such that the end deflection angles for this member may be determined from the character of the bending to which the member is subjected. Thus in Fig. 30, the center vertical 6-7 is undeformed under symmetrical loading. This is evident when we consider that the effect of the deformations of members to the left of 6-7 must from symmetry be equal and opposite to the effect of members to the right of 6-7. Hence 6-7 remains straight and its end deflection angles τ_{67} and τ_{76} are each equal to zero. Then, as shown below, values of τ_6 and τ_7 may readily be determined. Again, suppose the vertical 6-7 of Fig. 30 is not present and that the center portion of the truss is formed by the triangle 579. In this case, the bending of member 59 is the same at each end and the character of its curvature is as shown in Case II, Fig. 26, subject to the condition that $\tau_{59} = -\tau_{95}$. Also, $\tau_{75} = -\tau_{79}$. It is then possible to determine τ_7 and a value for τ_9 may be determined in terms of τ_5. The equations may then be solved as in the problem under consideration.

The formulation of equations for the truss of Fig. 30 will be explained in detail, using as an example joint 3. Noting that members 4-3, 5-3, 2-3, and 1-3 enter joint 3, Eq. (15) takes the form

$$\begin{aligned}
&2[\tau_3(\Sigma K \text{ for joint } 3) + \Sigma(K\Sigma d \angle) \text{ for joint } 3]\\
&+[\tau_4 K_{43} + K_{43}(\Sigma d \angle \text{ for member } 4\text{-}3)]\\
&+[\tau_5 K_{53} + K_{53}(\Sigma d \angle \text{ for member } 5\text{-}3)]\\
&+[\tau_2 K_{23} + K_{23}(\Sigma d \angle \text{ for member } 2\text{-}3)]\\
&+[\tau_1 K_{13} + K_{13}(\Sigma d \angle \text{ for member } 1\text{-}3)] = 0
\end{aligned}$$

All values listed above are given in Table C. Thus, (ΣK for joint 3) and $\Sigma(K\Sigma d\angle)$ are given respectively in columns 6 and 7 of Table C. The first term is then $2[19.1347\tau_3 + 291.986]$. The remaining terms are found under the joint whose number appears first in the member notation and in the line containing that member. Thus for member 4-3, we find under joint 4 for member 4-3 that $K = 10.6520$ and $K\Sigma d\angle =$ 2,063.825. Then the complete term is $10.6520\tau_4 + 2{,}063.825$. Values

for the remaining members are as follows: Member 5-3, $0.4978\tau_5 - 4.6492$, member 2-3, $0.3434\tau_2 - 77.563$; and member 1-3, $7.6415\tau_1 + 0$. On collecting these several terms, and combining the numerical quantities, we have finally, $7.6415\tau_1 + 0.3434\tau_2 + 38.2694\tau_3 + 10.652\tau_4 + 0.4978\tau_5 = -2{,}565.585$, which is the independent equilibrium equation for joint 3.

A convenient tabulation for the formulation of these equations is given below. The first line gives the joint under which the values are obtained, and the second line gives the member whose values are required. At each joint, the first quantity recorded is twice the summations given for the joint in question under columns 6 and 7 of Table C. The tabulation for all joints follows.

TABLE D.—FORMULATION OF EQUATIONS

	Joint	Member	τ Values	Absolute terms	
Joint No. 1	1	2Σ	$28.4294\tau_1$	+1,537 06	
	3	3-1	$7.6415\tau_3$	346 54	
	2	2-1	$6\ 5732\tau_2$		
				1,883 60	
	$28.4294\tau_1 + 6\ 5732\tau_2 + 7\ 6415\tau_3 = -\ 1{,}883\ 60$ ...... (1)				
Joint No. 2	2	2Σ	$26\ 9796\tau_2$		−3,028.534
	1	1-2	$6\ 5732\tau_1$	+ 768 53	
	3	3-2	$0\ 3434\tau_3$		− 21 840
	5	5-2		−2,281 844	
	$6\ 5732\tau_1 + 26\ 9796\tau_2 + 0\ 3434\tau_3 + 6\ 5732\tau_5 = +\ 2{,}281\ 844$ (2)				
Joint No. 3	3	2Σ	$38\ 2694\tau_3$	+ 583.972	
	4	4-3	$10\ 652\tau_4$	+2,063.825	
	5	5-3	$0\ 4978\tau_5$		− 4 6492
	2	2-3	$0\ 3434\tau_2$		− 77 563
	1	1-3	$7\ 6415\tau_1$	+2,565 585	
	$7.6415\tau_1 + 0.3434\tau_2 + 38.2694\tau_3 + 10.652\tau_4 + 0.4978\tau_5 = -2{,}565\ 585$.. ....(3)				
Joint No. 4	4	2Σ	$43.2948\tau_4$	+4,136.620	
	7	7-4	$10\ 6520\tau_7$	+2,269 835	
	5	5-4	$0.3434\tau_5$		− 42.719
	3	3-4	$10.6520\tau_3$		
				+6,363 736	
	$10.652\tau_3 + 43\ 2948\tau_4 + 0.3434\tau_5 + 10.6520\tau_7 = -\ 6{,}363.736$............................. . ...(4)				

TABLE D.—FORMULATION OF EQUATIONS.—(*Continued*).

	Joint	Member	τ Values	Absolute terms	
	5	2Σ	$38.462\tau_5$		$-2,290.428$
	2	2-5	$6.5732\tau_2$		$-1,436\ 355$
	3	3-5	$0.4978\tau_3$		$-\ \ 32.716$
	4	4-5	$0.3434\tau_4$	$+\ \ 4.485$	
	7	7-5	$1.7036\tau_7$	$+\ 320.246$	
Joint No. 5	6	6-5	$10.113\tau_6$		
				$-3,434\ 768$	
	$6\ 5732\tau_2 + 0\ 4978\tau_3 + 0\ 3434\tau_4 + 38.462\tau_5 + 10.113\tau_6 + 1.7036\tau_7 = +3,434\ 768$ ......(5)				

Tabulations for joints 6 and 7 have not been included in Table D, for as stated above, symmetry of form of truss and loading conditions place at our disposal the following condition equations: At joint 6, $\tau_{67} = 0$; at joint 7, $\tau_{76} = 0$. Values of τ_{67} and τ_{76} may be obtained from Eq. (13) and Table C, which gives in column 4, the values for any member corresponding to ΣdB of Eq. (13). Thus for member 6-7, we may write $\tau_{67} = \tau_6 - 162.835$ and for member 7-6, $\tau_{76} = \tau_7 + 106.545$. On placing equal to zero these values of τ_{67} and τ_{76} we have

$$\tau_6 = +162.835$$

and

$$\tau_7 = -106.545$$

Equations (4) and (5) of Table D contain τ_6 and τ_7 which are now known quantities. These known values may then be inserted in these equations and the resulting absolute terms transferred to the right-hand side of the equations. The revised equilibrium equations then become

$$8.4294\tau_1 + 6\ 5732\tau_2 + 7.6415\tau_3 = -1,883.60 \quad (1)$$
$$6\ 5732\tau_1 + 26\ 9796\tau_2 + 0\ 3434\tau_3 + 6\ 5732\tau_5 = +2,281\ 844 \quad (2)$$
$$7.6415\tau_1 + 0\ 3434\tau_2 + 38\ 2694\tau_3 + 10\ 652\tau_4 + 0\ 4978\tau_5 = -2,565.585 \quad (3)$$
$$10\ 652\ \tau_3 + 43\ 2948\tau_4 + 0\ 3434\tau_5 = -5,228\ 872 \quad (4)$$
$$6.5732\tau_2 + 0.4978\tau_3 + 0.3434\tau_4 + 38.462\tau_5 = +1,969\ 530 \quad (5)$$

19e. Solution of Equations.—Equations (1) to (5) given at the end of the preceding article form a set of linear simultaneous equations from which the several values of τ may be determined. The general process involved in the solution of any such set of equations is the elimination of the unknowns one by one until finally a single equation is obtained containing only one unknown. The value of this unknown, when determined, may be substituted back in the other equations and all of the unknowns may finally be determined.

An order of elimination which tends toward uniform accuracy throughout the calculations is shown in Table E. Select first all equations which begin with τ_1. These are Eqs. (1), (2), and (3). Divide all terms in each of these three equations by the coefficient of τ for its leading term. The resulting equations appear in Table E as Eqs. (1′), (2′) and (3′). To eliminate τ_1 from these three equations subtract the several terms of the equation having the least coefficients from the other equations of the group. An inspection of the equations will determine which equation is to be subtracted from the others. However, by a simple system of notation, the proper order of elimination can easily and quickly be determined. In addition to the notation given in the left-hand column of Table E, let each equation be identified by its original equation number, which has been placed in the second column of Table E. Thus Eqs. (1′), (2′), and (3′) are also given the numbers 1, 2, and 3. Let these numbers be called the *reference numbers* of the equations. Since the coefficients of Eq. (1′) are smallest, this equation is to be subtracted from Eqs. (2′) and (3′). The coefficients of Eq. (1′) being small, Eqs. (*a*) and (*b*) obtained by subtracting Eq. (1′) from Eqs. (2′) and (3′) will be quite similar to Eqs. (2′) and (3′). Therefore give Eqs. (*a*) and (*b*) the same reference numbers as the equations from which they were derived, that is 2 and 3, as shown in Table E. The next group of equations between which elimination is to be made are all equations beginning in τ_2. On examining Table E and the list of equations which are to be solved, we find that these are Eqs. (*a*), (*b*), and 5. Again divide by the leading coefficient, giving Eqs. (*a*′), (*b*′) and (5′), whose reference numbers as given in Table E are 2, 3, and 5. Here Eq. (*a*′) (reference number 2) has least coefficients and is to be subtracted from the others. Note that in each case where subtractions have been made, the equation with the least reference number was subtracted from the others. This will always be found to be the case. Therefore, the following rule may be stated: In eliminating between equations of any group, always subtract the equation with the smallest *reference number* from the other equations of the group. In this manner, the solution of the given equations has been carried out in Table E and the value of $\tau_5 = +36.022$ has been determined.

The other values of τ are obtained by successive resubstitution in the equations, beginning at the end of the table. In selecting the equation from which the value of any τ is to be calculated, choose an equation in which the coefficient of the required τ is large compared with the coefficients of τ for the other terms. In this way any errors made in previous calculations become smaller. Such an equation may be obtained by selecting the first equation, reading from the bottom of the table, which has the same reference number as the subscript of the required τ. We then select the following equations from Table E:

For value of τ_4, use Eq. (f'), reference number 4; for value of τ_3, use Eq. (c'), reference number 3; for value of τ_2, use Eq. (a'), reference number 2; and for τ_1, use Eq. (1′), reference number 1.

As an example of the methods used in determining the several values of τ, consider the calculation for value of τ_4. Equation (f') is $\tau_4 + 0.0021992\,\tau_5 = -116.22$. As given in Table E, $\tau_5 = +36.022$. Therefore, $\tau_4 = -116.22 - (0.0021992)(+36.022) = -116.30$. Other values, obtained in a similar manner are as follows:

$$\tau_5 = +36.022; \quad \tau_4 = -116.30;$$
$$\tau_3 = -19.361; \quad \tau_2 = +96.349; \quad \text{and } \tau_1 = -83.332.$$

In carrying out any set of calculations of the type shown in Table E, it is desirable to make frequent check on the accuracy of the calculations. This may be done by substituting values of τ as determined in one of the equations which has not been used in the resubstitution. Also, it is best to select some equation with large coefficients for the several values of τ, for in this manner any error which has been made will appear increased in magnitude. The desired check is obtained by comparing the value as obtained by substituting the several values of τ in the left-hand side of the equation with the absolute term as given in Table E. Thus, having calculated values of τ_2 to τ_5, a check may be had by substituting these values in Eq. (b'). Such a substitution gives 1,446.7, as compared to the absolute term given in Table E. This check would be considered as very satisfactory. If the two quantities differed by an amount exceeding say 5 per cent of the absolute term, a search should be made for the source of error before proceeding with the calculations.

In carrying out any extended set of calculations such as those given in Table E, it is necessary that the same relative accuracy be maintained throughout the work. Any number which is the result of slide rule calculation is generally subject to uncertainty regarding the value of the last significant figure. For example, the number 436,500 may have a value anywhere between 436,450 and 436,550. In reading such a number from the scale, we would probably decide to use four significant figures and read 436,500. Hence to maintain uniform accuracy, always retain the same number of significant figures, regardless of the position of the decimal point. Thus, the numbers 436,500: 43.65, 0.4365, and 0.00004365 have the same relative accuracy, which is one in 4,365. If these numbers be written 436,500, 43.7, 0.44, and 0.00004, their relative accuracies become respectively one in 4365, one in 437, one in 44, and one in 4. Hence any set of calculations involving these numbers has an accuracy equal to that of the lowest value, or one in 4. In the calculations given in Table E, five significant figures have been retained throughout the work.

TABLE E.—SOLUTION OF EQUATIONS

Eq.	Ref. No.	τ_1	τ_2	τ_3	τ_4	τ_5	Absolute term
1	1	28 429	6.5732	7.6415	....	....	− 1,883 6
2	2	6 5732	26.980	0 34340	....	6.5732	+ 2,281 8
3	3	7.6415	0.34340	38.269	10.652	0 49780	− 2,565.6
1′	1	1.0	0 23122	0.26880	....	....	− 66 258
2′	2	1 0	4.1048	0.052245	....	1 0	+ 347.17
3′	3	1 0	0 044938	5.0081	1.3940	0 065148	− 335 75
a	2	...	+ 3 8736	− 0.21655	....	+ 1 0	+ 413 43
b	3	...	− 0 18628	+ 4.7393	+ 1 3940	+ 0 065148	− 269.49
5	5		+ 6 5732	+ 0.49780	+ 0 34340	+ 38.462	+ 1,969.5
a′	2	.	+ 1 0	− 0.055906	....	+ 0 25815	+ 106.730
b′	3	.	− 1 0	+25 442	+ 7.4830	+ 0.34970	− 1,446 6
5′	5	.	+ 1 0	+ 0 075735	+ 0.052245	+ 5 8515	+ 299 62
c	3	......		+25 386	+ 7.4830	+ 0 60785	− 1,339.9
d	5	......	.. .	+ 0.13164	+ 0 052245	+ 5.5933	+ 192 89
4	4	....		+10.652	+43.295	+ 0 34340	− 5,228.9
c′	3	. .	..	+ 1 0	+ 0.29478	+ 0 023915	− 52 781
d′	5	......	....	+ 1.0	+ 0.39685	+ 42 488	+ 1,465 0
4′	4		.	+ 1.0	+ 4.0638	+ 0 032234	− 490 85
e	5	.			+ 0 10207	+ 42 464	+ 1,517.8
f	4				+ 3 7690	+ 0 008289	− 438 07
e′	5		.		+ 1.0	+416 05	+14,870 0
f′	4	....	...		+ 1 0	+ 0 0021992	− 116 22
g	5	.	....		.	+416 05	+14,986.0
g′	5				.	+ 1 0	+ 36 022

19*f*. Calculation of Secondary Stresses.—Having given the values of the reference τ at the several joints, we may determine the end deflection angles for each member. Then from Eq. (16), the moments at the ends of the members are readily determined, and from Eq. (17) the fiber stresses may be calculated. Since E was omitted in calculating angular changes, as noted in Art. **19*b***, E may also be omitted in substituting in Eqs. (16) and (17).

A convenient tabulation for calculation of values of end deflection angles, moments and secondary fiber stresses is given in Table F. Column 1 gives the members at the several joints listed in same order as in Table C. Column 2, Table F, gives the values of the reference τ for each joint as calculated in the preceding article. Column 3 gives the values of $\Sigma d \angle$ taken from Table C. Column 4 gives the values of the end deflection angle for each member. These are calculated from Eq. (13), which may be written in the form

$$\tau_{nm} = \tau_n + \Sigma d \angle$$

Thus for member 4-5 at joint 4, $\tau_n = \tau_4 = -116.30$ and for member 4-5, $\Sigma d\angle = +13.061$. Then $\tau_{45} = -116.30 + 13.061 = -103.24$. Other values are determined in the same manner.

Column 5 contains values of the term $(2\tau_{nm} + \tau_{mn})$ contained in Eqs. (16) and (17). To determine the value of this term for any member, as for example member 2-3 of joint 2, we may write the above expression in the form $(2\tau_{23} + \tau_{32})$. The value of τ_{23} is given for member 2-3, joint 2, of Table E as -129.52. The value of τ_{32} is found at joint 3 for member 3-2 as -82.961. Then

$$2\tau_{23} + \tau_{32} = 2(-129.52) - 82.961 = -342.00$$

the value which appears in the table. Other values are determined in the same manner. Column 6 contains values of K, which are taken from column 6 of Table A.

Column 7 gives values of the moments at the ends of the members as determined from Eq. (16). These values are twice the product of the quantities given in columns 6 and 7. We have in column 7 our first real check on the accuracy of our work. If the work has been correct, the sum of the moments at any joint will total zero. On examining the values given in column 7 for the several joints it will be found that the errors have been very small. Plus signs for the moments given in column 7 indicate positive, or *counter-clockwise* moments.

Column 8 gives values of $2\frac{c}{l}$, which have been taken from column 7 of Table A. Column 9 gives the values of secondary fiber stress f_s as determined from Eq. (17). These values are the products of the quantities given in columns 7 and 8 of Table F. Plus signs indicate *tension* and minus signs indicate *compression*. The fiber stresses given are for the fiber of any member which is first met in passing around any joint in a *clockwise* direction. Thus member 1-2 is shown to have a plus, or tensile fiber stress. In passing around joint 1 of Fig. 31 in a clockwise direction the fiber of member 1-2 which is met first is the top fiber. The top is then the tension side of 1-2 at joint 1 and the bottom is the compression side. Where the member is unsymmetrical, as in the case of 1-3, 3-4, and 4-7, two fiber stresses are given. The character of stress is determined by the rule given above. Column 10 gives values of the primary unit stresses f_p which are taken from the values given on Fig. 31 for the several members. Finally, column 11 gives the percentage of secondary stress for each member which is determined by dividing the value given in column 9 by the one given in column 10. The greatest percentage recorded is 28.8, which occurs in member 6-5 at joint 6.

Figure 32 shows, greatly exaggerated, the nature of the bending of the members due to the applied loads. While Fig. 32 shows clearly the bending of the individual members, it does not represent the true conditions

TABLE F.—VALUES OF SECONDARY STRESSES

Joint	Member (1)	τ_n (2)	$\Sigma d\angle$ (3)	τ_{nm} (4)	$2\tau_{nm} + \tau_{mn}$ (5)	K (6)	M (7)	$\frac{2c}{l}$ (8)	f_s (9)	f_p (10)	$\frac{f_s}{f_p} = \%$ (11)	Member
								0.035008T	− 4.925T		8.84	
1	1–3	− 83.332	0	−83.332	−140 675	7.6415	−2,149.6	0.054365B	+ 7.648B	−55.750	13.70	1–3
	1–2		+116.92	+33 588	+163 525	6.5732	+2,149.6	0.067190	+10.988	+62.741	17.50	1–2
	2–1		0	+96.349	+226 29	6.5732	+2,974.6	0.067190	+15.208	+62.741	24.25	2–1
2	2–3	+ 96 349	−225 87	−129.52	−342.00	0 34340	− 234.76	0.035562	−12.150	+51.760	23.50	2–3
	2–5		−218.57	−122.22	−208 42	6.5732	−2,739.6	0 067190	−14.000	+62.741	22 32	2–5
								0.05236T	+ 2.0278T		3.14	
	3–4		0	− 19.361	+ 38.728	10 652	+ 825.10	0.07967B	− 3 0850B	−64.597	4.77	3–4
	3–5		− 65.72	− 85.081	−143 57	0.49780	− 142 93	0.026704	− 3.8340	+60.431	6.35	3–5
3	3–2	− 19.361	− 63 60	− 82.961	−295.44	0 34340	− 202.90	0 035562	−10.506	+51.760	20.3	3–2
	3–1		+ 45.35	+ 25.989	− 31.354	7.6415	− 479.16	0 054365B	− 1.7048B	−55.750	3.06	3–1
								0 035008T	+ 1.0980T		1.97	
								0.05236T	− 6.600T		10.40	
	4–7		0	−116.30	−126.05	10.652	−2,685.4	0.07967B	+10 045B	−64.597	15.55	4–7
4	4–5	−116.30	+ 13 061	−103.24	−294.86	0.34340	− 202.48	0 035562	−10.485	0		4–5
	4–3		+193 75	+ 77.450	+135 54	10.652	+2,887 8	0.07967B	+10.798B	−64.597	16.70	4–3
								0 05236T	− 7.0980T		10.98	

5	5–2	+ 36.022	0	+ 36.022	− 50 176	6.5720	− 659.50	0.067190	− 3.3700	+62.741	5.37	5–2
	5–3		− 9 430	+ 26.592	− 31 897	0.49780	− 31.756	0.026704	− 0 85180	+60.431	1 41	5–3
	5–4		−124.40	− 88.378	−280.00	0.34340	− 192.316	0.035562	− 9.9580	0		5–4
	5–7		−162.57	−126.55	−171.66	1.7036	− 584.84	0.031730	− 5.4465	−23.088	23.60	5–7
	5–6		− 81 172	− 45 150	+ 72.535	10.113	+1,467.18	0.067190	+ 4.8740	+65.473	7.45	5–6
6	6–5	+162.835	0	+162 835	+280.52	10.113	+5,673.8	0 067190	+18 849	+65.473	28.8	6–5
	6–7		−162.835	0	0	0.34340	0	0 035562	0	+51.760		6–7
7	7–6	−106.54	+106 54	0	0	0 34340	0	0 035562	0	+51.760		7–6
	7–5		+187 98	+ 81.44	+ 36.330	1.7036	+ 123 81	0 031730	+ 1.1528	−23.088	5.0	7–5
	7–4		+213.09	+106.55	+ 96 80	10 652	+2,061.8	0 07967*B*	+ 7.711*B*	−61.597	11.92	7–4
								0.05236*T*	− 5.068*T*		7.84	

at the joints. For example, at joint 6, the tangents to the elastic curves for member 6-5 and the corresponding member to the right of joint 6 in

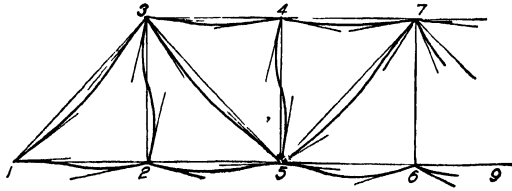



Fig. 32.

reality form a continuous horizontal line. Since the bending of the members and the deflection of the joints are so small compared to the

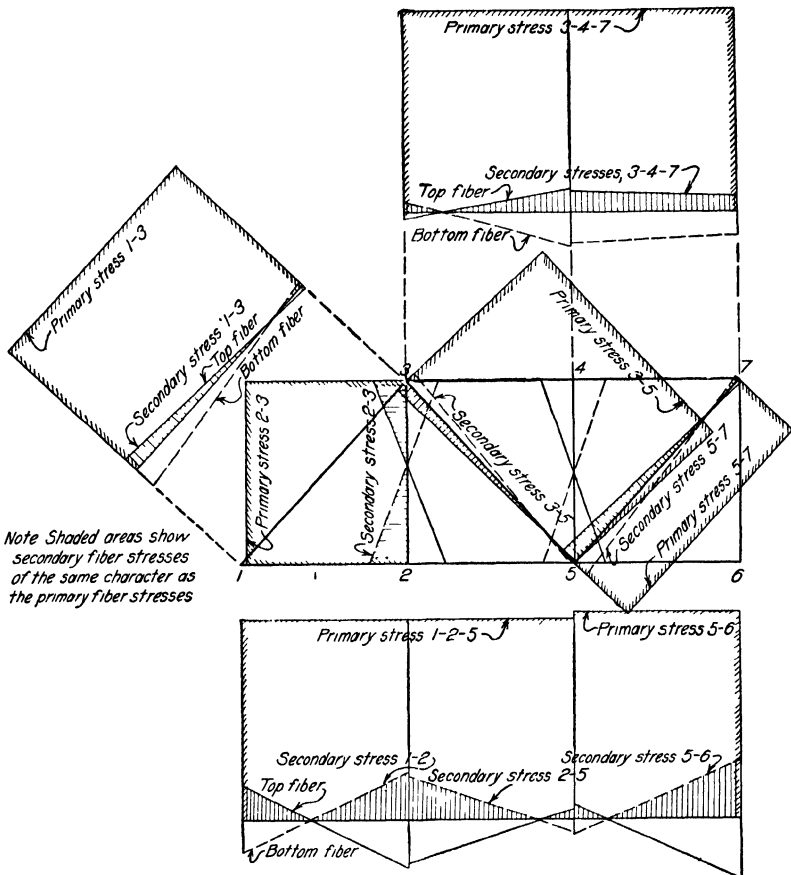



Fig. 33.

dimensions of the structure, it is impossible to represent the true conditions on a sketch. It therefore seems best to show the bending of the

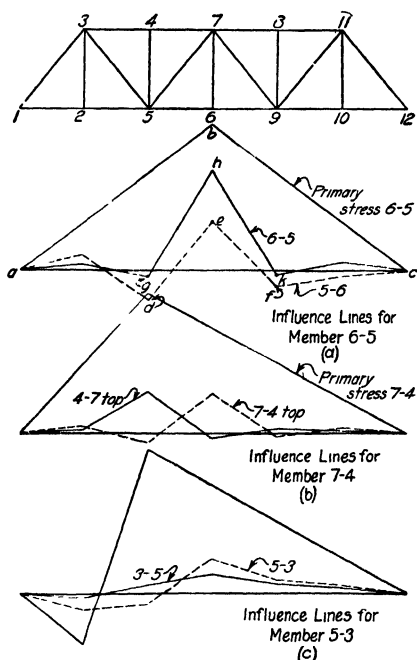
individual members, as in Fig. 32, and not to attempt to show the relation between the bending of members meeting at a joint.

Figure 33 shows the secondary stresses plotted to scale on the members in order to give some idea of the relative values in the several members. The primary unit stresses in the members are also plotted to the same scale as the secondary stresses.

20. Load Position for Maximum Combined Secondary and Primary Stresses.—The position of any set of applied loads which will cause maximum secondary stresses may be determined best by the use of influence lines. These influence lines are drawn for 1-lb. loads applied successively at the several panel points. Methods of calculation for these influence lines are similar to those used in the preceding article.

Fig. 34.

Figure 34 shows the influence lines for several members. Figure 34*a* is the influence line for bottom chord member 6-5 of Fig. 30. The triangle *abc* is the influence line for primary stress in 6-5. The dash line figure *adefc* is the influence line for secondary stress at joint 6 for member 6-5, and the full line figure *aghkc* is the influence line for secondary stress at 5 for member 5-6. As can be seen from these influence lines, the truss is to be fully loaded for maximum primary stress. For secondary stress of the same character as the primary stress, joints 2, 6 and 10 should be loaded for member 6-5, and for member 5-6, joints 2 and 6 should be loaded. However, the secondary stresses are generally small compared to the primary stresses and it will therefore be found that the combined maximum fiber stress due to the primary and secondary stresses will occur for the loading position which will give maximum primary stresses. Figure 34*b* shows the influence line for the compression fiber of top chord member 4-7 and Fig. 34*c* shows the influence line for stress in diagonal 3-5.

In general it will be found that the secondary stresses calculated for the truss fully loaded, as in the preceding article, will give a very good estimate of the secondary stresses likely to be encountered under any loading conditions. The secondary stress relation desired is generally the percentage of secondary to primary stress in any member.

Since this percentage is generally not much affected by the load position, the values given in Table F may be taken as giving the maximum probable percentages for any loading conditions.

21. Effect of Eccentric Connections.—In the calculations given in the preceding articles it has been assumed that the gravity axes of all members lie on the lines connecting the several joints and that the axes of all members at any joint intersect at a single point. Let Fig. 35 represent the conditions at a truss joint where the axes of the members do not meet in a common point. Suppose the gravity axis of the horizontal member ac to be located on the line a_1c_1 at a distance e_2 from its proper position at ac, and suppose the axis of the inclined member ab to be placed at a distance e_1 from its proper position. Let S_2 and S_1 be the stresses in these members. Since these are internal forces their moment about any point will be assumed as positive for clockwise moment. If M_e = moment of S_1 and S_2 about a, the joint center, we have

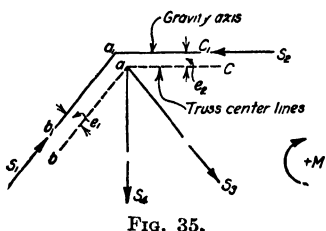



FIG. 35.

$$M_e = +S_1e_1 - S_2e_2 \tag{18}$$

In writing the equilibrium equations of Art. **14,** moments such as M_e of Eq. (18) must be included with the moments at the ends of the members. For eccentric joints in Fig. 27 Eq. (9) would then be written

$$\Sigma M_2 = M_{24} + M_{23} + M_{21} + M_e = 0$$

On substituting in place of the several moments, their values in terms of the end deflection angles for the several members, and following exactly the same process as given in Art. **16,** it will be found that Eq. (15), p. 412 takes the form

$$2[(\Sigma K)\tau_n + \Sigma(K\Sigma d\angle)] + \left[K_{mn}\tau_m + K_{mn}\sum_{mn}^{\tau m} d\angle\right]$$

$$+ \text{[similar terms for other members]} + \frac{M_e}{2E} = 0 \tag{19}$$

Note that Eq. (19) differs from Eq. (15) only in the addition of the term $\frac{M_e}{2E}$

To illustrate the methods of secondary stress calculation when certain truss joints are eccentric, let it be assumed that the gravity axes of all top chord members of the truss of Fig. 30 lie ½ in. above the joint centers, and that the gravity axis of the end post members

lie ¾ in. above the joint centers. Assume, as in Art. **19**, that loads of 1,000 lb. are applied at each lower chord joint.

Figure 36 shows the several joints at which the connections are eccentric. On each member is shown the stress as calculated for the applied loads and the eccentricity of the gravity axes. The eccentric moments at the several joints are as follows:

Joint 1. (Fig. 36*d*)

$$M_e = -(3{,}396.3)(0.75) = -2{,}547.2 \text{ in.-lb.}$$

Joint 3. (Fig. 36*a*)

$$M_e = +(3{,}396.3)(0.75) - (3{,}678.2)(0.5) = +708.10 \text{ in.-lb.}$$

Joints 4 and 7. (Figs. 36*b* and *c*)

$$M_e = 0$$

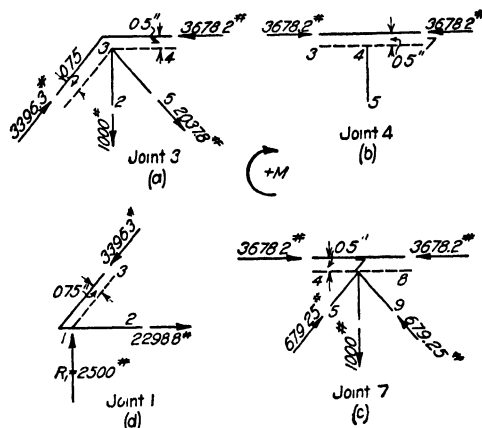



FIG. 36

Since the stresses in all members of the truss are the same as given in the problem of Art. **19**, all values of $\Sigma d\angle$, and also values of $K\Sigma d\angle$ are the same as given in Table C. It can also be seen that the formulation of the equations for values of τ due to the effect of primary stresses in the members will be the same as given in Table D. At joints where eccentricities exist, the moments calculated above must be taken into consideration.

In evaluating the term $\dfrac{M_e}{2E}$ of Eq. (19), the quantity E may be omitted as in the calculations of angular changes. Then to account for the effect of eccentricity at any joint, add one-half the eccentric moment to the values given in Table D. The equilibrium equations given on p. 419 may be modified to account for eccentricity by adding to the absolute term a quantity $-\dfrac{M_e}{2}$. For the conditions stated above only the equations for joints 1 and 3 will be modified. At joint 1, the quantity

to be added is $-(\frac{1}{2})(-2{,}547.2) = +1{,}273.6$ and at joint 3, $-(\frac{1}{2})(+708.10) = -354.05$ is to be added. The equations then become

$$8.4294\tau_1 + 6.5732\tau_2 + 7.6415\tau_3 = -610.00 \quad (1)$$
$$6.5732\tau_1 + 26.9796\tau_2 + 0.3434\tau_3 + 6.5732\tau_5 = +2{,}281.844 \quad (2)$$
$$7.6415\tau_1 + 0.3434\tau_2 + 38.2694\tau_3 + 10.652\,\tau_4 + 0.4978\tau_5 = -2{,}919.635 \quad (3)$$
$$10.652\,\tau_3 + 43.2948\tau_4 + 0.3434\tau_5 = -5{,}228.872 \quad (4)$$
$$6.5732\tau_2 + 0.4978\tau_3 + 0.3434\tau_4 + 38.462\,\tau_5 = +1{,}969.530 \quad (5)$$

The methods employed in the solution of these equations and in the determination of the resulting secondary stresses are exactly the same as given in the preceding articles. The complete solution of the problem will therefore not be given.

It will be found that the effect of external moments at any joint due to eccentric connections is small for all members except those entering the joint at which the moment exists. At the joint in question very large additional secondary stresses are likely to occur.

Fig. 37.

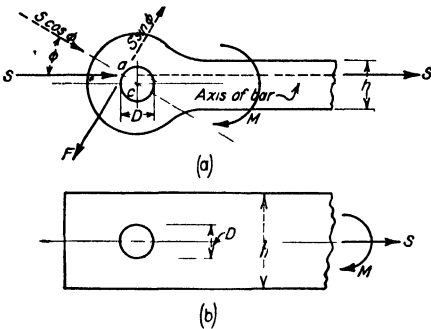
22. Secondary Stresses in Pin Connected Members.—Members which are pin connected at the ends are in general not free to turn about the pins due to the friction between the member and the pin. Figure 37*a* shows the conditions at a pin joint when the stress does not act along the axis of the member. At point *a* where the line of action of S intersects the pin, resolve S into its components tangent and normal to the pin surface. These components are $S \sin \phi$ and $S \cos \phi$. When the member is about to turn on the pin due to a moment M, the force $S \sin \phi = F$, the frictional resistance to turning. Angle ϕ is the friction angle between the two metal surfaces. Then as turning on the pin is about to occur, we have from moments about c the pin center,

$$M = \frac{SD}{2} \sin \phi$$

If coefficient of friction between pin and member $= k = \tan \phi$, we may write without appreciable error, $\sin \phi = \tan \phi = k$. Then

$$M = \frac{SD}{2} k \qquad (20)$$

The secondary stress in an extreme fiber of the member due to this moment is

$$f_s = \frac{Mc}{I} = \frac{SDkh}{4I}$$

in which h = depth of member which is assumed as symmetrical about its gravity axis so that $c = \frac{h}{2}$. If A = area of section and r = its radius of gyration, $I = Ar^2$, and we have

$$f_s = \frac{S}{A}\frac{Dkh}{4r^2} \tag{21}$$

The primary fiber stress in the member is $f_p = \frac{S}{A}$. Hence the percentage of secondary stress is

$$\frac{f_s}{f_p} = \frac{Dkh}{4r^2} \tag{22}$$

The coefficient of friction k may be taken as 0.2. For eyebar members, such as shown in Fig. 37a, $r = 0.29h$, and in general the ratio of pin diameter to depth of bar is approximately $\frac{D}{h} = \frac{3}{4}$. For riveted members, such as shown in Fig. 37b, $r = 0.4h$, approximately, and the ratio $\frac{D}{h}$ is about ⅓. On substituting these values in Eq. (22) we have

For eyebars	For riveted members
$\frac{f_s}{f_p} = 44.5$ per cent	$\frac{f_s}{f_p} = 10.4$ per cent

Thus, eyebars will not turn on the pins until the secondary fiber stresses become about 44.5 per cent of the primary stresses, and riveted members will turn if the secondary fiber stresses exceed about 10.4 per cent of the primary fiber stresses.

In general, eyebars are narrow flexible members, and it is highly improbable that their secondary stresses under ordinary service conditions would exceed about 15 per cent of the primary stresses. Hence it is not probable that an eyebar will turn on the pin. Riveted members are generally rather rigid, and it is probable that the secondary stresses in well designed members will be at least 20 per cent of the primary stresses. Therefore it is almost certain that riveted members will turn on the pins.

When it is uncertain whether members turn on the pins it will be best in calculating secondary stresses to assume first that the members do not turn. The moments at the ends of the members, or the percentages of secondary to primary fiber stresses should then be determined on this assumption. These values should then be compared with those given by Eqs. (20) or (22). The true conditions at the joints can then be determined. If it is found that some members do turn, then the moment, as given by Eq. (20), must be used in the calculations.

Where some of the members of a truss are free to turn at the joints, the secondary stresses in the truss members may be determined by

assuming that the moment of inertia of all members free to turn is zero. Hence, $K = 0$ for such members. However, in calculating angular changes for the several triangles, these members must be included with the others which are not free to turn.

The top chord member in pin connected trusses is generally made as a continuous riveted member over its entire length. Diagonals and verticals are generally pin connected to the top chord member. The top chord also is generally pin connected at the hip joints.

Illustrative Problem.—Assume that the truss of Fig. 30 is pin connected except for the top chord member, which is continuous from joints 3 to 11, at which points it is pin connected. Calculate the secondary stresses in the top chord member for the same loadings as given in Art. **19.** Assume first that the pins are concentric. It will later be assumed that the pins are eccentric.

(*a*) *Pins Concentric.*—For the given conditions, the secondary stresses may be calculated by assuming that all members turn at the joints except the top chord members. To account for this condition, assume $K = 0$ for all members except those of the top chord. Values of $d\angle$ and $\Sigma d\angle$ will be the same as given in Table C, p. 416. However, it will be necessary to consider such values only at joints 3, 4, and 7. The necessary values are given in Table G.

TABLE G.—VALUES OF $\Sigma d\angle$ AND $K\Sigma d\angle$

Joint (1)	$\angle$ (2)	$d\angle$ (3)	$\Sigma d\angle$ (4)	Member (5)	K (6)	$K\Sigma d\angle$ (7)
				3–4	10.6520	0
	435	− 65.72	− 65.72	3–5	0	
3	532	+ 2 124	− 63.60	3–2	0	
	231	+108.95	+ 45.35	3–1	0	
						0
				4–7	10 652	
4	745	+ 13.061	+ 13 061	4–5	0	
	543	+180.69	+193.75	4–3	10.652	+2,063 825
					21 304	+2,063 825
				7–8	10.652	0
	879	+ 25.108	+ 25.108	7–9	0	
7	976	+ 81.437	+106.545	7–6	0	
	675	+ 81.437	+187 982	7–5	0	
	574	+ 25.108	+213.090	7–4	10.652	+2,269 835
						+2,269 835

The formulation of equations is carried out as in Art. **19***d*, using values taken from Table G. At joint 3, member 3-4 is the only one considered. We then have

$$2K_{34}\tau_3 + K_{43}\tau_4 = 21.3040\tau_3 + 10.652\tau_4 + 2{,}063.825 = 0$$

TABLE H.—SECONDARY STRESSES IN TOP CHORD MEMBERS
PIN CONNECTED WARREN TRUSS
Joints Concentric

Joint	Member (1)	τ_n (2)	$\Sigma d\angle$ (3)	τ_{nm} (4)	$2\tau_{nm} + \tau_{mn}$ (5)	K (6)	M (7)	$\frac{2c}{l}$ (8)	f_s (9)	f_p (10)	$\frac{f_s}{f_p} = \%$ (11)	Member
3	3–4	− 40 135	0	− 40 135	0	10 652	0	0 05236*T*	0		0	3–4
								0 07967*B*	0		0	
4	4–7	−113 48	0	−113 48	−120 40	10 652	−2,565 0	0 05236*T*	−6 310	−64 597	9 78	4–7
								0 07967*B*	+9,610			
	4–3		+193 75	+ 80 27	+120 40	10 652	+2,565 0	0 07967*B*	+9 610			4–3
								0 05236*T*	−6 310	−64 597	9 78	
7	7–4	−106 545	+213 09	+106 545	+ 99 61	10 652	+2,122 1	0 07967*B*	+7 935			7–4
								0 05236*T*	−5,210	−64 597	8 07	

TABLE I.—SECONDARY STRESSES IN TOP CHORD MEMBERS
PIN CONNECTED WARREN TRUSS
Joints Eccentric

Joint	Member	τ_n	$\Sigma d\angle$	τ_{nm}	$2\tau_{nm} + \tau_{mn}$	K	M	$\frac{2c}{l}$	f_s	f_p	$\frac{f_s}{f_p} = \%$	Member
3	3–4	+ 9 195	0	+ 9 195	+ 86 35	10 652	+1 839 2	0 05236T	+ 4 510			3–4
								0 07967B	− 6 870	−64 597	10 63	
4	4–7	−125 81	0	−125 81	−145 07	10 652	−3 090 6	0 05236T	− 7 695	−64 597	11 90	4–7
								0 07967B	+11 560			
	4–3		+193 75	+ 67 940	+145 07	10 652	+3 090 6	0 07967B	+11 560			4–3
								0 05236T	− 7 695	−64 597	11 90	
7	7–4	−106 545	+213 09	+106 545	+ 86 28	10 652	+1 838 1	0 07967B	+ 6 860			7–4
								0 05236T	− 4 500	64 597	6 97	

from which

$$21.3040\tau_3 + 10.652\tau_4 = -2{,}063.825 \qquad (1)$$

At joint 4, members 4-3 and 4-7 are to be considered, and we have

$$(2)(21.3040)\tau_4 + (2)(2{,}063.825) + 10.6520\tau_3 + 10.652\tau_7 + 2{,}269.835 = 0$$

From symmetry, $\tau_{76} = 0$, as before. Hence $\tau_{76} = \tau_7 + 106.545 = 0$, and $\tau_7 = -106.545$. Placing this value of τ_7 in the equation for joint 4, we have

$$10.6520\tau_3 + 42.608\tau_4 = -5{,}262.621 \qquad (2)$$

On solving Eqs. (1) and (2), we have

$$\tau_3 = -40.135 \text{ and } \tau_4 = -113.48$$

Secondary stresses, as determined for these values of τ, are given in Table H.

(*b*) *Pins Eccentric.*—Assume that the top chord pins are located 0.5 in. above the truss center lines, as in Art. **21**, and calculate the secondary stresses in the top chord member.

For the assumed conditions, a moment of $-(3{,}678.2)(0.5) = -1{,}839.1$ in.-lb. exists at joint 3 (see Fig. 36*a*). As before, the moments at joints 4 and 7 are zero. Equation (1) above is then modified by the addition of a term $-(\frac{1}{2})(-1{,}839.1) = +919.55$ to the absolute term. The condition equations are then

$$21.304\tau_3 + 10.652\tau_4 = -1{,}144.275 \qquad (1)$$
$$10.6520\tau_3 + 42.608\tau_4 = -5{,}262.621 \qquad (2)$$

On solving these equations, we have $\tau_3 = +9.195$ and $\tau_4 = -125.81$. The resulting secondary stresses are given in Table I. Note that the moment in member 3-4 at joint 3 is equal and opposite to the moment due to eccentricity.

23. Secondary Stresses Due to Weight of Members.

—In Fig. 38, member 1-2 is acted upon by a uniform downward load of w lb. per ft.

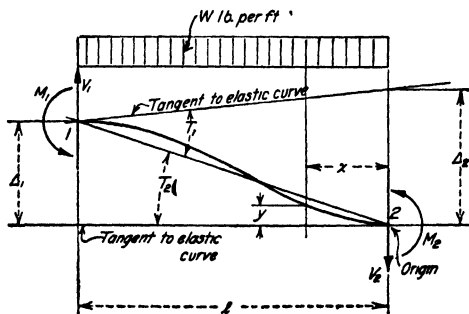



Fig. 38.

in addition to the end moments and shears shown in Fig. 22. The moment at any point distance x from the right end of the member is

$$M_x = M_2 - (M_1 + M_2)\frac{x}{l} + \frac{wlx}{2} - \frac{wx^2}{2}$$

Proceeding as in Art. **13**, it can readily be shown that

$$\tau_1 = \frac{\Delta_2}{l} = \frac{1}{EI}\left[(2M_1 - M_2)\frac{l}{6} - \frac{wl^2}{24}\right]$$

and

$$\tau_2 = \frac{\Delta_1}{l} = \frac{1}{EI}\left[(2M_2 - M_1)\frac{l}{6} + \frac{wl^4}{24}\right]$$

From these equations we have

$$M_1 = \frac{2EI}{l}(2\tau_1 + \tau_2) + \frac{1}{12}wl^2$$

and (23)

$$M_2 = \frac{2EI}{l}(2\tau_2 + \tau_1) - \frac{1}{12}wl^2$$

The last terms of Eq. (23) will be recognized as the end moments in a uniformly loaded fixed beam. If m = moment at end of a fixed beam uniformly loaded, considered as positive moment when it acts clockwise, we may write

$$M_{nm} = \frac{2EI}{l}(2\tau_{nm} + \tau_{mn}) - m_{nm} \quad (24)$$

On placing values similar to Eqs. (24) in Eq. (15), p. 412, we have

$$2[(\Sigma K)\tau_n + \Sigma(K\Sigma d\angle)] + [K_{mn}\tau_{mn} + K_{mn}\Sigma d\angle]$$
$$+ [\text{similar terms for other members}] - \Sigma\frac{m_{nm}}{2E} = 0 \quad (25)$$

The fiber stress at the end of the member due to the moment M_{nm} is

$$f_{nm} = 2\frac{c}{l}(2\tau_{nm} + \tau_{mn}) - m_{nm}\frac{c}{I} \quad (26)$$

Illustrative Problem.—Determine the secondary stresses in the top chord of the truss of Fig. 30, assuming that the top chord member is riveted, being pin connected only at joints 3 and 11, and that all web members are pin connected to the top chord. Assume concentric and eccentric pins, as in the problem of Art. **22.** Let the applied loads be taken as 80,000 lb. at each lower chord joint.

Since the applied loads are proportional to those used in the preceding problems, all values of $d\angle, \Sigma d\angle$, and $K\Sigma d\angle$ may be obtained by multiplying values given in Table G by 80.

In formulating the equilibrium equations after Eq. (25), the quantities involved may be taken from the problem of Art. **22.** The absolute terms there given must be multiplied by 80, and terms of the form m_{nm} must be added. For the truss under consideration, the area of the top chord section, as given in Table A, p. 414, is 56.94 sq. in. Allowing 15 per cent for weight of details, the weight per foot is $w = \left(\frac{56.94}{144}\right)(490)(1.15) = 223$ lb. per ft. The moment at the end of a fixed beam 320 in. long is $m_{nm} = -\frac{1}{12}wl^2 = -(\frac{1}{12})(\frac{223}{12})(320)^2 = -158{,}580$ in.-lb Therefore, the quantities to be added to the absolute terms are $-79{,}290$ for joint 3 and 0 for joint 4, since, as shown by Eqs. (23), the term m_{nm} or $\frac{1}{12}wl^2$ is equal and opposite for members 4-3 and 4-7. The equilibrium equations for concentric pins are therefore,

$$21.304\tau_3 + 10.652\tau_4 = (-2{,}063.825)(80) - 79{,}290$$
$$21.304\tau_3 + 10.652\tau_4 = -244\,396 \quad (1)$$

TABLE J—SECONDARY STRESSES IN TOP CHORD MEMBER DUE TO WEIGHT OF MEMBER

Pins Concentric

Joint	Member (1)	τ_n (2)	$\Sigma d\angle$ (3)	τ_{nm} (4)	$2\tau_{nm} + \tau_{mn}$ (5)	K (6)	M (7)	$\frac{I}{c}$ (8)	f_s (9)	f_p (10)	$\frac{f_s}{f_p} = \%$ (11)	Member
3	3-4	−7,465	0	−7,465	−7,445	10 652	0	406 88*T*	0			3-4
								267 40*B*	0			
4	4-7	−8,015	0	−8,015	−7,507	10 652	− 1,340	406 88*T*	− 3 29	−5,168	0 0637	4-7
								267 40*B*	+ 5 01			
	4-3		+15,500	+7,485	+7,505	10 652	+ 1,340	406 88*T*	− 3 29	−5,168	0 0637	4-3
								267 40*B*	+ 5 01			
7	7-4	−8,524	+17,047	+8,523	+9,031	10 652	+33,800	406 88*T*	− 83 08*T*	−5,168	1 61	7-4
								267 40*B*	+126 4*B*			

TABLE K.—SECONDARY STRESSES IN TOP CHORD MEMBER DUE TO WEIGHT OF MEMBER

Pins Eccentric

Joint	Member (1)	τ_n (2)	$\Sigma d\angle$ (3)	τ_{nm} (4)	$2\tau_{nm} + \tau_{mn}$ (5)	K (6)	M (7)	$\frac{I}{c}$ (8)	f_s (9)	f_p (10)	$\frac{f_s}{f_p}$ = % (11)	Member
3	3–4	−3,518	0	−3,518	− 538	10 652	+147 150	406 88*T*	+361 7*T*			3–4
								267 40*B*	−550 4*B*	−5,168	10 65	
4	4–7	−9,002	0	−9,002	−9,481	10 652	−43 420	406 88*T*	−106 80*T*	−5,168	2 07	4–7
								267 40*B*	+162 40*B*			
	4–3		+15,500	+6,498	+9,478	10 652	+43,420	406 88*T*	−106 80*T*	−5,168	2 07	4–3
								267 40*B*	+162 40*B*			
7	7–4	−8,524	+17,047	+8 523	+7,565	10 652	+ 3 320	406 88*T*	− 81 60*T*	−5,168	1 58	7–4
								267 40*B*	+124 15*B*			

and

$$10.6520\tau_3 + 42.608\tau_4 = (-5{,}262.621)(80)$$
$$10.6520\tau_3 + 42.608\tau_4 = -421{,}010 \qquad (2)$$

On solving these equations, we have

$$\tau_3 = -9{,}832 \qquad \tau_4 = -3{,}277$$
$$\text{Also } \tau_7 = -(106.545)(80) = -8{,}254$$

The resulting secondary stresses are given in Table J. In determining the values of M given in column 7, the product of the quantities given in columns 5 and 6 is to be combined with the term $-m_{nm}$, which in this case is $-(-158{,}580) = +158{,}580$. To conform to the signs given in Eqs. (23) a plus sign is to be used for members 3-4 and 4-7, and a minus sign for members 4-3 and 7-4. Note that this same sign notation has been observed in formulating the equilibrium equation given above. Values of the fiber stresses may be obtained from Eq. (26) or directly from Table J by multiplying the moments given in column 7 by the value of $\frac{I}{c}$ given in column 8.

Assuming the same eccentricity of pins as in Art. **22,** the equilibrium equations may be obtained from those given on p. 435. We then have

$$21.304\tau_3 + 10.652\tau_4 = (-1{,}144.275)(80) - 79.290 = -170{,}832 \qquad (1)$$
$$10.6520\tau_3 + 42.608\tau_4 = (-5{,}262.621)(80) = -421{,}010 \qquad (2)$$

On solving these equations we have

$$\tau_3 = -3{,}518 \qquad \tau_4 = -9{,}002$$

Values of secondary stresses are as given in Table K. These values are determined by the methods used in Table J.

On comparing the values of secondary stresses given in Tables H and I due to rigidity of joints and the values given in Tables J and K for rigidity of joints and weight of members combined, it will be noted that the effect of weight of members has been to reduce the secondary stresses at the joints. This is due to the fact that the deflection of the truss causes the rigid top chord to bend into a slight curve. This bending causes small compressive stresses in the upper fibers of the top chord member. When the weight of members is included the fiber stresses at the joints are reduced, but those at the center of the member are increased. Figure 39 shows the variation in fiber stress for the assumed conditions. It can be shown that for a beam fixed at the ends and uniformly loaded, the *total* moment at the end and center is $(\frac{1}{12} + \frac{1}{24})wl^2 = \frac{1}{8}wl^2$. Hence to plot the moment diagrams for the case under consideration, we may plot as base lines the moments taken from Tables J and K. These are shown by the dash lines *abc* of Fig. 39. On these lines as bases may be plotted the parabolas *adb* and *bec*. To determine the center ordinates, at *d* and *e*, we have $\frac{1}{8}wl^2 = (\frac{3}{2})(\frac{1}{12})\,wl^2 = (\frac{3}{2})(158{,}580) = 237{,}870$ in.-lb. The fiber stress in the top fiber due to this moment, which is equal for members 3–4 and 4–7 is

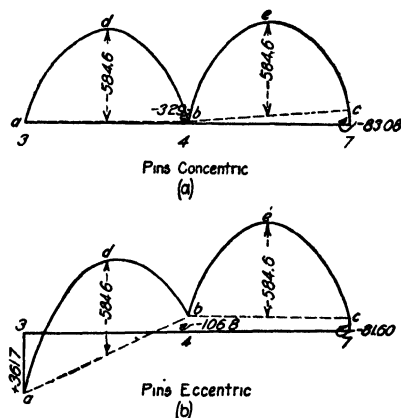



FIG. 39.—Effect of weight of members.

$$f = \frac{237,870}{\left(\frac{I}{c}\right)} = \frac{237,870}{406.88} = 584.6 \text{ lb. per sq. in. compression.}$$

Figure 39 shows the resulting fiber stress diagrams.

24. General Conclusions.—In the preceding articles, a few of the more important causes of secondary stresses have been considered and methods have been given for the determination of the stresses. On examining the formulas and the calculated secondary stresses given above, it can be seen that the width of a member has considerable effect on its secondary stress. As shown by Eq. (17), the fiber stress for a given moment varies directly with the width of member. Hence in general, the narrower the member in the plane of bending, the smaller will be its secondary stress. It will be found that a member whose width is not greater than one-tenth its length will seldom have secondary stresses in excess of 25 per cent of the primary stresses. Where very wide short members are used, as for example, the bottom chord members in through bridges with shallow floors riveted directly to the chords, secondary stresses equal to or greater than the primary stresses are not uncommon. In such cases, secondary stresses should be determined and provision made for them in designing the section.

It will generally be found that the secondary stresses in trusses with the simpler types of framing, such as the Pratt or Warren truss, will be less than in more complicated types of framing, such as trusses with subdivided panels. One of the principal sources of high secondary stresses in trusses with subdivided panels is the bending of chord members due to the elongation of the hangers. [This effect is shown to some extent in the Warren Truss with verticals which is shown in Fig. 32.] The presence of large deflection angles for the chord members meeting at joints 2 and 6 indicates that these members are bent out of the line connecting joints 1 and 5 and joints 5 and 9. This bending is due to the elongation of the tension verticals 2-3 and 6-7. Joint 4 of top chord member 3-4-7 is also subjected to rather large secondary stresses due to the presence of the vertical 4-5. Although this member has no primary stress and so is not distorted, it causes bending in 3-4-7 because it forces joint 4 to follow joint 5, thereby causing bending in the top chord.

These few examples will give some idea of the nature of secondary stresses and the causes contributing to high secondary stresses. For a more complete discussion of the general subject of secondary stresses the reader is referred to "Modern Framed Structures," Part II, by Johnson, Bryan and Turneaure.[1] The application of the method of moment distribution[2] for determining secondary stresses may be found in "Theory of Modern Steel Structures," Vol. II, by L. E. Grinter,[3] and "Continuous Frames of Reinforced Concrete" by Cross and Morgan.[1]

[1] John Wiley & Sons, Inc., New York.

[2] Section 6 presents the theory of moment distribution.

[3] The Macmillan Company, New York.

SECTION 6

STATICALLY INDETERMINATE FRAMES

There are many structures composed of frames that resist shear by virtue of the cross bending in the members of the structure. These structures are statically indeterminate since neither the reactions on the structures nor the stresses in the members can be determined from the conditions of static equilibrium.

Two outstanding methods are available for the determination of the moments in such frames when arching of members does not exist. The method of moment distribution, developed by Professor Hardy Cross,[1] is perhaps the most readily used method and can be adapted with assurance to members with variable section, such as haunched beams which do not develop arching action. The slope deflection method[2] may be used to determine the deflections and angular rotations at the ends of members in such frames. Equations for moments may be developed from the slope deflection method for a variety of commonly used frames. Such equations are presented in Appendix A, and their use may be understood after a study of the section on the slope deflection method. Those members which develop arching action may be analyzed by the "Ellipse of Elasticity" as presented in the volume on arches[3] or by the "Column Analogy" developed by Professor Hardy Cross.[4]

MOMENT DISTRIBUTION

If the joints of a frame are assumed to be fixed against *rotation and translation* during the application of frame loads, the moments at the ends of members entering any given joint (*i.e.*, fixed-end moments) will not total zero except for unusual cases. The algebraic total of such fixed-end moments at a joint may be changed in sign and distributed to each member in proportion to the resistance of the member to angular rotation (*i.e.*, stiffness) in order to obtain a balance. Such "rotational release"

[1] HARDY CROSS, "Analysis of Continuous Frames by Distributing Fixed-end Moments" *Trans. Am. Soc. Civil Eng.*, Vol. 96, pp. 1–156, 1932.

CROSS and MORGAN, "Continuous Frames of Reinforced Concrete," John Wiley & Sons, Inc., New York, 1932.

[2] *Bull.* 108, Engineering Experiment Station, University of Illinois.

[3] HOOL and KINNE, "Reinforced Concrete and Masonry Structures," McGraw-Hill Book Company, Inc., New York.

[4] *Bull.* 215, Engineering Experiment Station, University of Illinois.

would increase the moment at the opposite ends of all members entering a joint and this effect must be accounted for by "carrying-over" the increase, thus disturbing the balance at all joints but to a lesser degree than before. This "distributing" and "carrying-over" may be repeated until the desired degree of accuracy is obtained, *always stopping with a distributing operation.* "Translational release" does not occur during such a procedure and must be provided for by an auxiliary correction often termed the *correction for sidesway* or *side lurch.* It is frequently up to the judgment of the designer as to whether or not such sidesway or side lurch can actually occur in the structure under consideration. Conditions may be such that sidesway is prevented and therefore translational release is not required. If complete translational release is doubtful, it is best to design for the maximum stresses resulting from both extreme conditions (*i.e.*, no sidesway and complete sidesway).

Two distinct operations may be necessary, therefore, in the moment distribution method. These are (1) rotational release and (2) translational release of the joints of a rectangular frame.

1. Rotational Release.– Three factors, used in the release of a joint from complete rotational restraint, are (*a*) the fixed-end moments, (*b*) the stiffness, and (*c*) the carry-over factor for each end of all members. These factors are defined and discussed below.

(*a*) The *fixed-end moment* is that moment which would exist at the completely restrained ends of a beam (or of a column acting as a beam), owing to the transverse loads thereon. The sign may be taken as *positive* when the moment tends to *rotate the joint* in a *clockwise direction.* The fixed-end moments for beams of constant section resulting from various loadings are provided in Tables 2 and 3 of Appendix A and for haunched beams on charts of Appendix C.

(*b*) The *stiffness* of a member for rotational release[1] is a measure of its resistance to angular change at the end under consideration. For the case in which the opposite end to the applied moment is fixed (*i.e.*, release of a joint with all opposing joints fixed), and the member is of constant section and of the same material throughout, the stiffness $\frac{M}{\theta}$ is equal to $\frac{4EI}{L}$ (Fig. 1). If the opposite end to the applied moment is hinged, the stiffness is $\frac{3EI}{L}$ (Fig. 2). Hence, although $\frac{I}{L}$, commonly denoted as K, is only a *relative* measure of the resistance to angular change, it may be used as the stiffness, and the value for a hinged condition[2] may be taken as $\frac{3}{4}$ of that for a fixed condition. Other values may be devel-

[1] The stiffness of a member, for translational release, is a measure of its resistance to translation and will be discussed under the heading "Translational Release."

[2] The moment at the fixed end must be computed with opposite end hinged (see Prob. 1).

oped but are not generally convenient to use. If the moment of inertia is variable, as in haunched beams, the values for stiffness may be obtained from charts of Appendix C.

(c) The *carry-over factor* may be defined as the ratio of free end to fixed-end moments of a cantilever beam subjected to any elastic rotation at the free end, both ends being restrained against translation. That is, if one end is released rotationally after both ends have been assumed fixed, the moment at the fixed end will be changed by the amount of the carry-over factor multiplied by the moment causing the release. This factor is ½ for either end of a beam of constant section (see Fig. 1) but

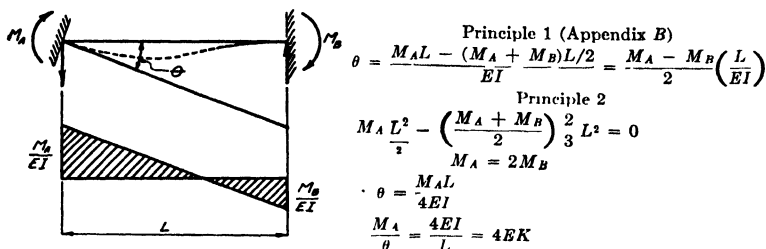



FIG. 1.

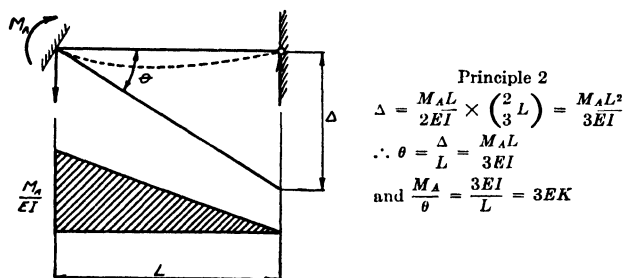



FIG. 2.

will be different for a haunched beam and for each end of an unsymmetrically haunched beam. Values of the carry-over factor for symmetrically and unsymmetrically haunched beams may be obtained from charts of Appendix C. It may be observed from (b) that there is no carry-over factor for a member hinged at one end *if the stiffness at the fixed end is taken as* $\frac{3}{4}\frac{I}{L}$ *and the moment is computed with the opposite end hinged.*

In using these factors for rotational release, the algebraic sum of the fixed-end moments is reversed in sign and distributed to the members in proportion to their stiffnesses; *i e.,* $\left(\frac{K_A}{\Sigma K \text{ at joint}}\right) \times (-\Sigma M \text{ at joint})$ would be added algebraically to the end A entering the joint. This moment which is distributed to A, however, must be multiplied by the

carry-over factor for end A and the product added algebraically to the opposite end B of the member. This must be done for both ends of all members of the frame and the entire process repeated until the desired accuracy is obtained, always stopping with a distributing operation. The procedure is illustrated in Figs. 3 and 7a.

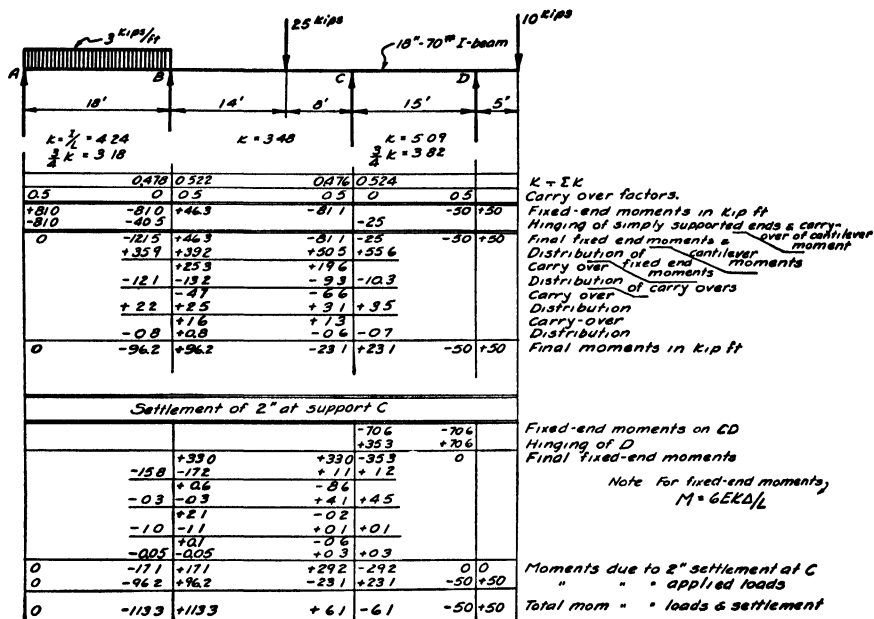



FIG. 3.

2. Translational Release.—For translational release, the joints are assumed to translate, without rotation, under the action of impressed shears opposite in direction to those produced by the restraint of translation. The joints are then released from rotational fixity in identically the same manner as described above. The moments and shears resulting from such a distribution may be uniformly increased or decreased in order to remove the forces of translational restraint by superposition. In allotting the impressed shears to the members, however, the stiffnesses used must be a measure of resistance to deflection at the end under consideration. For the case in which both ends are fixed and the member is of constant section and of the same material throughout, the stiffness is $\left(\frac{V}{\Delta}\right) = \frac{12EI}{L^3} = \frac{12EK}{L^2}$ (Fig. 4). If the opposite end to the applied moment is hinged, the stiffness is $\frac{3EK}{L^2}$ (Fig. 5). Hence, although $\frac{K}{L^2}$ is only a relative measure of resistance to deflection, it may be used as the stiffness factor in *proportioning the impressed shears*, and the value for a

hinged condition may be taken as ¼ of that for a fixed-end condition. After the fixed-end moments at the joints due to the impressed shears are

This should be used with the stiffness of *BA*, $\left(\frac{K}{L^2}\right)_{BA}$, in proportioning impressed shears between *BA* and *CD*. The shear allotted to *CD* is then proportioned between *DE* and *FG* in the usual manner

FIG. 6.

computed, the *joints are allowed to rotate in the usual manner by use of the stiffness and carry-over factors previously computed for rotational release*

The moments resulting from this final operation will yield the true shears producing an arbitrary deflection. The amount of this deflection does not need to be known, since the moments due to a shear equal and opposite to that of translational restraint may be obtained by direct proportion, as previously noted (sidesway moment/final moment of translatory release = unbalanced shear from rotational release/final shear of translatory release) and added algebraically to the moments

Moments and horizontal forces from rotational release of frame joints

Moments and horizontal forces required for translatory release. Obtained from Fig. 7b by proportion

True moments and horizontal forces for vertical loads. Obtained by superposition

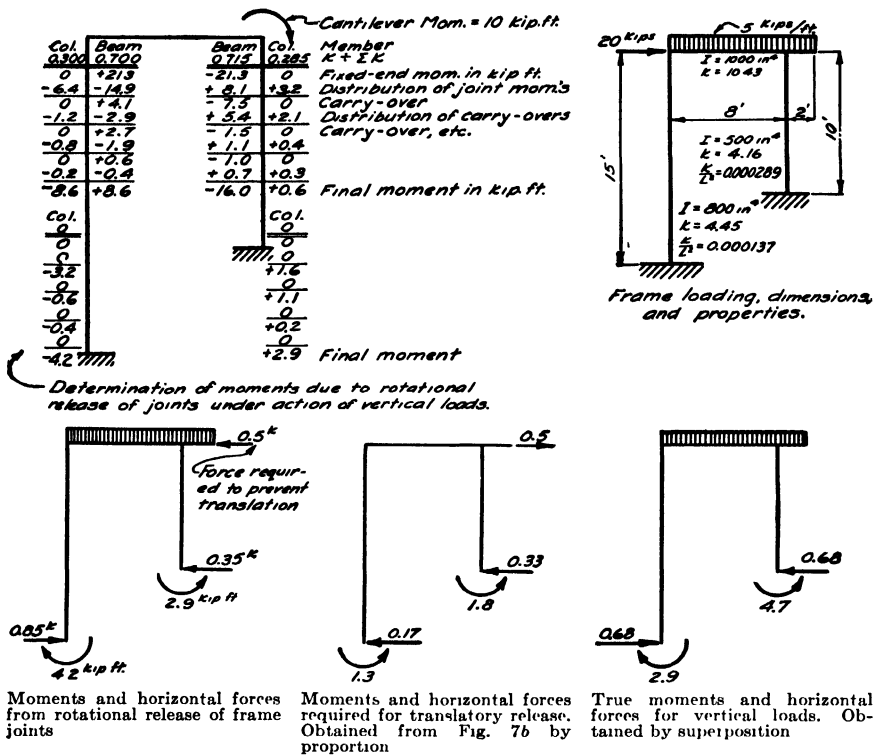



FIG. 7a.

previously obtained by rotational release. In the case of horizontal forces, the impressed shears should be taken in the same direction as the shears produced by the horizontal forces and the final moments and shears obtained by direct proportion. Multi-story frames involve the removal of additional restraining forces at stories other than at the applied load due to the effect of rotational release which disturbs the balance of the shears. The procedure is illustrated in Figs. 8*a* and *b*.

Care must be taken in distributing impressed shears in the case of beams framing into an intermediate level of a column as shown in Fig. 6. In such a case, the deflection at *D* will be inversely proportional to the sum of the translational stiffnesses of *ED* and *GF*, and the deflection of *C*

relative to D will be inversely proportional to the translational stiffness of CD. Hence the total deflection of the system will be proportional to

The columns have been subjected to fixed end moments which result from shears in the columns chosen such that they are proportional to the K/L^2 values. In this case the total shear has been arbitrarily taken as 17.25 kips (that is, 5.55 in left column and 11.7 in right) producing a total moment of 100 kip ft. (41.5 + 58.5). Any other total shear would be satisfactory, since the final moments and shears would be proportional to the former within the elastic range of stresses.

The joints are then released from rotational restraint by distributing and carrying over, resulting in a true translation of an arbitrary amount. Moments and horizontal forces are shown in the lower left-hand sketch.

In the case of one-story frames (all beams colinear), above procedure may be varied by using an arbitrary moment divided in proportion to the K/L values of the columns; that is,

$$\frac{K_1/L_1}{\frac{K_1}{L_1}+\frac{K_2}{L_2}} \div \frac{K_2/L_2}{\frac{K_1}{L_1}+\frac{K_2}{L_2}} = \frac{K_1/L_1^2}{\frac{K_1}{L_1^2}+\frac{K_2}{L_2^2}} \times L_1 - \frac{K_2/L_2^2}{\frac{K_1}{L_1^2}+\frac{K_2}{L_2^2}} \times L_2$$

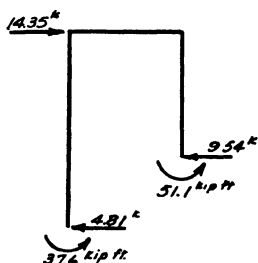
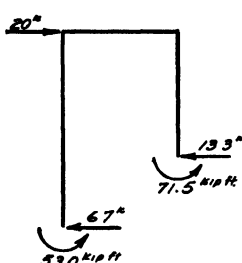
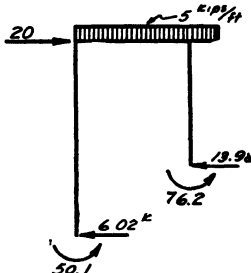
Moments and horizontal forces from translation of arbitrary amount.

Moments and horizontal forces for translation under 20^K load, by proportion.

True moments and horizontal forces for vertical and horizontal loads, obtained by superposition with Fig. 7a.

Fig. 7b.

$\left[\frac{1}{(K/L^2)_{CD}}\right] + \left[\frac{1}{(K/L^2)_{ED} + (K/L^2)_{GF}}\right]$. The stiffnesses to be used in proportioning impressed shears, therefore, may be obtained by taking the reciprocal of such sums (see Fig. 9a). The procedure is illustrated in the example of Figs. 9a, b, and c,

3. Settlement of Supports.—In the case of support settlement, an estimate must be made first of the magnitude of such a movement, or perhaps a possible maximum movement is known. For the vertical and horizontal components of such a movement, the fixed-end moments may be computed from the equations presented in the article on translational release (Figs. 4, 5). A vertical movement will induce these moments into the connecting horizontal members, while a horizontal movement will affect the vertical member which is undergoing the moment. For the

Frame loading, dimensions and properties.

Fig. 8a.

rotational component, the fixed-end moments may be computed from the equations in the article on rotational release (Figs. 1, 2). Such moments

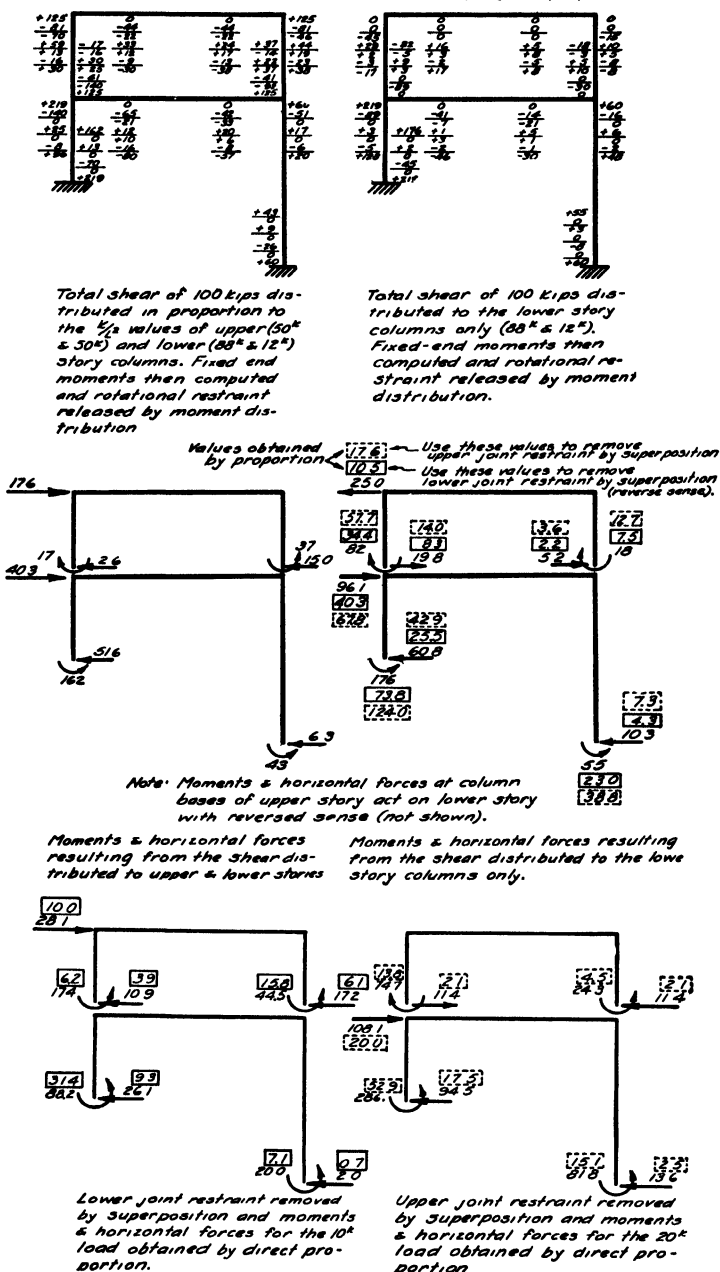



Fig. 8b.

will affect the vertical member undergoing the rotation. All such fixed-end moments should be balanced in the usual manner for rotational

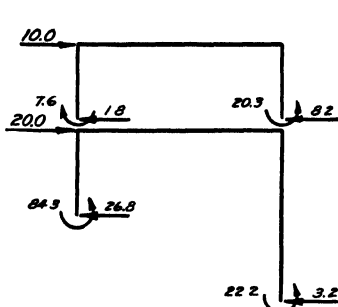
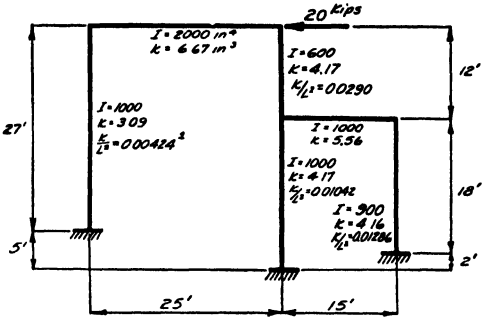
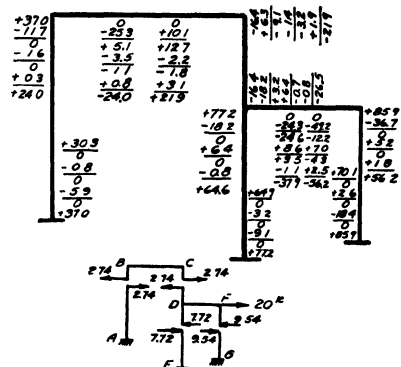
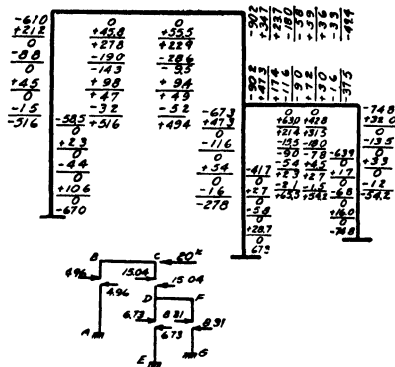
release correcting for sidesway if necessary. Such procedure is shown in Fig. 3.

Moments and horizontal forces for actual loading. Obtained by superposition of 10ᵏ & 20ᵏ moments & forces.

FIG. 8c.

k/L² values computed by use of L in feet & k in inches³

Frame loading, dimensions, & properties

FIG. 9a.

Shear stiffness of *CDEFG*

$$= \frac{1}{\frac{1}{0.0290} + \frac{1}{0.01042 + 0.01286}} = 0.0129$$

Shear stiffness of $AB = 0.00424$

Shear distributed to CD

$$= \left[\frac{0.0129}{0.0129 + 0.00424}\right] 20 = 15.04$$

Shear stiffness of *ABCD*

$$= \frac{1}{\frac{1}{0.00424} + \frac{1}{0.0290}} = 0.0037$$

Shear stiffness of $EDFG = 0.01042 + 0.01286$
$= 0.02328$

$$\text{Shear to } CD = \left[\frac{0.0037}{0.0037 + 0.02328}\right] 20 = 2.74$$

Distribution of arbitrary upper and lower story shears, followed by distribution of resulting fixed-end moments

FIG. 9b.

SLOPE DEFLECTION METHOD

The equations in the tables of Appendix A give the moments in the statically indeterminate frames commonly used when subjected to various loads. A study of the following material is essential to the proper use of these tables.

The sketches have been made a part of the tables so as to identify the different cases at a glance. In addition, an analytical index of the

tables is presented which, it is believed, will make possible a quick location of any table sought.

4. Derivation of Equations.[1]—(1) If a member carries no transverse load, and if one or both ends of the member are rotated, and if one end is deflected relative to the other end, moments are developed in the member and the moment at either end can be expressed in terms of the rotation and deflection. The equation for the moment can be

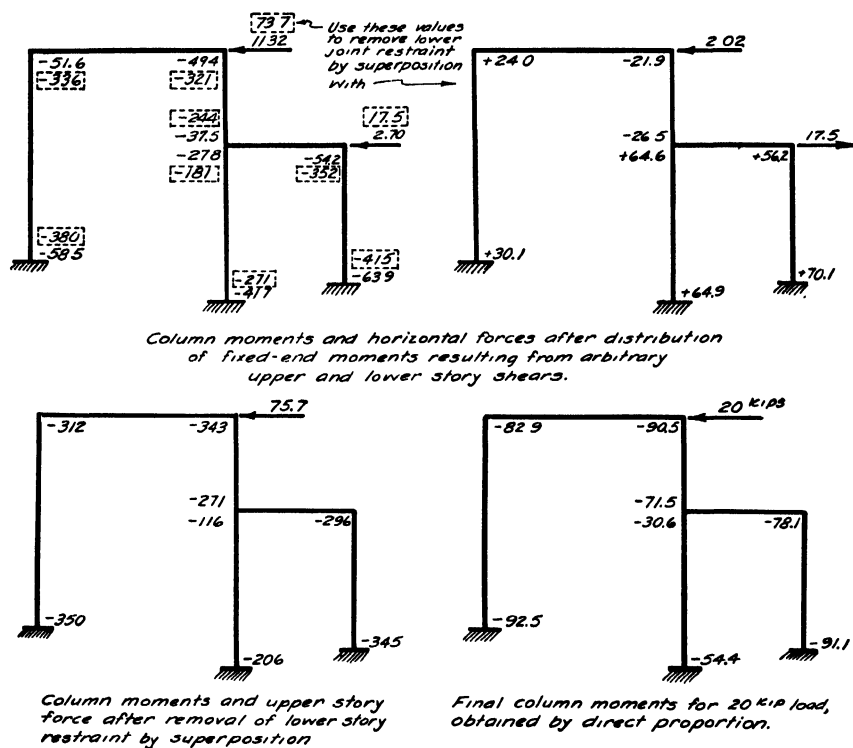



FIG. 9c.

written in the form, $M_{AB} = 2EK(2\theta_A + \theta_B - 3R)$, in which M_{AB} is the moment at A in the member AB, θ_A is the rotation of the end A from its unstrained position, θ_B is the rotation of the end B from its unstrained position $R = \frac{d}{l}$, l is the length of the member, and d is the deflection of one end relative to the other end.

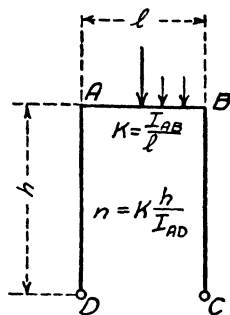
(2) If the ends of a member are fixed so that neither end rotates and so that neither end deflects relative to the other end, a transverse

[1] For complete derivation of the fundamental equations, see Appendix B.

load on the member will produce moments at the ends. Let the moment at the end A of the member AB be represented by C_{AB} and let the moment at the end B be represented by C_{BA}. Then, $M_{AB} = C_{AB}$ and $M_{BA} = C_{BA}$.

(3) If one or both ends of a member are rotated, and if one end is deflected relative to the other end, and if the member also carries transverse loads, then $M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - C_{AB}$. That is, if the ends of the member are restrained and if the member carries a transverse load, the resulting moment is made up of two parts, one part is due to rotation and deflection of the ends, and the other part is due to the load. Furthermore, these two parts composing the total moment maintain their separate identities—that is, the part of the moment due to the load is independent of the rotations and the deflection, and the parts of the moment due to the rotations and deflections are independent of the load.

Fig. 10.

These statements being true, it is apparent that the moment at the end of a member of any structure is composed of two parts, one part is a function of the geometrical properties of the structure and the physical properties of the material and the other part is a function of the load on the member. For example, in Fig. 10, the rectangular bent $ABCD$ carries a vertical load on top. If the bases of the columns are hinged and if the cross-sections of the two legs are identical, the moment at A in the member AB is given by the equation $M_{AB} = -\frac{3}{2}\frac{C_{AB} + C_{BA}}{2n + 3}$,[1] in which n is a function of the geometrical properties of the members, and C_{AB} and C_{BA} are functions of the load. This equation is true no matter what system of loads is applied to the member AB, and the equation may be used to obtain the moment at A in the member due to any load on AB providing the values of C_{AB} and C_{BA} for the given system of loads are known.

Values of C_{AB} and C_{BA} for various loads are given in Tables 2 and 3 of Appendix A. If the load on AB consists of a single load P at the center, then, from Table 3, $C_{AB} = C_{BA} = \frac{1}{8}Pl$ and the equation for the moment at A takes the form

$$M_{AB} = -\frac{3}{2}\frac{\frac{Pl}{8} + \frac{Pl}{8}}{2n + 3} = -\frac{3}{8}\frac{Pl}{2n + 3}$$

Similarly, if the load on AB is a uniformly distributed load, C_{AB} and C_{BA}, as given by Table 3, equal $\frac{Wl}{12}$ and $M_{AB} = -\frac{1}{4}\frac{Wl}{2n + 3}$. In a similar

[1] See Eq. 134, *Bull.* 108, Engineering Experiment Station, University of Illinois.

manner the moment at A can be determined for any load on AB if the values of C are known.

Equations for the moments in the members of a great variety of structures have already been published.[1] However, these equations are general—that is, they are applicable for any system of loads—and before they can be used in the design of a structure subjected to a particular load, values of C for the given load must be substituted in the general equation in order that the particular equation desired may be obtained. In the tables of Appendix A, equations are presented that give the moments in the members of a large variety of structures due to a large number of the specific types of loads usually encountered in structural design. With a few exceptions, these tables contain (1) the general equations applicable to the structure in question when subjected to any system of loads, and (2) a series of special equations, each applicable for a special load.

If equations for any desired special load are not presented, the moments may be obtained from the general equations at the beginning of the table by substituting the values of C, as given in Tables 2 and 3, for the special load. To illustrate how this is done consider the following illustrative problem.

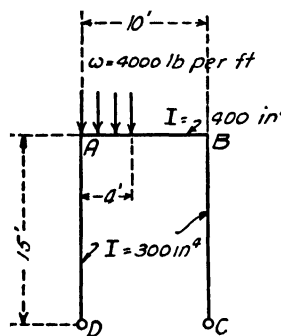



FIG. 11.

Illustrative Problem.—The rectangular bent of Fig. 11 carries a uniformly distributed load over a portion of the top. The cross-sections of the two legs are identical and the legs are hinged at the bases. It is required to find the moments at A and B.

Table 43 is applicable. Either of two methods of procedure may be followed. (1) Algebraic expressions for C_{AB} and C_{BA} may be obtained from Table 2 and substituted in the first equation of Table 43, thus deriving an algebraic equation for the moments at A and B. Or (2) the numerical values of C_{AB} and C_{BA} may be obtained and substituted in the first equation of Table 43, thus getting the numerical values of the moments direct. If a large number of similar problems are to be solved, the first method may be the shorter. If only one or two problems are to be solved, the second method may be the shorter. Both methods follow:

First Solution

From Table 2

$$C_{AB} = \frac{Wa}{12l^2}(3a^2 - 8al + 6l^2), \text{ and } C_{BA} = \frac{Wa^2}{12l^2}(4l - 3a)$$

Substituting these values in the general equation, gives

$$M_{AD} = -M_{BC} = \frac{3}{2}\,\frac{\frac{Wa}{12l^2}(3a^2 - 8al + 6l^2) + \frac{Wa^2}{12l^2}(4l - 3a)}{2n + 3}$$

[1] W. M. WILSON, F. E. RICHART, and CAMILLO WEISS, *Bulletin* 108, Engineering Experiment Station, University of Illinois.

Simplifying, gives

$$M_{AD} = -M_{BC} = \frac{1}{4l} \cdot \frac{Wa(3l - 2a)}{2n + 3} \qquad (1)$$

From the figure: $W = 16{,}000$ lb., $a = 48$ in., $l = 120$ in., $h = 180$ in., $n = {}^{400}\!/_{120}\,({}^{180}\!/_{300}) = 2$.

Substituting these values in Eq. (1), gives

$$M_{AD} = -M_{BC} = \frac{1}{(4)(120)} \cdot \frac{(16{,}000)(48)(3 \times 120 - 2 \times 48)}{(2)(2) + 3} = 60{,}100 \text{ in.-lb.}$$

Second Solution

From Table 2

$$C_{AB} = \frac{(16{,}000)(48)}{(12)(120)(120)}[(3)(48)(48) - (8)(48)(120) + (6)(120)(120)] = 209{,}400 \text{ in.-lb.}$$

$$C_{BA} = \frac{(16{,}000)(48)(48)}{(12)(120)(120)}[(4)(120) - (3)(48)] = 71{,}600 \text{ in.-lb.}$$

$$M_{AD} = -M_{BC} = \frac{3}{2} \cdot \frac{209{,}400 + 71{,}600}{(2)(2) + 3} = 60{,}100 \text{ in.-lb.}$$

The general equations at the beginning of each table have been taken from Bulletin 108 of the Engineering Experiment Station of the University of Illinois. The special equations have been derived from the general equations by the method illustrated in deriving Eq. (1), above.

5. Notation.—The notation used is as follows:

C_{AB} = moment at end A of a member AB that is fixed at both ends, and for which there is no deflection of one end relative to the other. Values of C_{AB} for various loads are given in Tables 2 and 3.

C_{BA} = moment at end B of a member AB that is fixed at the ends and for which there is no deflection of one end relative to the other. Values of C_{BA} for various loads are given in Tables 2 and 3.

d = deflection of one end of a member relative to the other end. Deflections are measured in a direction normal to the original position of the axis of the member.

E = modulus of elasticity.

H_{AB} = moment at end A of a member AB that is fixed at A and hinged at B and for which there is no deflection of one end relative to the other. Values of H_{AB} and H_{BA} for various loads are given in Tables 2 and 3.

h = length of a vertical member

I = moment of inertia of the cross-section of a member.

K = moment of inertia divided by length, $\frac{I}{l}$ or $\frac{I}{h}$.

l = length of a horizontal member.
M_{AB} = moment at end A of a member AB.
M_{BA} = moment at end B of a member AB.
n, p, and s = relations between the K's of the members of a structure. Defined in detail in each table in which they are used.
R = ratio of the deflection of a member to its length, $\frac{d}{h}$ or $\frac{d}{l}$.
θ_{AB} = rotation of the tangent to the elastic curve of a member at the point A of the member AB.
θ_{BA} = rotation of the tangent to the elastic curve of a member at the point B of the member BA.

6. Equations in the Tables.—The equations contained in the tables of Appendix A give the sign, or sense, of the moments as well as their magnitudes. The sense given is the sense of the couple acting on the portion of the member considered. If M_{AB} is found to be equal to $-180{,}000$ in.-lb., the internal stress couple acts in an anti-clockwise direction at A on the member AB.

The sense given is correct when the forces act in the direction shown in the figures. In Table 43, Type of Loading 2, if the force acts downward the resulting moment M_{AD} will be positive. But if the force is an upward force of, say, 2,000 lb., then $P = -2{,}000$ lb. and the moment M_{AD} is negative.

The quantity C_{AB} represents the moment at the end A of a one-span beam AB due to a given load when both ends of the beam are fixed. In a similar manner, H_{AB} represents the moment at the end A of a one-span beam AB due to a given load when the end A is fixed and the end B is free to rotate. For algebraic convenience, H_{AB} appears in the expression for the moments in some members even though the member in question is not hinged at either end. This has been possible because of the algebraic relation between C_{AB} and H_{AB}.

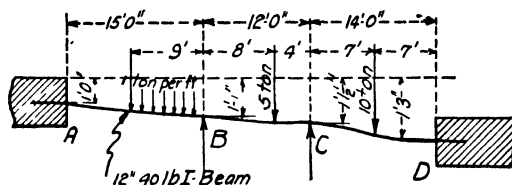



Fig. 12.

Illustrative Problem.—Figure 12 represents a 12-in., 40 8-lb. I-beam carrying loads as shown. The supports have settled so that they are not on the same level. However, there has been no rotation of the end supports, so that the tangents to the elastic curve at the ends are horizontal. It is required to find the moments in the girder

The equations of Table 36, Appendix A, are applicable.

$$l_0 = 180 \text{ in.} \qquad n_0 = \frac{1.868}{1.494} = 1.25$$
$$l_1 = 144 \text{ in.}$$
$$l_2 = 168 \text{ in.} \qquad n_1 = \frac{1.600}{1.868} = 0.8575$$
$$I_0 = I_1 = I_2 = 269 \text{ in.}^4$$
$$K_2 = \frac{269}{168} = 1.600 \text{ in.}^3 \qquad d_A = 12 \text{ in.}$$
$$d_B = 13 \text{ in.}$$
$$K_1 = \frac{269}{144} = 1.868 \text{ in.}^3 \qquad d_C = 13.5 \text{ in.}$$
$$d_D = 15.0 \text{ in.}$$
$$K_0 = \frac{269}{180} = 1.494$$
$$H_{AB} = \frac{(9)(2{,}000)(108)}{(8)(180)(180)}[(2)(180)(180) - (108)(108)] = 398{,}520 \text{ in.-lb.}$$
$$H_{BA} = \frac{(9)(2{,}000)(108)}{(8)(180)(180)}[(2)(180) - 108]^2 = 476{,}280 \text{ in.-lb.}$$
$$H_{BC} = \frac{(10{,}000)(96)(48)}{(2)(144)(144)}[144 + 48] = 213{,}333 \text{ in.-lb.}$$
$$H_{CB} = \frac{(10{,}000)(96)(48)}{(2)(144)(144)}[144 + 96] = 266{,}666 \text{ in.-lb.}$$
$$H_{CD} = H_{DC} = (3/16)(20{,}000)(168) = 630{,}000 \text{ in.-lb.}$$

Substituting these values in the equations of Table 36, gives

$$M_{BC} = \frac{(6)(30{,}000{,}000)(1.868)[(4 \times 0.8575 + 3)(144)(168)(13 - 12) - 2(2 \times 0.8575 + 1)(13.5 - 13)(180)(168)]}{(180)(144)(168)[4(1.25 + 1)(0.8575 + 1) - 1.25]}$$
$$+ \frac{(6)(30{,}000{,}000)(1.868)[(2)(0.8575)(180)(144)(15 - 13.5)]}{(180)(144)(168)[4(1.25 + 1)(0.8575 + 1) - 1.25]}$$
$$- \left\{\frac{2}{3[4(1.25 + 1)(0.8575 + 1) - 1.25]}\right\}\left\{[(4 \times 0.8575 + 3)(2)(1.25)(476{,}280) - (1.25)(398{,}520) + (2)(213{,}333)] + 2[630{,}000 - (2)(630{,}000) - (2)(0.8575)(266{,}666)]\right\} = 481{,}000 \text{ in.-lb.}$$
$$M_{BC} = 481{,}000 \text{ in.-lb.}$$

$$M_{CD} = \left\{\frac{(6)(30{,}000{,}000)(1.6)}{(180)(144)(168)[4(1.25 + 1)(0.8575 + 1) - 1.25]}\right\}\left\{2(1.25 + 2)(180)(168)(13.5 - 13) - (3 \times 1.25 + 4)(180)(144)(15 - 13.5) - (2)(144)(168)(13 - 12)\right\} + \left\{\frac{2}{3[4(1.25 + 1)(0.8575 + 1) - 1.25]}\right\}\left\{(3 \times 1.25 + 4)[630{,}000 - (2)(630{,}000) - (2)(0.8575)(266{,}666)] + (2)(0.8575)[(2)(1.25)(476{,}280) - (1.25)(398{,}520) + (2)(213{,}333)]\right\}$$
$$M_{CD} = -1{,}354{,}500 \text{ in.-lb.}$$

$$M_{AB} = -\frac{481{,}000}{2} - 398{,}520 - \frac{(3)(30{,}000{,}000)(1.868)}{1.25}\left[\frac{13 - 12}{180}\right]$$
$$M_{AB} = -1{,}386{,}220 \text{ in.-lb.}$$
$$M_{DC} = -\frac{1{,}354{,}500}{2} + 630{,}000 - (3)(30{,}000{,}000)(1.6)\left[\frac{15 - 13.5}{168}\right]$$
$$M_{DC} = -1{,}331{,}250 \text{ in.-lb.}$$

With M_{AB}, M_{BC}, M_{CD}, and M_{DC} known the structure is statically determinate. Considering the forces on the portion CD and taking moments about C determines R_D, the reaction at D. It is a downward force of 6,000 lb. In a similar manner the other reactions can be determined. These reactions are found to be R_C = 45,300 lb. upward, R_B = 7,050 lb. downward, and R_A = 15,750 lb. upward. The summation of the vertical forces is found to be zero, showing that the calculations are correct.

Illustrative Problem.—Figure 13 represents a continuous girder of four spans. The ends of the girder are restrained in the position shown. It is required to determine the

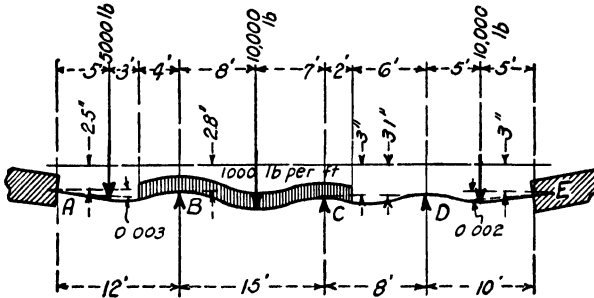



Fig. 13.

moments at the supports. The problem will be solved by applying the equations of three moments.

The equations of Table 40, Appendix A, are applicable. The constants dependent upon the loading and upon the dimensions of the girder are as follows:

$$K_0 = \frac{I}{(12)(12)} = \frac{I}{144}\text{ in.}^3 \qquad n_1 = \frac{K_1}{K_0} = \frac{4}{5} = 0.800 \qquad l_0 = (12)(12) = 144\text{ in.}$$

$$K_1 = \frac{I}{(12)(15)} = \frac{I}{180}\text{ in.}^3 \qquad n_2 = \frac{K_2}{K_1} = \frac{15}{8} = 1.875 \qquad l_1 = (12)(15) = 180\text{ in.}$$

$$K_2 = \frac{I}{(12)(8)} = \frac{I}{96}\text{ in.}^3 \qquad n_3 = \frac{K_3}{K_2} = \frac{4}{5} = 0.800 \qquad l_2 = (12)(8) = 96\text{ in.}$$

$$K_3 = \frac{I}{(12)(10)} = \frac{I}{120}\text{ in.}^3 \qquad l_3 = (12)(10) = 120\text{ in.}$$

$$d_A = 2.5\text{ in.} \qquad d_B - d_A = 2.8 - 2.5 = +0.3\text{ in.} \qquad \theta_A = +0.003$$

$$d_B = 2.8\text{ in.} \qquad d_C - d_B = 3.0 - 2.8 = +0.2\text{ in.} \qquad \theta_E = -0.002$$

$$d_C = 3.0\text{ in.} \qquad d_D - d_C = 3.1 - 3.0 = +0.1\text{ in.} \qquad E = 30{,}000{,}000\text{ lb. per sq. in.}$$

$$d_D = 3.1\text{ in.} \qquad d_E - d_D = 3.0 - 3.1 = -0.1\text{ in.}$$

$$d_E = 3.0\text{ in.}$$

$$H_{AB} = \frac{(5{,}000)(5)(7)(19)(12)}{(2)(144)} + \frac{(12{,}000)(16)[(2)(144) - 16]}{(8)(144)} = 183{,}900\text{ in.-lb.}$$

$$H_{BA} = \frac{(5{,}000)(5)(7)(17)(12)}{(2)(144)} + \frac{(4{,}000)(4)(12)}{(8)(12)(12)}(2\times 12-4)^2$$

$$= 190{,}700\text{ in.-lb.}$$

$$H_{BC} = \frac{(10{,}000)(8)(7)(22)(12)}{(2)(225)} + \left(\frac{12{,}000}{8}\right)(225) = 665{,}500\text{ in.-lb.}$$

$$H_{CB} = \frac{(10{,}000)(8)(7)(23)(12)}{(2)(225)} + \left(\frac{12{,}000}{8}\right)(225) = 681{,}500\text{ in.-lb.}$$

$$H_{CD} = \frac{(2,000)(2)(12)}{(8)(8)(8)} 2 \times 8 - 2)^2 = 18,400 \text{ in.-lb.}$$

$$H_{DC} = \frac{(2,000)(2)(12)}{(8)(8)(8)}(2 \times 8 \times 8 - 2 \times 2) = 11,600 \text{ in.-lb.}$$

$$H_{DE} = \frac{(10,000)(5)(5)(15)(12)}{(2)(100)} = 225,000 \text{ in.-lb.}$$

$$H_{ED} = \frac{(10,000)(5)(5)(15)(12)}{200} = 225,000 \text{ in.-lb.}$$

For the spans AB and BC use Case 4 of Table 40. The equation is

$$M_{BC}(4 + 3n_1) + 2M_{CD} = \frac{6EK_1}{l_0 l_1}[3l_1(d_B - d_A) - 2l_0(d_C - d_B)] + 2n_1H_{AB} - 4[H_{BC} + n_1H_{BA}] - 6EK_1\theta_A$$

Substituting the values of the constants in this equation, gives:

$$M_{BC}(4 + 3 \times 0.8) + 2M_{CD} = \frac{6EI}{(144)(180)(180)}[(3)(180)(0.3) - (2)(144)(0.2)] + (2)(0.8)(183,900) - 4[(665,500) + (0.8)(190,700)] - \frac{(6E)(I)(0.003)}{180}$$

$$6.4M_{BC} + 2M_{CD} = 1,028I - 2,978,000 \qquad (A)$$

For the portion BC and CD, Case 1 of Table 40 is applicable In this equation, substituting subscript B for A, C for B and D for C, and substituting the subscript 1 for 0, 2 for 1, and 3 for 2, gives:

$$n_2M_{BC} + 2M_{CD}(n_2 + 1) + M_{DE} = \frac{6EK_2}{l_1 l_2}[l_2(d_C - d_B) - l_1(d_D - d_C)] - 2[H_{CD} + n_2H_{CB}]$$

Substituting the values of the known constants in these equations, gives

$$1.875M_{BC} + 2M_{CD}(1.875 + 1) + M_{DE} = \frac{6EI}{(96)(180)(96)}[(96)(0.2) - (180)(0.1)] - (2)(18,400) - (2)(1.875)(681,500)$$

$$1.875M_{BC} + 5.75M_{CD} + M_{DE} = 130.209I - 2,592,425 \qquad (B)$$

For the portion CD and DE, Case 5 of Table 40 is applicable. In this equation, substituting subscript C for A, D for B, and E for C and substituting the subscript 2 for 0, and 3 for 1, gives:

$$2n_3M_{CD} + M_{DE}(4n_3 + 3) = \frac{6EK_3}{l_2 l_3}[2l_3(d_D - d_C) - 3l_2(d_E - d_D)] - 4[H_{DE} + n_3H_{DC}] + 2H_{ED} + 6EK_3\theta_E$$

Substituting the values of the known constants in these equations, gives:

$$(2)(0.8)(M_{CD}) + (4 \times 0.8 + 3)(M_{DE}) = \frac{6EI}{(120)(96)(120)}[(2)(120)(0.1) - (3)(96)(-0.1)] - (4)(225,000) - (4)(0.8)(11,600) + (2)(225,000) + \frac{6EI}{120}(-0.002)$$

$$1.6M_{CD} + 6.2M_{DE} = 3,875I - 487,120 \qquad (C)$$

Equations A, B, and C contain three unknown quantities M_{BC}, M_{CD} and M_{DE}.

Rewriting equations A, B, and C in tabular form—that is, with the unknown quantities at the tops of the columns and the coefficients below—gives:

No. of equation	Left hand member of equation			Right-hand member of equation
	M_{BC}	M_{CD}	M_{DE}	
A	6.4	2		$1{,}028I - 2{,}978{,}000$
B	1.875	5 75	1	$130{,}209I - 2{,}592{,}425$
C		1 6	6.2	$3{,}875I - 487{,}120$
$A \div 6.4 = D$	1	0 3125		$160\ 60I - 465{,}000$
$B \div 1.875 = E$	1	3 0670	0 5330	$69\ 42I - 1{,}383{,}000$
$E - D = F$		2.7545	0.5330	$-91\ 18I - 918{,}000$
$F \div 2.7545 = G$		1	0.1935	$-33\ 09I - 336{,}000$
$C \div 1\ 6 = H$		1	3.8740	$2{,}420I - 304{,}400$
$H - G = J$			3.6805	$2{,}453\ 09I + 32{,}000$
$J \div 3\ 6805$			1.0	$666I + 8{,}700$

$$M_{DE} = (666I + 8{,}700) \text{ in.-lb.}$$

Substituting value of M_{DE} in Eq. H, gives

$$M_{CD} = -(160I + 338{,}100) \text{ in.-lb.}$$

Substituting value of M_{CD} in A, gives

$$M_{BC} = (210.7I - 359{,}350) \text{ in.-lb.}$$

To determine M_{AB}, use the equation of Case 1, Table 40, with subscripts as written. Substituting the values of M_{BC} and M_{CD} just determined, gives

$$0.8M_{AB} + 2(0.8 + 1)(210.7I - 359{,}350) - (160I + 338{,}100) =$$

$$\frac{6EI}{(180)(144)(180)}[(180)(0.3) - (144)(0.2)] - 2[665{,}500 + (0.8)(190{,}700)]$$

$$M_{AB} = 467.1I - 5{,}450$$

To determine M_{ED}, use the same equation as above, changing the subscripts as follows: substitute C for A, D for B, E for C, F for D, 2 for 0, and 3 for 1. M_{ED} will be $-M_{EF}$. Making these changes in the equation gives

$$-M_{ED} = -n_3M_{CD} - 2M_{DE}(n_3 + 1) + \frac{6EK_3}{l_2l_3}[l_3(d_D - d_C) - l_2(d_E - d_D)] - 2[H_{DE} + n_3H_{DC}$$

Substituting the values of the known quantities in this equation, gives

$$M_{ED} = -(560I - 229{,}400)$$

With M_{AB}, M_{BC}, M_{CD}, M_{DE}, and M_{ED} known, the structure is statically determinate. By applying the usual equations for static equilibrium and making $I = 1{,}000$ in.^{4}, the reactions are found to be as follows.

R_A = 685 lb. downward $\qquad$ R_B = 19,910 lb. upward

R_C = 28,741 lb. upward $\qquad$ R_D = 9,839 lb. downward

R_E = 7,873 lb. upward

The algebraic sum of all reactions and loads equals zero, showing that the calculations are correct.

7. Analytical Index of Tables.—The following index is given so as to make it possible to locate any desired table in Appendix A quickly.

GENERAL EQUATIONS

APPLICATION OF GENERAL EQUATIONS

CONTINUOUS GIRDERS

TABLE

SECTION 7

WIND STRESSES IN HIGH BUILDINGS

The modern high building with skeleton construction and light curtain walls presents the problem of determining wind stresses and making proper provision to resist them. The wind pressure assumed and the stresses therefrom often play an important part in the design, especially when the width of a building is small compared with its height.

High buildings are usually located in cities having building codes and should be designed to withstand the wind pressure stated in the local code. A wide range of pressures may be found specified in the various codes.

In those cases permitting a choice of wind pressure, the recommendations of Sub-Committee 31, Committee on Steel, Structural Division of the American Society of Civil Engineers may be followed. The first 10 recommendations have been reproduced in Sec. 11*b* on "Wind Load" in this volume.

Although the commonly used approximate methods of analysis are presented in this section, it would be well worth the reader's time to cover those additional methods discussed and recommended in the subcommittee report referred to above. Particular attention should be paid to the Cross, Goldberg, and Grinter methods as well as the Witmer method of K-percentages. These are thoroughly discussed in the various *Proceedings* of the American Society of Civil Engineers referred to in the report and should prove of value for those cases requiring special attention because of unusual conditions existing in the proportions of the bays or stories.

It is evident that for horizontal wind pressure a transverse bent of the structure is a cantilever beam or truss, with its fixed end at the foundation. All stresses due to wind are carried by the steel frame to the foundation. (The inertia of the building resists a portion of the wind pressure but for present purposes this is not considered).

A variety of opinion exists as to the best method of "routing the stresses." Whatever method is employed, the shear across any horizontal section will be equal to the total wind pressure on the structure above the section considered.

The architects of early high office buildings developed four types of wind bracing: (1) *Vertical sway rod* type, (2) *portal* type, (3) *deep girder* type, and (4) *knee-brace* type. Each type served its purpose but they have all largely given way to the *gusset plate* type which is a system of wind bracing without diagonals.

The *vertical sway rod* type is effective and where it can be used, comparatively inexpensive. With connections properly made there is little bending in either columns or lateral struts. The calculations are for a tower bent extending from basement to roof.

The *portal* type consists of a series of solid-webbed portals or arches one above the other. The type was expensive but because it allowed arched openings between columns it was sometimes used. An analysis of stresses in the portal type may be found in an article by C. T. Purdy, *Engineering News*, Vol. XXVI, Dec. 26, 1891, p. 605. (The names of Purdy and Jenney are to be honored as pioneers in steel skeleton construction.)

The *deep girder* type consists of girders, often latticed, in certain bents between columns, which are designed to carry both vertical and wind loads. The columns are designed to resist the bending moment due to the wind stress in the flanges of the girders.

The *knee-brace* type is a series of knee-braces between columns and floor girders, producing bending moments in both.

An objection common to all of these types is their limitations. They can only be used in certain bents of a building. Quite often they interfere with doors and windows; they cut up the masonry; the knee-braces extend into rooms and corridors; they require additional fireproofing and they hamper the architect in developing his architectural features.

Much study and thought has been given of late years to wind bracing without diagonals, especially for high office buildings and hotels. Different methods of calculating the wind stresses have been developed. A broad classification of these methods may be: (1) exact methods, and (2) approximate methods.

1. Exact Methods.—Of the exact methods four will be noted.

1*a*. Jonson's Method.—Ernst F. Jonson has a paper, "The Theory of Frameworks with Rectangular Panels and Its Application to Buildings Which Have to Resist Wind" in the *Transactions Am. Soc. C. E.*, Vol. LV, December, 1905, p. 413. Jonson begins his paper: "Frame works with rectangular panels and stiff joints, such as are generally used in modern high buildings, are, of course, statically undetermined structures. The stresses in each member, therefore, are functions of those in all the other members, so that no member can be figured except in relation to all the others, and only on the basis of a preliminary and more or less incorrect design, from which, by successive modifications, a practically correct one may be evolved."

It may be stated here that the necessity of assuming column and girder sections before stress calculations can be made is common to all exact methods.

Jonson's method is the pioneer of the exact methods. It is based upon the fundamental proposition: In a beam with uniform cross-section, the deflection of one point measured from the tangent at another point is the sum of the moments of the bending moment on each unit of length between the two points, around the former point, divided by the modulus of elasticity and the moment of inertia, or, expressed in the language of the calculus

$$d = \int \frac{M}{EI} x dx$$

Jonson determines an equation that he calls "the theorem of four moments." He then writes it in two more forms, one for each of the two pairs of opposite corners of a rectangular panel. By means of these three equations the moments may be found in any framework composed of quadrangular panels. The labor of determining the unknowns is great, so much so as to make the method impracticable for ordinary use.

1*b*. Melick's Method.—Cyrus A. Melick presented as a thesis for his doctor's degree at the Ohio State University: "Stresses in Tall Buildings," which was published June, 1912 as *Bulletin* **8** of the College of Engineering, a monograph of 227 pages. The equations for determining wind stresses are staggering. Dr. Melick concludes that it is impracticable to design a high building by any correct theoretical method; empirical rules must be deduced and followed.

1*c*. Smith's Method.—Professor Albert Smith of Purdue University: "Wind Stresses in the Steel Frames of Office Buildings," in in the *Journal* of the Western Society of Engineers, Vol. XX, April, 1915, p. 341, presents a method by which the wind stresses are computed in accordance with the theory of least work. Like the preceding method the calculations are very lengthy, and the method is not adapted to actual practice.

1*d*. Wilson and Maney's Method.—By far the nearest approach to an exact method that is workable is the slope-deflection method of Professors Wilson and Maney as given in their "Wind Stresses in the Steel Frames of Office Buildings," *Bulletin* **80** (June, 1915), Engineering Experiment Station of the University of Illinois. As they state in their outline: "The method is based upon the proposition in mechanics used by Mr. Jonson, but the method which the writers have developed differs from the one used by him in that the changes in the slopes and the deflections have been used as the unknown quantities instead of the direct stresses and the moments. Four members, two

columns and two girders, intersect in a point. Each member is subjected to a different stress and a different moment, whereas all of the members are subjected to the same change in slope and all the columns in a story are subjected to the same deflection. It is therefore apparent that there are fewer unknown slopes and deflections than moments and direct stresses. The large reduction in the number of unknowns very much simplifies the solution of the equations."

2. Approximate Methods.[1]—From the foregoing it is seen that for determining wind stresses and wind moments in skeleton construction where diagonals are not used, resort must be had for purposes of practical design to approximate methods. Four methods in current use will be given.

Considering a single bent: It will be assumed that all columns in any given story have the same sectional area and the same moment of inertia, that all girders of the same floor have the same moment of inertia, and that the joints are perfectly rigid. It is obvious that if the forces in the several members of the frame are small in relation to the stiffness of the members, the longitudinal distortions may be neglected; hence the adjacent joints occupying the corners of a rectangle will after distortion occupy the corners of an oblique parallelogram.

It is assumed that the point of contraflexure of each column is at midheight of the story. The first method described further involves the tacit assumption that the girders have their points of contraflexure at midlength. Specific assumptions as to the distribution of column shears and direct stresses are made in the several methods. In none of the four methods are the assumptions strictly consistent. For example, in Method I the assumption as to location of points of contraflexure would make the distorted shape of panel constant in any given story, and from this would follow that the column shears must be equal; but the calculation gives column shears of different amount.

The resistance to overturning will cause a direct stress in tension on the windward side of the neutral axis, taken by all or some of the columns on that side according to the method used, and a direct stress in compression on the leeward side taken by the columns on that side.

Figures 1, 8, 10 and 14 give results obtained from calculations according to Methods I, II, II-A and III respectively. Loads and stresses are given in thousands of pounds and bending moments in thousands of foot-pounds. Direct stresses are given in parentheses ().

[1] Revision and extension of an article, "Wind Bracing Without Diagonals for Steelframe Office Buildings," R. Fleming, *Engineering News*, Vol. LXIX March 13, 1913, p. 492.

See also R. Fleming, "Wind Stresses in Tall Buildings," by John Wiley & Sons, Inc., New York, 1930; and Henry V. Spurr, "Wind Bracing," McGraw-Hill Book Company, Inc., New York, 1930.

2a. Method I.—This may be called the *cantilever method* and is a restatement with some modifications of an article entitled, "Wind Bracing with Knee-braces or Gusset-plates," by A. C. Wilson, in *Engineering Record*, Sept. 5, 1908. A section or bent of the building is considered similar to a beam loaded as a cantilever.

If a beam of rectangular section be loaded as a cantilever with concentrated loads, it is possible by the theory of flexure to find the internal stresses at any point. If, however, rectangles be cut out of the beam between the loads, there will then be a different condition of stress. What was the horizontal shear of the beam will now be a shear at the point of the contraflexure of the floor girders, causing bending, and, as in the beam, the nearer the neutral axis the greater the shear. The vertical shear in the beam would be taken up by the columns as a shear at the points of contraflexure and the amount of this shear taken by each column would, as in the beam, increase toward the neutral axis. The direct stresses of tension or compression in the beam would act on the columns as a direct load of either tension or compression, and as in the beam would decrease toward the neutral axis.

FIG. 1.—Rectangular building frame. Direct and bending stresses calculated by approximate Method I.
144 0 = Bending moment of 144,000 ft.-lb.
(4.0) = Direct stress of 4,000 lb.

Each intersection of column with floor girders would be held in equilibrium by forces acting at the points of contraflexure; and to find all the forces acting around a joint at any floor the bending moments of the building at the points of contraflexure of the columns above and below the floor in question are found as will be explained later.

It is assumed that if a beam of constant, symmetrical cross-section and homogeneous material is fixed at both ends, and that if forces tend to move those ends from a position in the same straight line to a position to one side with the ends still parallel, reversed bending will occur with the point of contraflexure in the center of the unsupported span. And since this condition exists in all columns and floor girders it will be necessary to find the shears at the points of contraflexure as well as the direct stresses in all members.

Figure 1 gives stresses and maximum moments in all members of a section of the building in accordance with the above statement.

The calculation of stresses and bending moments in members about the sixth floor will be given in detail. The direct stress in any column is assumed to be proportional to its distance from the neutral axis of the cross-section of the building. In the cross-section considered, the neutral axis coincides with the center line of the building. The total

moment of the wind loads above the sixth floor about the line of inflection of the sixth-story columns must equal the moment of the direct stresses in these columns about the neutral axis. Let $8X$ be the direct stress in each of the sixth-story columns B and C, then $24X$ will be the direct stress in each of the sixth-story columns A and D. Hence we have

$$(4{,}000)(30) + (6{,}000)(18) + (6{,}000)(6) = (24X)(24) + (8X)(8) + (8X)(8) + (24X)(24)$$

From which $8X = 1{,}650$ and $24X = 4{,}950$.

In the same way for the fifth-story columns we have the equation

$$(4{,}000)(42) + [(6{,}000)(30 + 18 + 6)] = 2[(24X)(24) + (8X)(8)]$$

From which $8X = 3{,}075$, the direct stress in the fifth-story columns B and C; and $24X = 9{,}225$, the direct stress in the fifth-story columns A and D.

The total horizontal shear on any line across the building is the sum of the wind loads above that line. The shear taken by any column in any story is proportional to the total horizontal shear in that story.

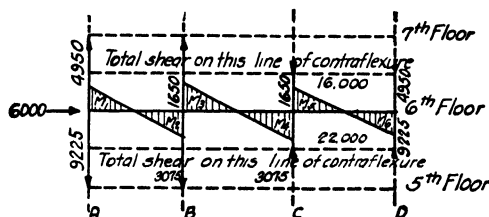



Fig. 2.—Column shears and girder moments at sixth floor, calculated by Method I.

In Fig. 2, if X = shear of any fifth-story column at its point of inflection, then $\frac{16{,}000}{22{,}000}X$ or $\frac{8}{11}X$ = shear at point of inflection of the sixth-story length of the same column, and $\frac{6{,}000}{22{,}000}X$ or $\frac{3}{11}X$ = increment of shear taken by the column at the floor girder.

We are now ready to consider the forces about the first joint, or the intersection of column A with the sixth-floor girder, sketched separately as Fig. 3.

The difference between 9,225 and 4,950 = 4,275 is taken up as a shear in the floor girder between columns A and B. The moments of the shears must hold the joint in equilibrium. Taking moments about the lower point of inflection we have

$$(8/11X)(12) + (3/11X)(6) = (4{,}275)(8)$$

from which $X = 3{,}300$, $8/11X = 2{,}400$ and $3/11X = 900$.

The bending moment M_1 for the floor girder is (4,275)(8) = 34,200 ft.-lb. The bending moment for the fifth-story column is (3,300)(6) = 19,800 ft.-lb., and that for the sixth-story column is (2,400)(6) = 14,400 ft.-lb. The direct thrust on the floor girder is 6,000 − 900 = 5,100.

Proceeding to the second joint, sketched in Fig. 4: The difference between 3,075 and 1,650 = 1,425 acts as a shear in the girder between columns B and C. This added to the 4,275 shear continued from the girder between A and B makes a total shear of 5,700 in the girder. The equation of moments is

$$(\tfrac{8}{11}X)(12) + (\tfrac{3}{11}X)(6) = (4,275)(8) + (5,700)(8)$$

From which X = 7,700; $\tfrac{8}{11}X$ = 5,600; and $\tfrac{3}{11}X$ = 2,100 are the shears taken by column B to hold the joint in equilibrium.

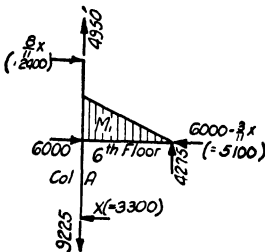
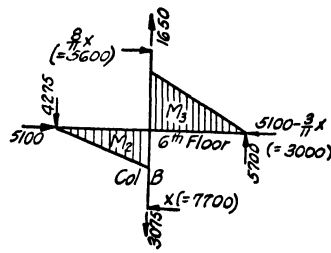



Fig. 3. Fig. 4.

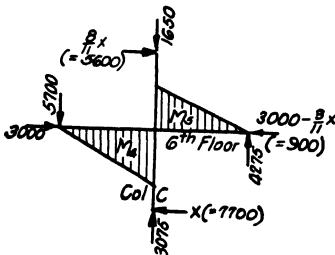
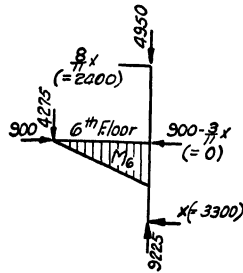



Fig. 5. Fig. 6.

Sixth-floor joints of building frame, with stresses corresponding to Method I.

The bending moment M_2 of the girder from A to B at column B is the same as at column A with an opposite sign; M_3 of the girder from B to C, is (5,700)(8) = 45,600 ft.-lb. The bending moment of the fifth-story column is (7,700)(6) = 46,200 ft.-lb., and that of the sixth-story column is (5,600)(6) = 33,600 ft.-lb. The direct thrust on the girder between B and C is 5,100 − 2,100 = 3,000.

At the third joint, Fig. 5, the shear taken by the girder between C and D is 5,700 − (3,075 − 1,650) = 4,275. From the equation of moments

$$(\tfrac{8}{11}X)(12) + (\tfrac{3}{11}X)(6) = (5,700)(8) + (4,275)(8)$$

whence

$$X = 7{,}700,\ \tfrac{8}{11}X = 5{,}600,\ \tfrac{3}{11}X = 2{,}100$$

As expected, the moments in column C are numerically equal to those in column B, and the girder moments $M_4 = M_3$, and $M_5 = M_2$. The compression in the floor girder between C and D is $3{,}000 - 2{,}100 = 900$.

At the fourth joint, Fig. 6, we have

$$(\tfrac{8}{11}X)(12) + (\tfrac{3}{11}X)(6) = (4{,}275)(8)$$

the same equation as at the first joint, and hence the same numerical values for moments and shears.

If bays are of unequal span lengths, the direct stresses in columns will vary as their distances from the neutral axis. The column stresses will also vary as their sectional areas.

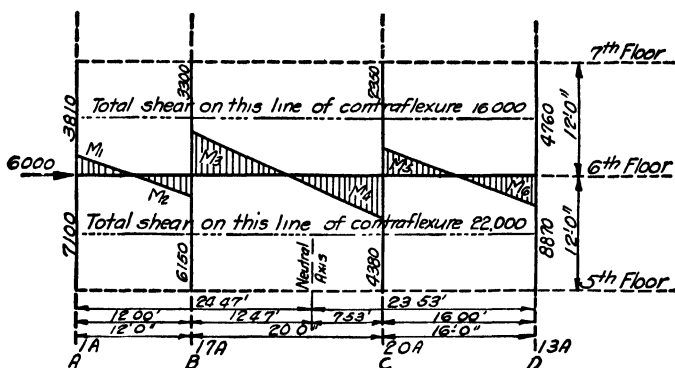



FIG. 7.—Column shears and girder moments at sixth floor, unequal bays, varying sectional areas, calculated by Method I

An illustration of unequal spacing of columns with varying sectional areas is given in Fig. 7. The widths of bays are 12, 20, and 16 ft. respectively, the sectional area of column A is assumed $1A$; of column B, $1.7A$; of column C, $2.0A$; and of column D, $1.3A$. The portion of transverse bent illustrated in Fig. 2 is shown in Fig. 7.

The distance of the neutral axis from left-hand side of the bent equals

$$[(1A)(0) + (1.7A)(12.) + (2.0A)(32.) + (1.3A)(48)] \div (1A + 1.7A + 2.0A + 1.3A) = 24.47 \text{ ft.}$$

The bending moment of wind loads above the sixth floor about the line of inflection of the sixth-story columns equals, as before,

$$(4{,}000)(30) + (6{,}000)(18) + (6{,}000)(6) = 264{,}000 \text{ ft.-lb.}$$

The bending moment of wind loads above the fifth floor about the line of inflection of the fifth-story columns, equals, as before

$$(4{,}000)(42) + (6{,}000)(30) + (6{,}000)(18) + (6{,}000)(6) = 492{,}000 \text{ ft.-lb.}$$

These moments must equal the moments of the direct stresses in the columns about the neutral axis. Since the direct stress in each column varies both as its distance from the neutral axis and its sectional area, we have,

Y = stress in column A (area $1A$ and distance 24.47 from neutral axis)

$\frac{(1.7)(12.47)}{24.47}Y$ = stress in column B (area $1.7A$ and distance 12.47 from neutral axis)

$\frac{(2.0)(7.53)}{24.47}Y$ = stress in column C (area $2.0A$ and distance 7.53 from neutral axis)

$\frac{(1.3)(23.53)}{24.47}Y$ = stress in column D (area $1.3A$ and distance 23.53 from neutral axis)

From the equality of moments, we have for the sixth-story columns,

$$(Y)(24.47) + \frac{(1.7)(12.47)}{24.47}(Y)(12.47) + \frac{(2.0)(7.53)}{24.47}(Y)(7.53) + \frac{(1.3)(23.53)}{24.47}(Y)(23.53) = 264{,}000$$

From this equation the stress in $A = 3{,}810$, $B = 3{,}300$, $C = 2{,}350$, and $D = 4{,}760$.

For the fifth-story columns the left-hand member of the above equation = 492,000 from which is found the stress in $A = 7{,}100$, $B = 6{,}150$, $C = 4{,}380$, and $D = 8{,}870$.

Having the direct stresses in columns the bending moments in columns and floor girders and the direct stresses in girders are determined precisely as before. The bending moments in girders are found to be as follows:

$$M_1 = +19{,}740,\ M_2 = -19{,}740,\ M_3 = +61{,}390$$
$$M_4 = -61{,}390,\ M_5 = +32{,}870,\ M_6 = -32{,}870$$

The bending moments in sixth-story columns are, A, 8,300, B, 34,160; C, 39,700; D, 13,840. The bending moments in fifth-story columns are, A, 11,420; B, 46,970; C, 54,580; D, 19,030. The direct compression stresses in the sixth-floor girders from the load of 6,000 are, A to B, 5,480; B to C, 3,340; C to D, 865.

The designer with whatever method he follows will find short cuts and ways of checking his work. For instance, the numerical sum of the bending moments in any row of floor girders equals the numerical sum of the moments of column sections immediately above and below the

girders. For the sixth-floor girders the numerical sum of the bending moments is

$$(16{,}000 + 22{,}000)(6) = 228{,}000$$

for all the cases illustrated.

2b. Method II.—This may be called the *method of equal shears.* It is assumed that the horizontal shear on any plane is equally distributed among the columns cut by that plane. The stresses and

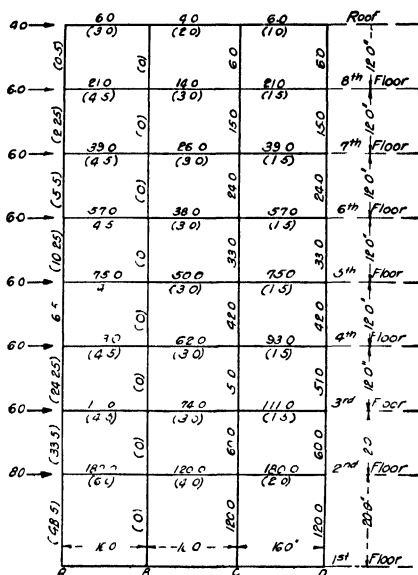



FIG. 8.—Rectangular building frame. Direct and bending stresses calculated by approximate Method II.
120 0 = Bending moment of 120,000 ft.-lb.
(4 0) = Direct stress of 4,000 lb.

maximum bending moments for a cross-section of the building are as given in Fig. 8.

Taking any aisle we find the direct stress in the fifth-story columns to be

$$\left(\frac{4{,}000}{3}\right)(42) + \left(\frac{6{,}000}{3}\right)(30) + \left(\frac{6{,}000}{3}\right)(18) + \left(\frac{6{,}000}{3}\right)(6) \div 16 = 10{,}250$$

The direct stresses coming upon any interior column from the adjacent aisles are equal in amount but opposite in direction. Hence their

algebraic sum is zero and only the outside columns have direct stresses. This may be found directly for any story, say the sixth,

$$(4{,}000)(30) + (6{,}000)(18) + (6{,}000)(6) \text{ divided by } 48 = 5{,}500$$

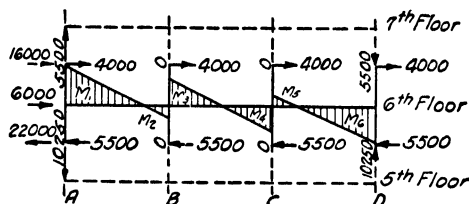



FIG. 9.—Column shears and girder moments at sixth floor, calculated by Method II.

Considering in detail, as in Method I, the sixth floor, we have in Fig. 9 the direct stresses and shears in the columns.

The shear in each girder is 10,250 − 5,500 = 4,750. The equations for bending moments in the girders are as follows:

$$M_1 = [(4{,}000)(6) + (5{,}500)(6)] = +57{,}000 \text{ ft.-lb.}$$

$$M_2 = [(4{,}000)(6) + (5{,}500)(6)] - [(10{,}250 - 5{,}500)(16)] = -19{,}000 \text{ ft.-lb.}$$

$$M_3 = 2[(4{,}000)(6) + (5{,}500)(6)] - [(10{,}250 - 5{,}500)(16)] = +38{,}000 \text{ ft.-lb.}$$

$$M_4 = 2[(4{,}000)(6) + (5{,}500)(6)] - [(10{,}250 - 5{,}500)(32)] = -38{,}000 \text{ ft.-lb.}$$

$$M_5 = 3[(4{,}000)(6) + (5{,}500)(6)] - [(10{,}250 - 5{,}500)(32)] = +19{,}000 \text{ ft.-lb.}$$

$$M_6 = 3[(4{,}000)(6) + (5{,}500)(6)] - [(10{,}250 - 5{,}500)(48)] = -57{,}000 \text{ ft.-lb.}$$

The bending moment at the sixth-floor girder of each sixth-story column is (4,000)(6) = 24,000 ft.-lb., and of each fifth-story column is (5,500)(6) = 33,000 ft.-lb.

The compression in the floor girders is 6,000 − 1,500 = 4,500 between columns A and B, 4,500 − 1,500 = 3,000 between B and C, and 3,000 − 1,500 = 1,500 between C and D. General equations can easily be deduced which will simplify the calculation of stresses and moments for other floors.

If the spaces between columns are unequal and it is assumed that each transverse bay takes equal wind loading, the direct stresses in an interior column from adjacent bays will be unequal. It is better to assume the wind loading taken by each bay to be proportional to its span length and thus have all direct stresses resisted by the outside rows of columns. (For horizontal shears in columns assumed proportional to their moments of inertia, see Method II-A.)

2c. Method II-A.—This is a special case of Method II and may be called the *portal method.* The structure is regarded as equivalent to a series of independent portals. The total horizontal

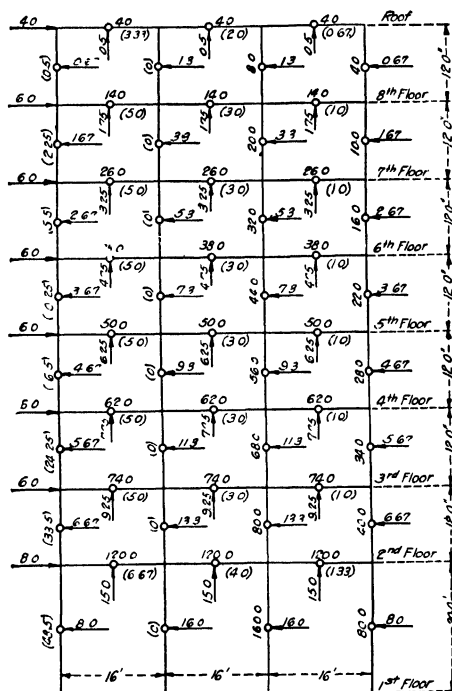



Fig. 10.—Rectangular building frame. Direct and bending stresses calculated by approximate Method II-A.

Loads and shears at points of contraflexure are given in thousands of pounds.

120 0 = Bending moment of 120,000 ft.-lb.

(4 0) = Direct stress of 4,000 lb.

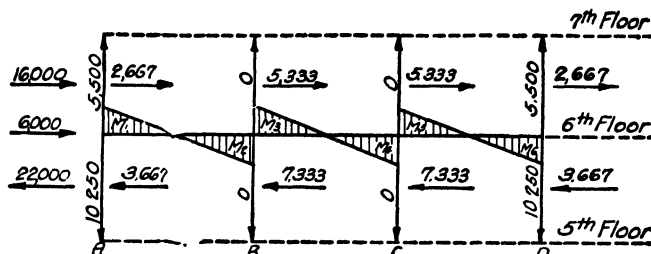



Fig. 11.—Column shears and girder moments at sixth floor, calculated by Method II-A.

shear on any plane is divided by the number of aisles instead of by the number of columns as in II. An outer column thus takes but one-half the shear of an interior column. The stresses and maximum bending moments for a cross-section of the building are as given in Fig. 10.

For equal spacing the direct or vertical axial stress due to the overturning moment of the wind is all taken by the outside columns and is the same in amount as in Method II.

Considering in detail the sixth floor, we have in Fig. 11 the direct stresses and shears in the columns.

The shear in each girder is 10,250 − 5,500 = 4,750. The equations for bending moments in the girders are as follows:

$$
\begin{aligned}
& & & \text{ft.-lb.} \\
M_1 &= [(2{,}667)(6) + (3{,}667)(6) = & & +38{,}000 \\
M_2 &= [(2{,}667)(6) + (3{,}667)(6) - (10{,}250 - 5{,}500)(16)] = & & -38{,}000 \\
M_3 &= [(2{,}667)(6) + (3{,}667)(6) + (5{,}333)(6) + (7{,}333)(6) & & \\
&\qquad - (10{,}250 - 5{,}500)(16)] = & & +38{,}000 \\
M_4 &= [(2{,}667)(6) + (3{,}667)(6) + (5{,}333)(6) + (7{,}333)(6) & & \\
&\qquad - (10{,}250 - 5{,}500)(32)] = & & -38{,}000 \\
M_5 &= [(2{,}667)(6) + (3{,}667)(6) + 2(5{,}333)(6) + 2(7{,}333)(6) & & \\
&\qquad - (10{,}250 - 5{,}500)(32)] = & & +38{,}000 \\
M_6 &= [(2{,}667)(6) + (3{,}667)(6) + 2(5{,}333)(6) + 2(7{,}333)(6) & & \\
&\qquad - (10{,}250 - 5{,}500)(48)] = & & -38{,}000
\end{aligned}
$$

The bending moment at the sixth-floor girder of each outer sixth-story column is (2,667)(6) = 16,000 ft.-lb., and of each inner sixth-story column is (5,333)(6) = 32,000 ft.-lb. At the fifth-floor girder the bending moment of each fifth-story outer column is (3,667)(6) = 22,000 ft.-lb. and of each fifth-story inner column is (7,333)(6) = 44,000 ft.-lb.

The compression in the floor girders is 6,000 − 1,000 = 5,000 between columns *A* and *B*, 5,000 − 2,000 = 3,000 between *B* and *C*, and 3,000 − 2,000 = 1,000 between *C* and *D*.

It is noted from the above that the bending moment in an outer column is one-half that in an interior column; that the point of contraflexure of each girder is at its center; and the bending moments due to wind for all girders of any transverse bent on the same floor are alike. This is an ideal condition for the detailer and the shop. The designer finds this method very simple and his work easily checked. The bending moment in a girder is the mean between the bending moments in the interior column above and below the girder.

If the spaces between columns are unequal, the wind loading taken by each bay should be assumed proportional to its span. This brings all direct stresses to the outside columns. The bending moment at each end of a floor girder if moments of inertia of columns are the same will be the same though they will not be alike for all the girders of a floor.

Horizontal shears in columns are proportional to their respective moments of inertia. An illustration of unequal spacing of columns and varying moments of inertia is given in Fig. 12.

The widths of bays are as in Fig. 7; 12, 20, and 16 ft., and the moments of inertia are $1.0I$ of column A, $1.7I$ of column B, $2.0I$ of column C, and $1.3I$ of column D. Assuming each single bent to take wind loading proportional to its width the direct stresses coming upon any interior columns from adjacent bays will still be equal in amount but opposite in sign and their algebraic sum will be zero. The outside columns only

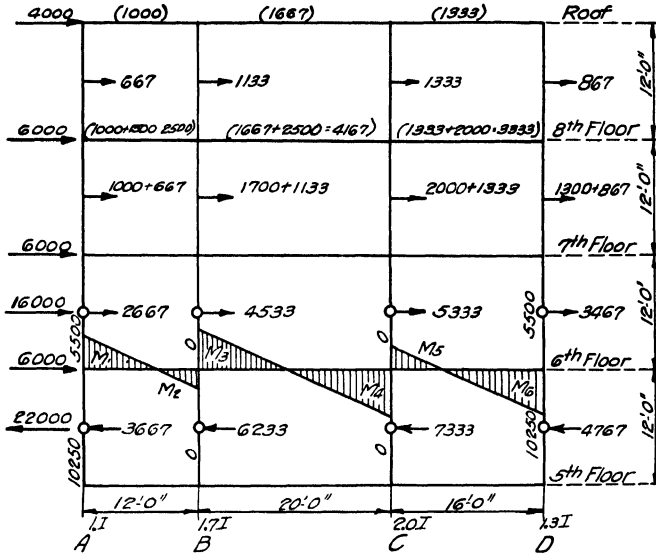



Fig. 12.—Column shears and girder moments at sixth floor, unequal bays, varying moments of inertia, calculated by Method II-A.

will have direct stresses from the wind and of same amount as before.

The bending moments in the sixth-floor girders are as follows:

M_1 = (2,667)(6) + (3,667)(6) = +38,000 ft.-lb.
M_2 = [(2,667 + 3,667)(6)] − [(10,250 − 5,500)(12)] = −19,000 ft.-lb.
M_3 = [(2,667 + 3,667)(6) + (4,533 + 6,233)(6)] − [(10,250 − 5,500) (12)] = +45,600 ft.-lb.
M_4 = [(2,667 + 3,667)(6) + (4,533 + 6,233)(6)] − [(10,250 − 5,500) (32)] = −49,400 ft.-lb.
M_5 = [(2,667 + 3,667)(6) + (4,533 + 6,233)(6) + (5,333 + 7,333)(6)] − [(10,250 − 5,500)(32)] = +26,600 ft.-lb.
M_6 = [(2,667 + 3,667)(6) + (4,533 + 6,233)(6) + (5,333 + 7,333)(6)] − [(10,250 − 5,500)(48)] = −49,400 ft.-lb.

The bending moments in columns are the shears given at points of contraflexure multiplied by one-half the column lengths. The direct compression stresses in the sixth-floor girders are, A to B, 6,000 − 1,000 = 5,000; B to C, 5,000 − 1,700 = 3,300; C to D, 3,300 − 2,000 = 1,300.

Methods I and II-A are specially adapted to transverse bents when the columns are turned as in Fig. 13; also when the outer columns carry floor loads only and the stresses are but one-half those of the inner columns.

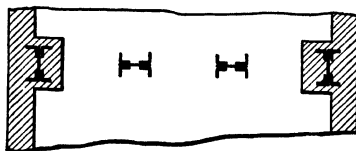



Fig. 13.—Cross-section of columns in transverse bent.

2d. Method III.—This may be called the *continuous portal method.* The direct stresses in the columns are assumed to vary as their distances from their neutral axis, and the horizontal shear on any plane is equally distributed among the columns cut by that plane. Stresses and maximum bending moments for a cross-section of the building are as given in Fig. 14.

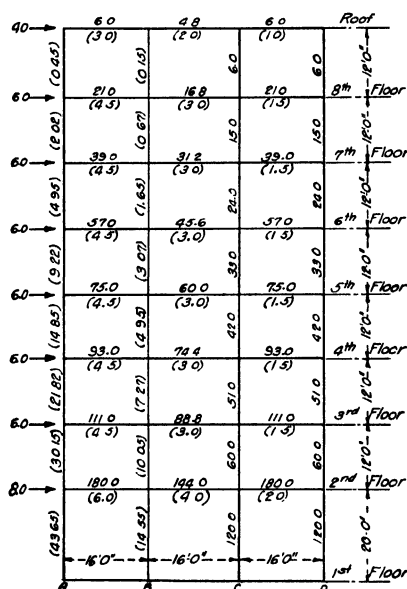



Fig. 14.—Rectangular building frame. Direct and bending stresses calculated by approximate Method III.

144.0 = Bending moment of 144,000 ft.-lb.
(4 0) = Direct stress of 4,000 lb.

The direct stresses in the columns are found the same way and are the same in amount as in Method I.

Considering in detail the sixth floor, we have in Fig. 15 the direct stresses and shears in the columns. The shear in the girder A to B and the girder C to D is 9,225 − 4,950 = 4,275. The shear in the girder B to C is (9,225 − 4,950) + (3,075 − 1,650) = 5,700.

The equations for bending moments in the girders are as follows:

M_1 = [(4,000)(6) + (5,500)(6)] = +57,000 ft.-lb.
M_2 = [(4,000)(6) + (5,500)(6)] − [(9,225 − 4,950)(16)] = −11,400 ft.-lb.
M_3 = 2[(4,000)(6) + (5,500)(6)] − [(9,225 − 4,950)(16)] = +45,600 ft.-lb.
M_4 = 2[(4,000)(6) + (5,500)(6)] − [(9,225 − 4,950)(32)] − [(3,075 − 1,650)(16)] = −45,600 ft.-lb.
M_5 = 3[(4,000)(6) + (5,500)(6)] − [(9,225 − 4,950)(32)] − [(3,075 − 1,650)(16)] = +11,400 ft.-lb.
M_6 = 3[(4,000)(6) + (5,500)(6)] − [(9,225 − 4,950)(48)] − [(3,075 − 1,650)(32)] + [(3,075 − 1,650)(16)] = −57,000 ft.-lb.

The bending moment at the sixth-floor girder of each sixth-story column is (4,000)(6) = 24,000 ft.-lb., and of each fifth-story column is

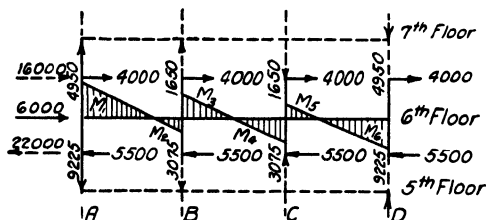



Fig 15.—Column shears and girder moments at sixth floor, calculated by Method III.

(5,500)(6) = 33,000 ft.-lb. The compression in the floor girders is 6,000 −1,500 = 4,500 lb. between columns A and B, 4,500 − 1,500 = 3,000 between B and C, and 3,000 − 1,500 = 1,500 between C and D.

If the columns are unequally spaced or their sectional areas are different, the location of the neutral axis must first be found. The direct stresses in the columns will vary both as their distances from the neutral axis and their sectional areas. The horizontal shears taken by the columns will vary as their moments of inertia. This may be illustrated by assuming direct stresses as in Fig. 7 and horizontal shears as in Fig. 12. The bending moments in sixth-floor girders will then be:

M_1 = (2,667)(6) + (3,667)(6) = +38,000 ft.-lb.
M_2 = [(2,667 + 3,667)(6)] − [(7,100 − 3,810)(12)] = −1,470 ft.-lb.
M_3 = [(2,667 + 3,667)(6) + (4,533 + 6,233)(6)] − [(7,100 − 3,810)(12)] = +63,110 ft.-lb.
M_4 = [(2,667 + 3,667)(6) + (4,533 + 6,233)(6)] − [(7,100 − 3,810)(32) + (6,150 − 3,300)(20)] = −59,670 ft.-lb.
M_5 = [(2,667 + 3,667)(6) + (4,533 + 6,233)(6) + (5,333 + 7,333)(6)] − [(7,100 − 3,810)(32) + (6,150 − 3,300)(20)] = +16,310 ft.-lb.
M_6 = [(2,667 + 3,667)(6) + (4,533 + 6,233)(6) + (5,333 + 7,333)(6)] − [(7,100 − 3,810)(48) + (6,150 − 3,300)(36) − (4,380 − 2,350)(16)] = −49,440 ft.-lb.

The bending moments in the columns are the shears at the points of contraflexure, as shown in Fig. 12, multiplied by one-half the column lengths. The direct compression stresses in the sixth-floor girders are, from *A* to *B*, 6,000 − 1,000 = 5,000; *B* to *C*, 5,000 − 1,700 = 3,300; *C* to *D*, 3,300 − 2,000 = 1,300.

Method III is limited to transverse bents of not more than four bays. In bents of five bays the points of contraflexure of girders in the leeward bay fall outside of the girders and the method fails. This is readily shown by Fig. 16. Two bays and two columns are added to Fig. 15. The same total shear is divided between six columns instead of four and

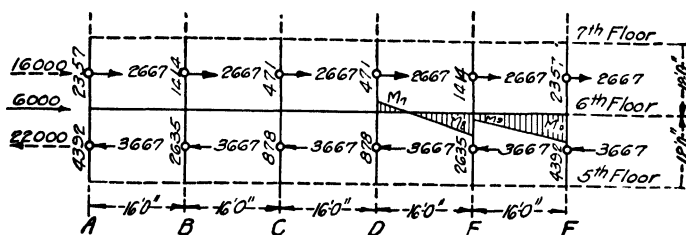



FIG. 16.—Illustrating failure of Method III.

direct stresses in columns are determined in accordance with the increased width. From equations of moments

$$M_7 = +8{,}750 \text{ ft.-lb.}$$
$$M_8 = -43{,}450 \text{ ft.-lb.}$$
$$M_9 = [5(2{,}667 + 3{,}667)(6)] - [(4{,}392 - 2{,}357)(64) + (2{,}635 - 1{,}414)(48) + (878 - 471)(32) - (878 - 471)(16)] = -5{,}340 \text{ ft.-lb.}$$
$$M_{10} = [5(2{,}667 + 3{,}667)(6)] - [(4{,}392 - 2{,}357)(80) + (2{,}635 - 1{,}414)(64) + (878 - 471)(48) - (878 - 471)(32) - (2{,}635 - 1{,}414)(16)] = -37{,}920 \text{ ft.-lb.}$$

Moments M_9 and M_{10} have the same sign, bringing the point of contraflexure between columns *D* and *E* outside the girder considered.

2*e*. Choice of Methods.—Each of the foregoing approximate methods has its advocates. At the present time Method II and II-A are probably more used than any other. The writer followed Method I in designing the 20-story Finance Building in Philadelphia and Method II-A for the 18-story Hurt Building in Atlanta. He believes that I is more in accordance with the true distribution of stresses but because II-A simplifies and duplicates connections he is often inclined to use it. To quote Professor W. H. Burr: "So long as the stresses found by one legitimate method of analysis are provided for, the stability of the structure is assured."

The designer wishes to calculate stresses before column and girder sections are determined. He can do so by assuming in Method I the same

sectional areas for all columns in a transverse bent, in Method II the same moments of inertia, or in Method II-A the moments of inertia of columns in the outside rows when turned as in Fig. 13, to be one-half those of the interior. The error is usually small but some thought given at the outset will enable him to foresee marked differences, if any, and make his calculations accordingly.

It is not always practicable, or even advisable, to hold rigorously to one system of bracing throughout. A considerable portion of wind pressure may sometimes be carried to the ends of a building and thence transferred to the foundation by connections that are not permissible in interior bents.

Buildings like the Fuller, the Singer, the Woolworth or the Metropolitan Tower in New York City, are each in a class by itself and for such structures no general rules can be laid down for wind bracing. For unusual structures like these, careful study must be made and special methods often be devised.

A building should be examined for wind in a longitudinal as well as in a transverse direction and provision made if necessary. This is quite obvious but in some instances it has been neglected.

3. Working Stresses.—Because wind loads are intermittent and seldom reach their maximum, greater working stresses are permitted than for the roof, floor, and dead loads. The New York and Chicago codes allow for the combined stresses due to dead, wind, and other live loads an increase of 33⅓ per cent in the working stress over that allowed for the dead and other live loads alone, provided the section thus obtained is not less than that required if wind force be neglected. This is equivalent to neglecting wind stress when it does not exceed 33⅓ per cent of the stress due to the combined dead and other live loads.

Fig. 17—Graphical combination of moments from vertical loads and wind loads in floor girder.

It is often convenient to assume wind loads on the basis of using the same working stresses as for dead and other live loads. A number of building codes call for a horizontal wind pressure of 30 lb. per sq. ft. and allow working stresses to be increased 50 per cent for wind bracing. A wind load of 30 lb. per sq. ft. with working stresses of 24,000 lb. per sq. in. is equivalent to a load of 20 lb. per sq. ft. with a working stress of 16,000 lb. per sq. in.—the working stress generally used for live and dead loads. A wind load of 20 lb. per sq. ft. with working stress of 24,000 lb. per sq. in. is equivalent to 13⅓ lb. per sq. ft. with working stress of 16,000 lb. per sq. in.

The diagrams of moments for any floor girder can easily be combined in one figure (see Fig. 17), and the total moment at any point read by scaling. For floor loading moment Fig. 17 is drawn for beam with ends supported.

4. Details.—The part that belongs to the detailer in making wind bracing efficient is often underrated. It is folly to add material to columns or floor girders to resist stresses and moments due to wind and then neglect details. Yet this is sometimes done. Care should be taken that column splices are strong enough for bending and that connections of floor girders to columns can resist the bending moments coming upon them. To do the latter and keep within prescribed architectural limits will often tax the ingenuity of the detailer. In fact, this particular phase should be borne in mind by the designer. It may influence him in proportioning members. In one particular instance the bending moments at the ends of a 15-in. floor beam were so large that connections could not be made to columns without encroaching on corridor space. The difficulty was obviated by substituting for the beam two 15-in. channels, which were connected to opposite sides of the columns. Other devices may be required elsewhere.

SECTION 8

RECTANGULAR TOWER STRUCTURES

1. Statically Determinate and Indeterminate Structures.—Framed tower structures have to be classified and treated as structures in space—that is, the principal stress-carrying members are not all located in one plane. Before proceeding with determination of the stresses in a framed structure of any kind, it is necessary to investigate the rigidity of the framing and ascertain whether or not the structure is statically determinate.

Considering an arbitrary structure in space, we will assume

m = total number of joints.
n = number of members.
r = number of rigid supports.

We then have $n + 3r$ unknown quantities. It is possible to form $3m$ equilibrium equations, hence we have $n + 3r = 3m$, or

$$n = 3(m - r) \tag{1}$$

which gives us the number of necessary members to form a statically determinate structure, providing an inspection of the frame work shows each separate joint to be stable.

2. Stress Diagrams.—If we have a rectangular tower structure with straight corner posts (not necessarily vertical), we can determine the stresses by considering the four sides in turn as cantilever trusses fixed at the bottom. The external loads are divided into components, which lie in the planes of the tower sides. The stresses in the corner posts are obtained by adding up the adjacent cord stresses in the trusses, and the web stresses will appear directly in the stress diagrams.

If the corner posts are not straight, we can take each panel, or number of panels, which have straight corner posts, separately, and determine the stresses in the same manner as above. The stresses in the top section are computed first, and the stresses in the bottom members of this section are then considered as external forces acting on the section immediately below, repeating this procedure until the bottom of the tower is reached.

This method, however, is very tedious, and we will, therefore, show how it is possible to draw a continuous stress diagram where the stress

in each member is represented by its projections on the two vertical symmetry planes of the tower.

Figure 1 represents an arbitrary tower structure with rectangular cross-sections. The sides of the tower are labelled A, B, C, and D, and each side is shown in elevation, looking from the outside towards the center of the tower. The two symmetry planes will in the following be referred to as the YZ- and the XZ-planes. Two external forces, P and Q, are acting at the top and are represented in the diagram by their com-

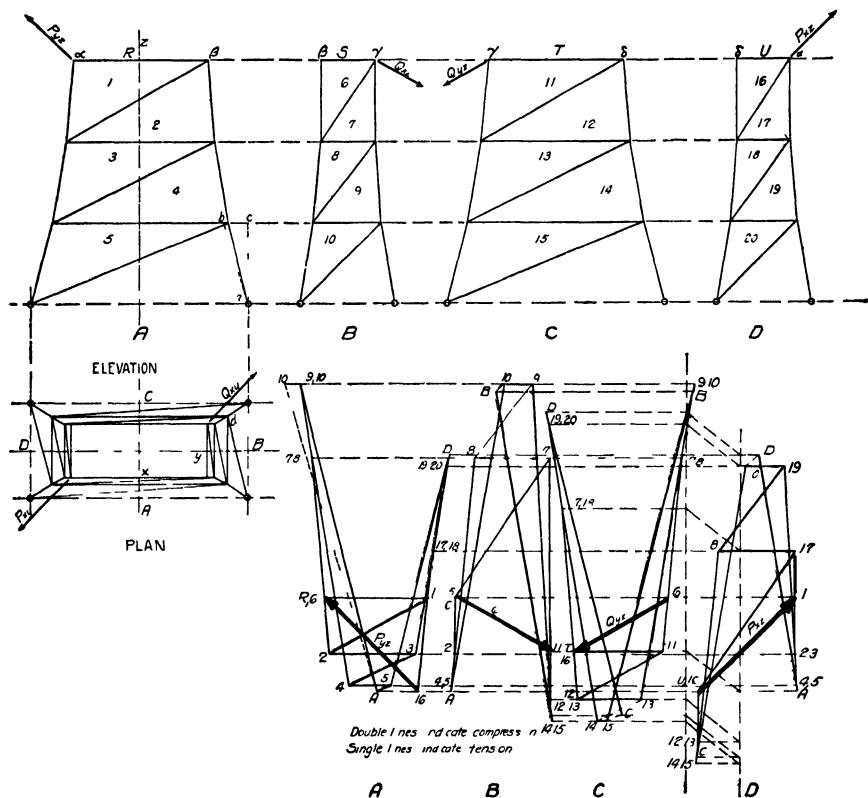



Fig. 1.

ponents P_{yz}, Q_{xz}, Q_{yz}, and P_{xz}, acting on the sides A, B, C, and D respectively. The stress diagram is divided into four parts, one for each side of the tower, which parts are also labelled A, B, C, and D.

If we first consider the side A, we find the force P_{yz} acting at the joint α, resisted by the members *R-1*, *1-17*, and *16-17*. We draw in the diagram the force polygon *R-16-1-R*, where *R-1* is the final stress in this member but *1-16* is the projection in the YZ-plane of the resultant to *1-17* and *17-16*. We cannot at this stage determine these stresses but proceed to the joint β and draw the force polygon *1-6-2-1*, giving us the stresses *2-6* and *2-1*.

We now move to the side B and investigate the XZ-projection of the joint β. *R-1* has no projection in this plane, and the XZ-projection of *1-2* is equal and reversed to *2-6*. Hence S-*6* = 0.

At the joint γ the external force Q_{xz} is acting, and we draw the polygon *6-11-7-6*, giving us the stresses *6-7* and *11-7*.

Proceeding to the YZ-projection of the joint γ, side C, we draw the polygon *6-T-11-6*, and, if we project *6-7* and *7-11* into this plane from B, we get the complete polygon for this joint, *6-T-11-7-6*. Next we draw the polygon for the joint δ and get the stresses *11-12* and *12-16*.

Moving to the side D and investigating the XZ-projection of the joint δ, we find U-*16* = 0. The next step brings us to the XZ- projection

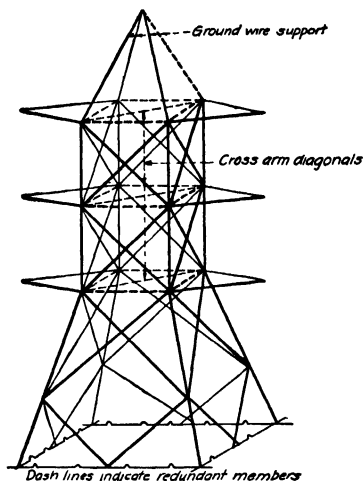



FIG. 2.

of the joint α, and we draw the polygon *16-1-17-16*, giving us the stresses *16-17* and *1-17*. If we project the stress *1-17* into diagram A, we can complete the polygon for the YZ-projection of the joint α; in the diagram we get *16-6-1-17-16*.

As may be seen from the diagram, it is necessary to make a vertical adjustment of the projecting lines between C and D. This would not be necessary if the vertical components of the two forces P and Q were equal.

It has to be remembered that each stress is represented by its two projections, which have to be combined to get the actual stress in the member in question.

If we draw the line *ac*, in elevation A, normal to the base line of the tower, and measure off the distance *bc* equal to *ad* in the plan view, *ba* represents the actual slope of the member *A-10*. If in the diagram, we draw *A-10′* parallel to *ab*, *A-10′* is the actual stress in the member. As may be seen, the percentage of error is very small if we assume *A-10* to be the actual stress instead of *A-10′*. In general, it is not considered necessary to make this correction unless the slope of the corner posts is

very great. The stresses in the horizontal struts are, of course, shown in their actual length in the diagram, and the error in the stresses in the diagonals is negligible if we take the long projection as the actual stress.

3. Redundant Members in Transmission Towers.—If we investigate the static condition of a common transmission tower, we generally find that the structure has too many members to be determinate. We will in this article look into the most common forms of redundancy in this type of structure.

3a. Horizontal Bracing.—It is the usual practice to provide horizontal bracing of the cross arms, as is shown in Fig. 2. It is obvious that these horizontal diagonals will receive practically no stress when the loading is symmetrical, but, when the tower is subjected to torsion they materially change the stress distribution.

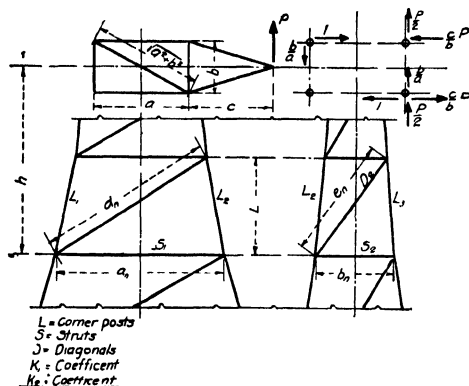



Fig. 3.

We will assume that

S = stress in any of the necessary members.
S_1 = stress in the cross arm diagonal.
S' = stress in any of the necessary members with the cross arm diagonal removed.
u = stress in any of the necessary members due to a unit tension in the cross arm diagonal.
E, A, and l = modulus of elasticity, area of cross-section, and length of any member.

We have then from the theory for statically indeterminate structures

$$S = S' + u \cdot S_1 \qquad (2)$$

and

$$S_1 = -\frac{\sum \frac{S'ul}{AE}}{\sum \frac{u^2 l}{AE}} \qquad (3)$$

Figure 3 represents any such rectangular tower with a diagonal brace of the cross arm. The external loading consists of an unbalanced pull, P, which can be replaced by a couple, $\frac{c}{b}P \times b$, and a shear force, P, acting at the right side of the body of the tower. The unit tension in the diagonal we assume to be $\sqrt{\left(\frac{b}{a}\right)^2 + 1}$, which can be replaced by its components, $\frac{b}{a}$ and 1. Selecting any intermediate panel of the tower, we get the values of S', u, l, A, etc., as follows:

Member	S'	u	$\frac{l}{A}$	$\frac{S'ul}{A}$	$\frac{u^2l}{A}$
L_1	$+\frac{hc}{ba_n}P$	$-\frac{h}{a_n}-\frac{bh}{ab_n}$	$\frac{L}{A_1}$	$-\frac{LP}{A_1}\left[\frac{h^2c}{a_n^2b}+\frac{h^2c}{aa_nb_n}\right]$	$\frac{h^2L}{A_1}\left[\frac{1}{a_n^2}+\frac{b^2}{a^2b_n^2}+\frac{2b}{aa_nb_n}\right]$
L_2	$-\frac{hc}{ba_n}P+\frac{h}{b_n}P$	$+\frac{h}{a_n}+\frac{bh}{ab_n}$	$\frac{L}{A_1}$	$-\frac{LP}{A_1}\left[\frac{h^2c}{a_n^2b}+\frac{h^2c}{aa_nb_n}-\frac{h^2}{a_nb_n}-\frac{h^2b}{ab_n^2}\right]$	$\frac{h^2L}{A_1}\left[\frac{1}{a_n^2}+\frac{b^2}{a^2b_n^2}+\frac{2b}{aa_nb_n}\right]$
L_3	$+\frac{hc}{ba_n}P-\frac{h}{b_n}P$	$-\frac{h}{a_n}-\frac{bh}{ab_n}$	$\frac{L}{A_1}$	$-\frac{LP}{A_1}\left[\frac{h^2c}{a_n^2b}+\frac{h^2c}{aa_nb_n}-\frac{h^2}{a_nb_n}-\frac{h^2b}{ab_n^2}\right]$	$\frac{h^2L}{A_1}\left[\frac{1}{a_n^2}+\frac{b^2}{a^2b_n^2}+\frac{2b}{aa_nb_n}\right]$
L_4	$-\frac{hc}{ba_n}P$	$+\frac{h}{a_n}+\frac{bh}{ab_n}$	$\frac{L}{A_1}$	$-\frac{LP}{A_1}\left[\frac{h^2c}{a_n^2b}+\frac{h^2c}{aa_nb_n}\right]$	$\frac{h^2L}{A_1}\left[\frac{1}{a_n^2}+\frac{b^2}{a^2b_n^2}+\frac{2b}{aa_nb_n}\right]$
S_1	$-k_1\frac{c}{b}P$	$+k_1$	$\frac{a_n}{A_2}$	$-\frac{k_1^2P}{A_2}\times\frac{a_nc}{b}$	$k_1^2\frac{a_n}{A_2}$
S_2	$-k_2P$	$-k_2\frac{b}{a}$	$\frac{b_n}{A_2}$	$+\frac{k_2^2P}{A_2}\times\frac{bb_n}{a}$	$k_2^2\frac{b^2b_n}{a^2A_2}$
S_3	$-k_1\frac{c}{b}P$	$+k_1$	$\frac{a_n}{A_2}$	$-\frac{k_1^2P}{A_2}\times\frac{a_nc}{b}$	$k_1^2\frac{a_n}{A_2}$
S_4	0	$-k_2\frac{b}{a}$	$\frac{b_n}{A_2}$	0	$k_2^2\frac{b^2b_n}{a^2A_2}$
D_1	$+k_1\frac{cd_n}{ba_n}P$	$-k_1\frac{d_n}{a_n}$	$\frac{d_n}{A_3}$	$-\frac{k_1^2P}{A_3}\times\frac{cd_n^3}{ba_n^2}$	$k_1^2\frac{d_n^3}{a_n^2A_3}$
D_2	$+k_2\frac{e_n}{b_n}P$	$+k_2\frac{be_n}{ab_n}$	$\frac{e_n}{A_3}$	$+\frac{k_2^2P}{A_3}\times\frac{be_n^3}{ab_n^2}$	$k_2^2\frac{b^2e_n^3}{a^2b_n^2A_3}$
D_3	$+k_1\frac{cd_n}{ba_n}P$	$-k_1\frac{d_n}{a_n}$	$\frac{d_n}{A_3}$	$-\frac{k_1^2P}{A_3}\times\frac{cd_n^3}{ba_n^2}$	$k_1^2\frac{d_n^3}{a_n^2A_3}$
D_4	0	$+k_2\frac{be_n}{ab_n}$	$\frac{e_n}{A_3}$	0	$k_2^2\frac{b^2e_n^3}{a^2b_n^2A_3}$

After having added up the values of $\frac{S'ul}{A}$ and $\frac{u^2l}{A}$, assuming the summations to extend over the entire structure, we get

$$\frac{S_1}{\sqrt{1+\left(\frac{b}{a}\right)^2}} = P\frac{\sum\frac{Lh^2}{A_1}\left(\frac{4c}{a_n^2b}+\frac{4c}{aa_nb_n}-\frac{2}{a_nb_n}-\frac{2b}{ab_n^2}\right)+\sum\frac{1}{A_2}\left(2k_1^2\frac{a_nc}{b}-k_2^2\frac{bb_n}{a}\right)+\sum\frac{1}{A_3}\left(2k_1^2\frac{cd_n^3}{a_n^2b}-k_2^2\frac{be_n^3}{ab_n^2}\right)}{4\sum\frac{Lh^2}{A_1}\left(\frac{1}{a_n^2}+\frac{b^2}{a^2b_n^2}+\frac{2b}{aa_nb_n}\right)+2\sum\frac{1}{A_2}\left(k_1^2a_n+k_2^2\frac{b^2b_n}{a^2}\right)+2\sum\frac{1}{A_3}\left(k_1^2\frac{d_n^3}{a_n^2}+k_2^2\frac{b^2e_n^3}{a^2b_n^2}\right)} \tag{4}$$

This is the general expression for S_1, which may be simplified for special cases.

If we have a square tower with all four sides of equal rigidity, we get $a = b$, $a_n = b_n$, $k_1 = k_2$, and

$$S_1 = P\frac{2c - a}{4a}\sqrt{2} \tag{5}$$

If we have a rectangular tower with one side of uniform width and the other side tapering from the bottom towards the top, and if we further assume that $\frac{a_n}{b_n} = \frac{d_n}{e_n}$, b_n = constant = the average width of the tapering side, and that A_1 is very large in comparison to A_3 and A_2, we get $a = a_n$, $k_1 = 1$, $k_2 = \frac{b}{b_n}$, and finally

$$\frac{S_1}{\sqrt{1 + \left(\frac{b}{a}\right)^2}} = P\frac{2\left(\frac{a}{b}\right)^3 c - \frac{b}{b_n}a}{2b\left[\left(\frac{a}{b}\right)^3 + \frac{b}{b_n}\right]} \tag{6}$$

and the component of S_1, parallel to the side a,

$$S_a = P\frac{2\left(\frac{a}{b}\right)^3 c - \frac{b}{b_n}a}{2b\left[\left(\frac{a}{b}\right)^3 + \frac{b}{b_n}\right]} \tag{7}$$

and the component of S_1, parallel to the side b,

$$S_b = P\frac{2\left(\frac{a}{b}\right)^3 c - \frac{b}{b_n}a}{2a\left[\left(\frac{a}{b}\right)^3 + \frac{b}{b_n}\right]} \tag{8}$$

If all four corner posts are vertical, that is $a = a_n$, $b = b_n$, and $k_1 = k_2 = 1$, we get

$$\frac{S_1}{\sqrt{1 + \left(\frac{b}{a}\right)^2}} = P\frac{2\left(\frac{a}{b}\right)^3 c - a}{2b\left[\left(\frac{a}{b}\right)^3 + 1\right]} \tag{9}$$

and the two components, S_a and S_b,

$$S_a = P\frac{2\left(\frac{a}{b}\right)^3 c - a}{2b\left[\left(\frac{a}{b}\right)^3 + 1\right]} \tag{10}$$

$$S_b = P\frac{2\left(\frac{a}{b}\right)^3 c - a}{2a\left[\left(\frac{a}{b}\right)^3 + 1\right]} \tag{11}$$

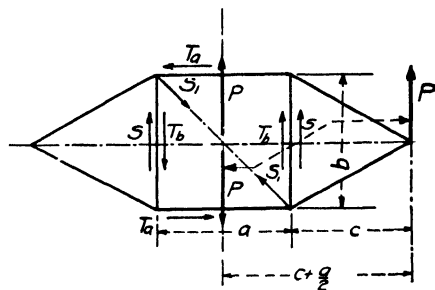
Expressions for other special shaped towers may be arrived at from the general Eq. (4) as required.

After the stress in the cross arm diagonal is determined, we can replace this member by two external forces of the magnitude S_1 and proceed with the stress diagram as shown previously.

3b. Ground Wire Support.—The common square type of transmission tower generally has its ground wire support formed by four members coming together to a point at the extreme top (Fig. 2). Only three members are, of course, necessary to form a rigid support, resulting in one redundant member. As the four members are arranged symmetrically, the stresses can, however, be determined on the basis of equal distribution.

3c. Double Diagonal Systems.—It is often found economical to use a double diagonal compressive system of framing for the sides of a tower when its width is small in comparison to its height. All horizontal struts will then become redundant, as is indicated in Fig. 2. Theoretical investigations show that this form of redundancy can be taken care of, with a small percentage of error, by assuming that each diagonal system carries one-half of the load and that the struts carry the necessary amount of horizontal shear to bring about the equal load distribution.

Fig. 4.

4. Distribution of Torsion in Transmission Towers.—The force P, acting at the end of a cross arm, can be replaced by a torsion moment, $P\left(c + \frac{a}{2}\right)$, and a shear force, P, acting at the center of the tower. We will assume that the influence on the four tower sides is expressed by the forces S, T_a, T_b, and S_1, located as shown in Fig. 4. We now get

$$T_a b + T_b a = P\left(c + \frac{a}{2}\right)$$

$$S = \frac{P}{2}$$

and, if the cross arm is not provided with diagonal bracing, we get

$$T_a = \frac{c}{b}P, \quad T_b = \frac{P}{2}, \text{ and } \frac{T_a}{T_b} = \frac{2c}{b}$$

If we assume $a = b$ and provide the tower with diagonal bracing, we get, applying Eq. (5),

$$T_b = P\frac{2c - a}{4a} + \frac{P}{2}$$

$$T_a = \frac{c}{a}P - P\frac{2c - a}{4a}$$

and $\frac{T_a}{T_b} = 1$, which shows that, if we have diagonal bracing of the cross arm and make the four tower sides of equal rigidity, the torsion effect is divided equally between the four sides of a square tower.

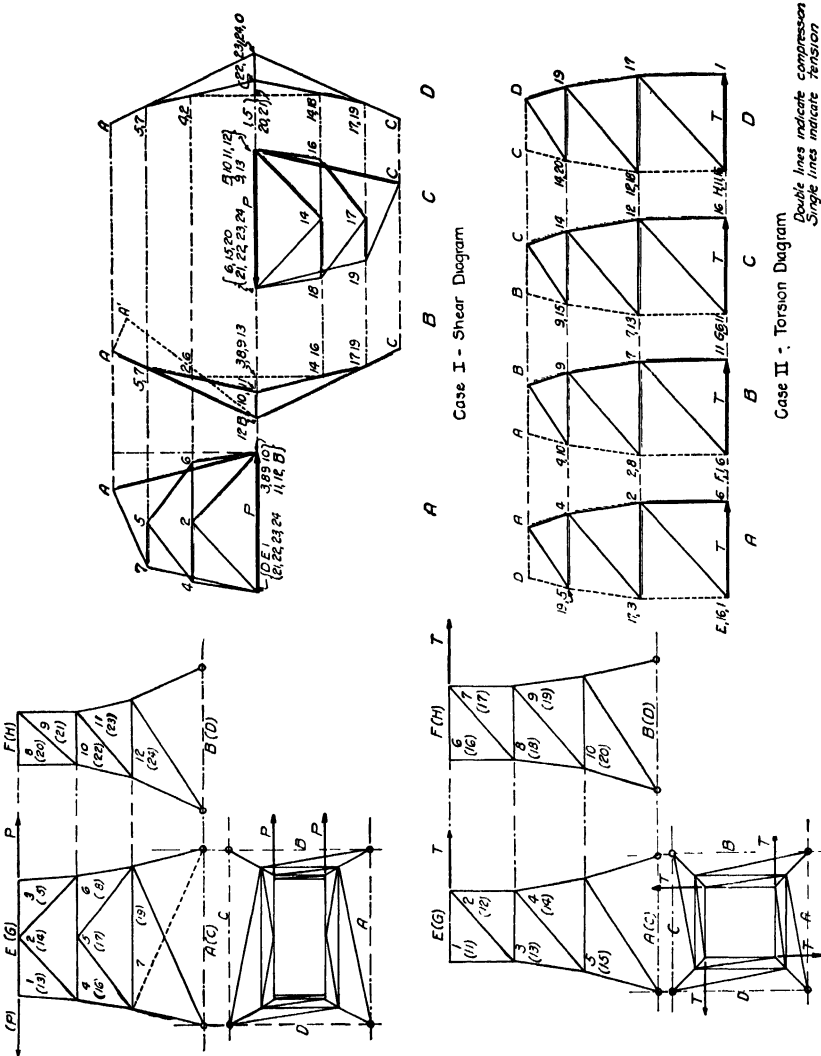



FIG. 5.

If we investigate the influence of the cross arm diagonal on the torsion distribution of a rectangular tower with vertical corner posts, we find, using Eq. (9),

$$\frac{T_a}{T_b} = \frac{b^2}{a^2} \tag{12}$$

Using Eq. (6), which applies on a rectangular tower with one side of uniform width and the other side tapering towards the top, we get

$$\frac{T_a}{T_b} = \frac{b^3}{a^2 b_n} \tag{13}$$

5. Simplified Stress Diagrams for Special Loadings.—The stress diagram described in Art. **2** can be greatly simplified under certain conditions of loading.

If the tower is subjected to straight overturning (no torsion), we can treat the two sides in which the forces are acting as cantilever trusses and draw the diagrams accordingly (Case I, Fig. 5). The only effect produced in the two other sides is a stress in the struts at such points where the corner posts change direction. A diagram of this type is generally called a shear diagram.

If we have torsion without overturning on a square tower (Case II Fig. 5), and the effect is evenly distributed over the four sides, the four parts of the diagram become identical, making it necessary to draw only one part. This is called a torsion diagram. If the tower is rectangular, but not square, it is necessary to include two adjacent sides in the diagram, the stresses in the opposite sides being similar.

Vertical loads acting symmetrically produce no web stresses except in the horizontal struts at such points where the corner posts change direction.

6. Loads on Transmission Towers.—The usual loads on a transmission tower consist of unbalanced wire pull, wind load on the wires and the structure itself, weight of wires, insulators and other hardware, and dead load of the tower structure.

The wire pull may be caused by broken wires, change in the direction of the line, change of wire size, etc. The parabolic equation, $y = cx^2$, is generally used as a basis for computing wire pulls and may be considered sufficiently accurate when the sag is small in comparison to the span.

We will assume

S = span.
d = sag below lower wire support.
w = load per lin. ft. of horizontal projection of wire (weight of wire, ice coating, wind).
h = difference in level of wire supports.
x_1 = horizontal distance from high support to lowest point of wire.
x_2 = horizontal distance from low support to lowest point of wire.
H = horizontal component of wire pull.
α_1, α_2 = angles of wire with horizontal at high and low wire supports respectively.
L = length of wire between supports.

The following formulas are of interest in connection with wire pull computations:

$$H = \frac{Sw}{2}\sqrt{\frac{S}{6(L - S) - 3\frac{h^2}{S}}} \text{ (approx.)} \tag{14}$$

$$x_1 = \frac{S}{2} + \frac{Hh}{Sw} \tag{15}$$

$$x_2 = \frac{S}{2} - \frac{Hh}{Sw} \tag{16}$$

$$L - S = \frac{2}{3} \times \frac{(d + h)^2}{x_1} + \frac{2}{3} \times \frac{d^2}{x_2} = \frac{S^3w^2}{24H^2} + \frac{h^2}{2S} \text{ (approx.)} \tag{17}$$

$$d + h = \frac{wx_1^2}{2H} \tag{18}$$

$$d = \frac{wx_2^2}{2H} \tag{19}$$

$$\tan \alpha_1 = \frac{Sw}{2H} + \frac{h}{S} \tag{20}$$

$$\tan \alpha_2 = \frac{Sw}{2H} - \frac{h}{S} \tag{21}$$

If the two wire supports are at the same level, that is $h = 0$, the formulas will be reduced to

$$H = \frac{Sw}{2}\sqrt{\frac{S}{6(L - S)}} \tag{22}$$

$$x_1 = x_2 = \frac{S}{2} \tag{23}$$

$$L - S = \frac{S^3w^2}{24H^2} \tag{24}$$

$$d = \frac{S^2w}{8H} \tag{25}$$

$$\tan \alpha_1 = \tan \alpha_2 = \frac{Sw}{2H} \tag{26}$$

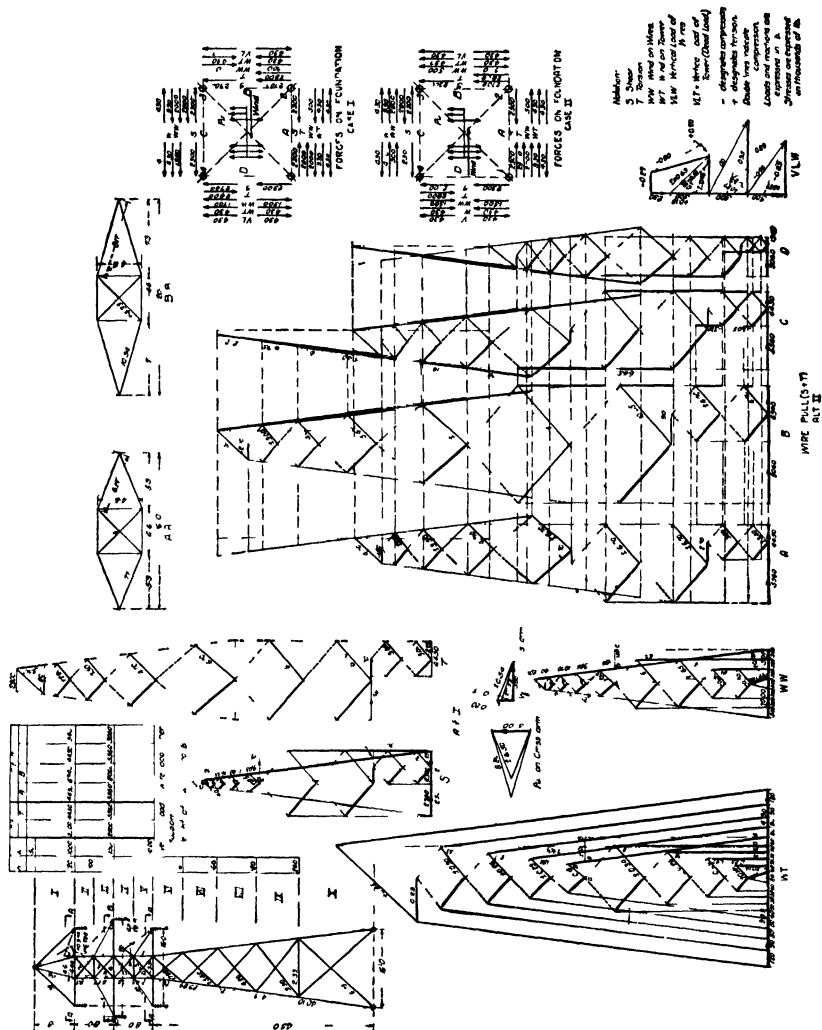
There is great diversity of opinion among engineers as to specifications for longitudinal strength of tower structures, due largely to the conflicting interests of the different public utilities concerned with the construction of overhead lines. The National Electrical Safety Code, Handbook Series of the Bureau of Standards, No. 3, is generally considered to contain the most authoritative digest of loading requirements for different types of transmission line construction.

7. Illustrative Problem.

7a. **Outlines and Loading.**—Figure 6 shows a square double circuit transmission tower of the suspension type with one ground wire,

which is to be designed in accordance with the following loading specification:

Wire pull: 5,000 lb. at each of any two conductor supports at the same side of the tower (two broken conductors).

Fig. 6.

Wind on wires: 1,000 lb. at each wire support normal to direction of line.

Wind on tower: 20 lb. per sq. ft. of exposed area of each face, normal to direction of line.

Vertical loads: 1,000 lb. at each wire support. Dead load of tower.

7b. Redundant Members.—We will assume that the double diagonal compressive system of framing is used throughout the tower except in the bottom panel.

Investigating the static condition of the structure, we find 135 members, 41 joints, and 4 rigid supports. According to Art. **1**, the number of necessary members is $3(41 - 4) = 111$, hence we have 24 redundant members. These consist of 20 horizontal struts, 1 ground wire support member, and 3 cross arm diagonals.

7c. Wire Pull.—The maximum stresses due to wire pull will, in this case, be produced when the two upper conductors are broken (except as noted in the following). To arrive at these stresses we can either divide the effect of the pull into shear and torsion, as is explained in Art. **4** (Alt. *I*), or we can draw a diagram as shown in Art. **2**, which will give the final stress due to wire pull directly (Alt. *II*). Diagrams of both types have been shown in Fig. 6, but all stress tables are based on Alt. *I*.

As the tower is square and of symmetrical design, the torsion will be divided equally between the four sides. Using Eq. (5), we get for the upper and lower cross arms

$$S_1 = (5{,}000)\,\frac{(11.5 - 4.5)}{(4)(4.5)}\sqrt{2} = 1{,}940\sqrt{2}$$

and for the middle cross arm

$$S_1 = (5{,}000)\,\frac{(15.5 - 4.5)}{(4)(4.5)}\sqrt{2} = 3{,}060\sqrt{2}$$

Alt. *I*.

Upper and lower cross arm

$$S = (\tfrac{1}{2})(5{,}000) = 2{,}500$$
$$T = \frac{(5{,}000)(8)}{(2)(4.5)} = 4{,}450$$

Middle cross arm

$$S = (\tfrac{1}{2})(5{,}000) = 2{,}500$$
$$T = \frac{(5{,}000)(10)}{(2)(4.5)} = 5{,}560$$

Alt. *II*.

Upper and lower cross arm

Face A, $P_a = \left(\frac{5.75}{4.5}\right)(5{,}000) - 1{,}940 = 4{,}450$ $(= T)$

Face B, $P_b = 5{,}000 + 1{,}940 = 6{,}940$ $(= S + T)$

Face C, $P_c = \left(\frac{5.75}{4.5}\right)(5{,}000) - 1{,}940 = 4{,}450$ $(= T)$

Face D, $P_d = 1{,}940$ $(= T - S)$

Middle cross arm

$$\text{Face } A,\ P_a = \left(\frac{7.75}{4.5}\right)(5{,}000) - 3{,}060 = 5{,}560 \quad (= T)$$
$$\text{Face } B,\ P_b = 5{,}000 + 3{,}060 = 8{,}060 \quad (= S + T)$$
$$\text{Face } C,\ P_c = P_a = 5{,}560 \quad (= T)$$
$$\text{Face } D,\ P_d = 3{,}060 \quad (= T - S)$$

7d. Stress Diagrams.—Alt. *I* shows separate shear (S) and torsion (T) diagrams, drawn as explained in Art. **5.** To find the stress in the strut at right angles to the face in which the forces are acting, we have in the shear diagram drawn the line *ac*, where *bc* is the stress in question.

It is of interest to note that, when the double compressive diagonal system of framing is used, the corner posts receive no stress due to torsion.

In Alt. *II* we have, for the sake of comparison, drawn a complete diagram for the wire pull, using the method demonstrated in Art. **2.** The stresses in the four sides A, B, C, and D are here shown directly, corresponding to T, $T + S$, T, and $T - S$ respectively.

The wind diagrams are of identical construction to the shear diagram (*W.W.* and *W.T.*).

In the diagram for vertical load of wires (*V.L.W.*, only lower half shown), the dash lines indicate the actual stresses in the cross arm members and also the stresses in the struts at right angles to the plane of the diagram.

In general, dash lines have been used in all the diagrams to indicate projected or repeated stresses, shown elsewhere in the diagram previously, or any stress which does not form part of the diagram proper.

7e. Cross Arm Stresses.—If we assume that the forces due to wire pull are all acting on the left side of the tower face, we get the following stresses in the horizontal struts which do not appear in the diagrams:

$$\text{Face } A,\ \frac{c}{a}P - \frac{S_1}{\sqrt{2}}$$
$$\text{Face } B,\ +\frac{P}{2}$$
$$\text{Face } C,\ -\frac{S_1}{\sqrt{2}}$$

Substituting the numerical values we get:

Upper and lower cross arm

$$\text{Face } A,\ \left(\frac{5.75}{4.5}\right)(5{,}000) - 1{,}940 = +4{,}450$$
$$\text{Face } B,\ +\ 2{,}500$$
$$\text{Face } C,\ -\ 1{,}940$$

Middle cross arm

$$\text{Face } A, \left(\frac{7.75}{4.5}\right)(5{,}000) - 3{,}060 = +5{,}560$$

$$\text{Face } B, + 2{,}500$$

$$\text{Face } C, - 3{,}060$$

Other cross arm stresses need no special explanation.

7f. Stress Tables.—The stress tables present no special difficulties except in connection with the horizontal struts. It has to be remembered that a horizontal strut which is located at a point where the corner posts change direction, will be stressed from forces occurring in the tower face at right angles to the one in which the strut is located. In the example on hand, struts *I-II* and *V-VI* will receive such stresses due to shear, wind load, and vertical load. In addition to this, the cross arms will produce stresses in the struts at the points where they join the tower, *e.g.*, a vertical load at the end of a cross arm will cause tension in the strut between the upper ends of the hangers and compression in the strut below which joins the ends of the horizontal cross arm members.

In Fig. 6 we have considered only one case of wire pull—that is, the two upper conductors broken. To get the maximum load in the strut *V-VI* and the lower cross arm, it is necessary to consider the two lower conductors broken. The stresses caused by this case of loading may easily be arrived at from the first case.

CORNER POSTS

Panel	*I*	*II*	*III*	*IV*	*V*	*VI*	*VII*	*VIII*	*IX*	*X*
S	....	−1 05	−3 30	−6.65	−11 00	−14 05	−15.40	−16 35	−16 95	−17 75
T	....	. .	.....	....	... ..	....	....	.... .	......	− 3 50
W. W.	−1.00	−1 55	−2.85	− 4.55	− 6 75	− 8.65	− 9.85	−10.70	−11.40	−12 00
W. T.	−0.16	−0.25	−0 47	− 0.78	− 1.18	− 1 56	− 1.84	− 2.17	− 2.53	− 3 35
V. L. W	−0.80	−0.75	−1.25	− 1.25	− 1.75	− 1.75	− 1.75	− 1.75	− 1.75	− 1.75
V. L. T.	−0.10	−0.25	−0.40	− 0.60	− 0 75	− 0.90	− 1.00	− 1.25	− 1.50	− 1.75
	−2 06	−3 85	−8.27	−13.83	−21 43	−26 91	−29.84	−32 22	−34 13	−40 10

DIAGONALS—FACE *A*

Panel	*II*	*III*	*IV*	*V*	*VI*	*VII*	*VIII*	*IX*	*X*
T	±3.00	±3 00	±6.70	±6.70	±6.70	±4.70	±3.50	±2.60	+4.70
W.W.	±0.95	±0.95	±1.65	±1 65	±1.00	±0.70	±0.50	±0.40	+0.73
W.T.	±0.14	±0.20	±0.26	±0.34	±0.18	±0 22	±0.26	±0.30	+0.85
	±4.09	±4.15	±8.61	±8.69	±7 88	±5.62	±4 26	±3.30	+6 28

DIAGONALS—FACE *B*

Panel	*II*	*III*	*IV*	*V*	*VI*	*VII*	*VIII*	*IX*	*X*
S	±1 70	±1 70	± 3 35	± 3 35	±1 05	±0 75	±0 50	±0 40	+0 75
T	±3 00	±3 00	± 6 70	± 6 70	±6 70	±4 70	±3 50	±2 60	+4 70
	±4 70	±4 70	±10 05	±10 05	±7 75	±5 45	±4 00	±3 00	+5 45

STRUTS—FACE *A* (*C*)

Panel	*I-II*	*II-III*	*III-IV*	*IV-V*	*V-VI*	*IX-X*
S					±1 70	
T	−2 23		−2 70			−1 80
C A	−1 94		−3 06			
W W.						−0 30
W T.						−0 23
V L W	−0 27	+0 95	−0 95	+0 69	−0 69	
V L T					−0 10	
	−4 44	+0 95	−6 71	+0 69	−2 49	−2 33

STRUTS—FACE *B* (*D*)

Panel	*I-II*	*II-III*	*III-IV*	*IV-V*	*V-VI*	*IX-X*
S	±1 25		±1 20			±0 30
T	−2 23		−2 70			−1 80
C A	+2 50		+2 50			
W W	±0 20		±0 15		±0 20	
W W	∓0 20				±1 00	
W T	±0 04				±0 18	
V L W.	+0 38	−0 28	+0 28	−0 28	+0 28	
V L T					−0 10	
	−0 64	−0 28	−1 27	−0 28	−1 20 +1 56	−2 10

Panel	Cross arms		Hangers		
	A-A	*B-B*	*I*	*III*	*V*
C A	±6 50	± 8 80			
W W	±0 53	± 0 54			
V L W	−0 74	− 1 00	+1 38	+1 11	+0 89
	−7 77	−10 34	+1 38	+1 11	+0 89

Strut I-VI

Face	*A*	*B*	*C*	*D*	
S	+0 55	−1 20	−0 55	+1 20	2 lower conductors broken
T	−2 23	−2 23	−2 23	−2 23	2 lower conductors broken
C A	+4 45	+2 50	−1 94		2 lower conductors broken
W W		±0 20		∓0 20	
W W		±1 00		+1 00	
W T		±0 18		∓0 18	
V L W	−0 69	+0 28	0 69	+0 28	
V L T	−0 10	−0 10	−0 10	−0 10	
	+1 98	−2 13	−5 51	−2 23	

To bear out the original assumption with respect to the torsion distribution, viz., that all four sides of the tower are of equal rigidity, we use for designing purposes only the maximum stress, which may occur in any of the four sides of a certain panel. These stresses are underlined in the tables.

7g. **Forces on Foundations.**—We have in Fig. 6 shown a convenient method of arriving at the forces acting on the tower foundations. Two cases of loading have been taken into account, the wind load being reversed in the second case.

In the tables and diagrams + indicates a shear force acting inwards and − a force acting outwards on the foundation.

VERTICAL FORCES ON FOUNDATIONS

	Case I				Case II			
	1	2	3	4	1	2	3	4
S	+17,600	+17,600	−17,600	−17,600	+17,600	+17,600	−17,600	−17,600
W W	−12,000	+12,000	+12,000	−12,000	+12,000	−12,000	−12,000	+12,000
W T	− 3,300	+ 3,300	+ 3,300	− 3,300	+ 3,300	− 3,300	− 3,300	+ 3,300
V L	− 3,500	− 3,500	− 3,500	− 3,500	− 3,500	− 3,500	− 3,500	− 3,500
	+ 1,200	+29,400	− 5,800	−36,400	+29,400	+ 1,200	−36,400	− 5,800

SHEAR FORCES ON FOUNDATIONS—CASE I

	Direction of line				Across line			
	1	2	3	4	1	2	3	4
S	+2,300	+2,700	−2,300	−2,700	+2,300	+2,300	−2,300	−2,300
T		+2 800		+2,800	+2,800		+2,800	
W W	−1,500	+1,500	+1,500	−1,500	−2,000	+1,500	+2,000	−1,500
W T.	− 430	+ 430	+ 430	− 430	− 930	+ 430	+ 930	− 430
V L	− 430	− 430	− 430	− 430	− 430	− 430	− 430	− 430
	− 60	+7,000	− 800	−2,260	+1,740	+3,800	+3,000	−4,460

SHEAR FORCES ON FOUNDATIONS—CASE II

	Direction of line				Across line			
	1	2	3	4	1	2	3	4
S	+2,300	+2,700	−2,300	−2,700	+2,300	+2,300	−2,300	−2,300
T		+2,800		+2,800	+2,800		+2,800	
W W	+1,500	−1,500	−1,500	+1,500	+2,000	−1,500	−2,000	+1,500
W T	+ 430	− 430	− 430	+ 430	+ 930	− 430	− 930	+ 430
V L	− 430	− 430	− 430	− 430	− 430	− 430	− 430	− 430
	+3,800	+3,140	−4,460	+1,600	+7,600	− 60	−2,860	− 800

We find maximum compression in post 3, Case II:

$$V = -36{,}400$$
$$S_d = -\ 4{,}660$$
$$S_a = -\ 2{,}860$$

Maximum uplift occurs in post 1, Case II:

$$V = +29{,}400$$
$$S_d = +\ 3{,}800$$
$$S_a = +\ 7{,}600$$

V = vertical force,
S_d = shear in direction of line,
S_a = shear across line.

APPENDIX A

TABLES USED WITH SECTION 6

TABLE 1

GENERAL EQUATIONS FOR THE MOMENTS AT THE ENDS OF A MEMBER *AB*

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - C_{AB} \qquad (A)$$

$$M_{BA} = 2EK(2\theta_B + \theta_A - 3R) + C_{BA} \qquad (B)$$

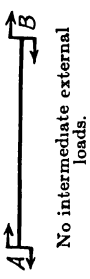
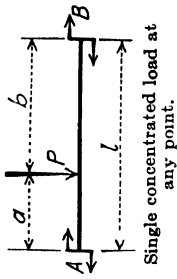
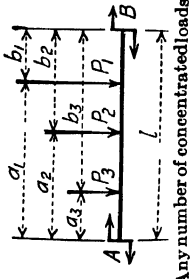
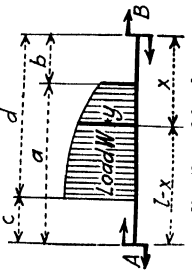
If end B is hinged,

$$M_{AB} = EK(3\theta_A - 3R) - H_{AB} \qquad (C)$$

If end A is hinged,

$$M_{BA} = EK(3\theta_B - 3R) + H_{BA} \qquad (D)$$

NOTE.—The signs of the quantities used in these equations are determined by the following rules:

θ is positive (+) when the tangent to the elastic curve is turned in a clockwise direction.

R is positive (+) when the member is deflected in a clockwise direction.

The moment of the internal stresses on a section is positive (+) when the internal couple acts in a clockwise direction upon the *portion of the member considered.*

C and H are constants whose values depend upon the load

Values of C and H for various loads are given in Tables 2 and 3.

Table 2
Values of Constants C and H to be Used in the Equations of Table 1

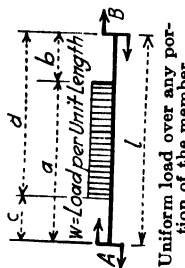
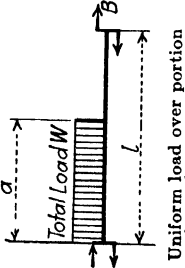
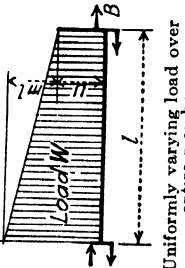
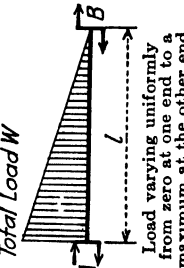
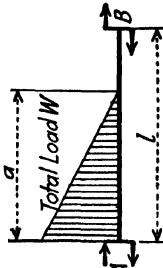
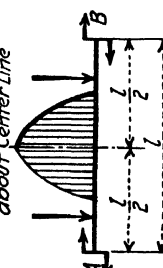
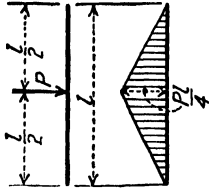
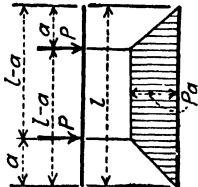
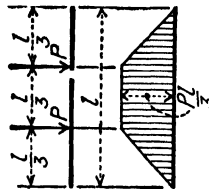
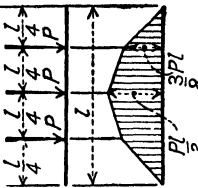
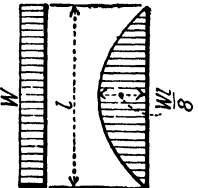
No.	Condition of loading	C_{AB}*	C_{BA}†	H_{AB}	H_{BA}
1	No intermediate external loads.	0	0	0	0
2	Single concentrated load at any point.	$\frac{Pab^2}{l^2}$	$\frac{Pba^2}{l^2}$	$\frac{Pab}{2l^2}(l+b)$	$\frac{Pab}{2l^2}(l+a)$
3	Any number of concentrated loads.	$\frac{1}{l^2}\sum Pab^2$	$\frac{1}{l^2}\sum Pba^2$	$\frac{1}{2l^2}\sum Pab(l+b)$	$\frac{1}{2l^2}\sum Pab(l+a)$
4	Any distributed load over any portion of the member.	$\frac{1}{l^2}\int^{d} yx^2(l-x)dx$	$\frac{1}{l^2}\int^{d} yx(l-x)^2dx$	$\frac{1}{2l^2}\int_{b}^{d} yx(l^2-x^2)dx$	$\frac{1}{2l^2}\int_{b}^{d} yx(l-x)(2l-x)dx$

5	c, d, a, b, w=Load per Unit Length, A, B, l Uniform load over any portion of the member.	$\frac{w}{12l^2}\left[d^3(4l - 3d) - b^3(4l - 3b)\right]$	$\frac{w}{12l^2}\left[a^3(4l - 3a) - c^3(4l - 3c)\right]$	$\frac{w}{8l^2}(d^2 - b^2)(2l^2 - b^2 - d^2)$	$\frac{w}{8l^2}(a^2 - c^2)(2l^2 - a^2 - c^2)$
6	a, Total Load W, A, B, l Uniform load over portion of member at one end	$\frac{Wa}{12l^2}(3a^2 - 8al + 6l^2)$	$\frac{Wa^2}{12l^2}(4l - 3a)$	$\frac{Wa}{8l^2}(2l - a)^2$	$\frac{Wa}{8l^2}(2l^2 - a^2)$
7	Load W, ml, u, A, B, l Uniformly varying load over entire member.	$\frac{l^2}{60}(5u + 3ml)$	$\frac{l^2}{60}(5u + 2ml)$	$\frac{l^2}{120}(15u + 8ml)$	$\frac{l^2}{120}(15u + 7ml)$
8	Total Load W, A, B, l Load varying uniformly from zero at one end to a maximum at the other end.	$\frac{Wl}{10}$	$\frac{Wl}{15}$	$\frac{2}{15}Wl$	$\frac{7}{60}Wl$

TABLE 2 (*Continued*)

No.	Condition of loading	C_{AB}*	C_{BA}†	H_{AB}	H_{BA}
9	Total Load W; a; l; A; B. Load varying uniformly from zero at any point to a maximum at one end.	$\frac{Wa}{30l^2}(3a^2 - 10al + 10l^2)$	$-\frac{Wa^2}{30l^2}(5l - 3a)$	$\frac{Wa}{60l^2}(3a^2 - 15al + 20l^2)$	$\frac{Wa}{60l^2}(10l^2 - 3a^2)$
10	Any Load Symmetrical about Center Line; $\frac{l}{2}$; $\frac{l}{2}$; l; A; B. Any load symmetrical about center of member.	$\frac{F}{l}$‡	$\frac{F}{l}$	$\frac{3}{2} \cdot \frac{F}{l}$	$\frac{3}{2} \cdot \frac{F}{l}$

* C_{AB} = fixed-end moment at A.

† C_{BA} = fixed-end moment at B

‡ F is the area of the moment diagram of a simple beam having the same length and carrying the same load as the member in question.

Values of $\frac{F}{l}$ for different loads are given in Table 3.

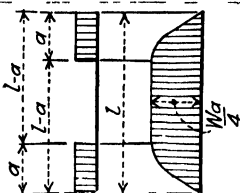
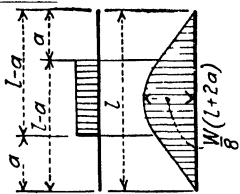
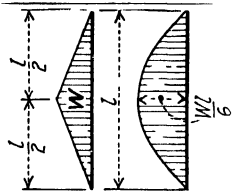
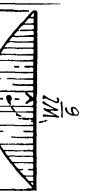
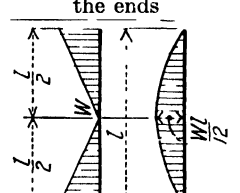
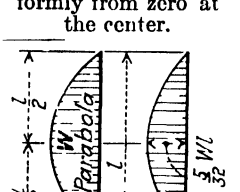



Table 3

Values of Constants C and H to be Used in the Equations of Table 1

(Load symmetrical about center of member)

No.	Condition of loading / Moment diagram	$C_{AB}^* = C_{BA}^* = \frac{F}{l}$	$H_{AB} = H_{BA} = \frac{3}{2}\frac{F}{l}$
1	Single load at the center.	$\frac{1}{8}Pl$	$\frac{3}{16}Pl$
2	Two equal loads.	$\frac{Pa}{l}(l - a)$	$\frac{3}{2}\frac{Pa}{l}(l - a)$
3	Equal loads at the third points.	$\frac{2}{9}Pl$	$\frac{1}{3}Pl$
4	Equal loads at the quarter points.	$\frac{5}{16}Pl$	$\frac{15}{32}Pl$
5	Uniform load over entire span.	$\frac{1}{12}Wl$	$\frac{1}{8}Wl$

* $C_{AB} = C_{BA}$ = fixed-end moment.

TABLE 3 (*Continued*)

No.	Condition of loading / Moment diagram	$C_{AB}^* = C_{BA}^* = \frac{F}{l}$	$H_{AB} = H_{BA} = \frac{3}{2}\frac{F}{l}$
6	Equal uniform loads at the ends. (l-a, a, $\frac{Wa}{4}$)	$\frac{Wa}{12l}(3l - 2a)$	$\frac{Wa}{8l}(3l - 2a)$
7	Uniform load at the center. (l-a, a, $\frac{W(l+2a)}{8}$)	$\frac{W}{12l}(l^2 + 2al - 2a^2)$	$\frac{W}{8l}(l^2 + 2al - 2a^2)$
8	Load increasing uniformly from zero at the ends ($\frac{l}{2}$, W, $\frac{Wl}{6}$)	$\frac{5}{48}Wl$	$\frac{5}{32}Wl$
9	Load increasing uniformly from zero at the center. ($\frac{l}{2}$, W, $\frac{Wl}{12}$)	$\frac{1}{16}Wl$	$\frac{3}{32}Wl$
10	Load varying as the ordinates of a parabola. ($\frac{l}{2}$, W, Parabola, $\frac{5Wl}{32}$)	$\frac{1}{10}Wl$	$\frac{3}{20}Wl$

* $C_{AB} = C_{BA}$ = fixed-end moment.

TABLE 4

CONTINUOUS GIRDER. ONE SPAN

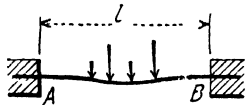
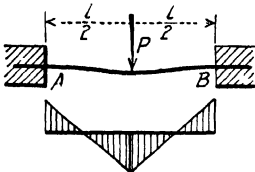
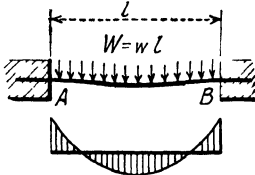
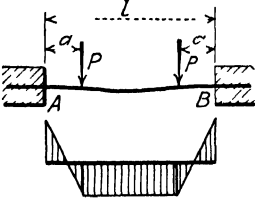
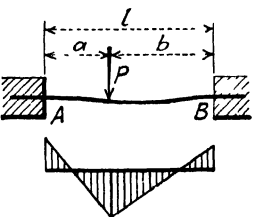
Supports all on same level
Girder fixed at the ends

Type of loading	Moments
1. Any system of loads.	$M_{AB} = -C_{AB}$ $M_{BA} = +C_{BA}$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{BA} = -M_{AB} = \frac{1}{8} Pl$
3. Uniform load.	$M_{BA} = -M_{AB} = \frac{1}{12} Wl$
4. Two equal symmetrically spaced loads.	$M_{AB} = -M_{BA} = -\frac{Pa}{l}(l - a)$
5. Single load at any point.	$M_{AB} = -\frac{Pab^2}{l^2}$ $M_{BA} = \frac{Pa^2b}{l^2}$

TABLE 5

CONTINUOUS GIRDER. ONE SPAN

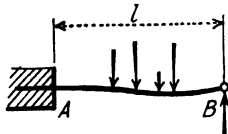
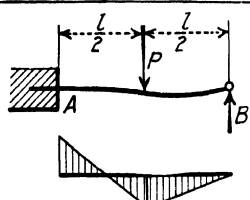
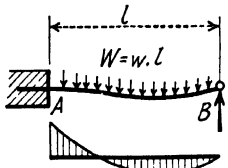
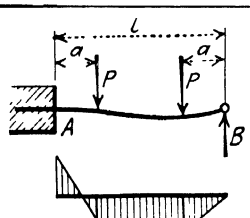
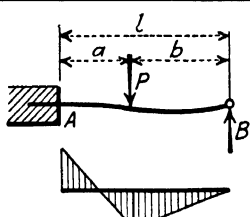
Supports all on same level
Girder fixed at A and hinged at B

Type of loading	Moments
1. Any system of loads.	$M_{AB} = -H_{AB}$ $M_{BA} = 0$ Values of H for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AB} = -\frac{3}{16}Pl$
3. Uniform load.	$M_{AB} = -\frac{1}{8}Wl$
4. Two equal loads symmetrically spaced.	$M_{AB} = -\frac{3}{2}\frac{Pa}{l}(l - a)$
5. Single load at any point.	$M_{AB} = -\frac{Pab}{2l^2}(l + b)$

TABLE 6

CONTINUOUS GIRDER. ONE SPAN

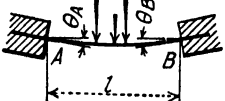
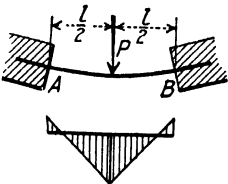
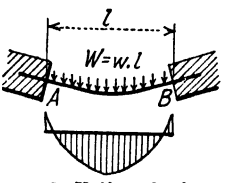
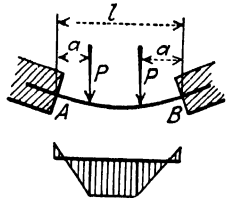
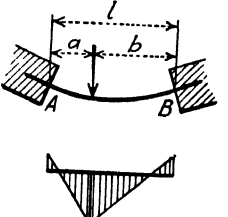
Supports all on same level
Girder restrained at A and B but not fixed

Type of loading	Moments and slopes
1. Any system of loads.	$M_{AB} = 2EK(2\theta_A + \theta_B) - C_{AB}$ $M_{BA} = 2EK(2\theta_B + \theta_A) + C_{BA}$ $\theta_A = \frac{2M_{AB} - M_{BA} + 2C_{AB} + C_{BA}}{6EK}$ $\theta_B = \frac{2M_{BA} - M_{AB} - 2C_{BA} - C_{AB}}{6EK}$ Values of C for various loads are given in Tables 2 and 3
2. Single load at center.	$M_{AB} = 2EK(2\theta_A + \theta_B) - \frac{1}{8}Pl$ $M_{BA} = 2EK(2\theta_B + \theta_A) + \frac{1}{8}Pl$ $\theta_A = \frac{2M_{AB} - M_{BA} + \frac{3}{8}Pl}{6EK}$ $\theta_B = \frac{2M_{BA} - M_{AB} - \frac{3}{8}Pl}{6EK}$
3. Uniform load.	$M_{AB} = 2EK(2\theta_A + \theta_B) - \frac{1}{12}Wl$ $M_{BA} = 2EK(2\theta_B + \theta_A) + \frac{1}{12}Wl$ $\theta_A = \frac{2M_{AB} - M_{BA} + \frac{1}{4}Wl}{6EK}$ $\theta_B = \frac{2M_{BA} - M_{AB} - \frac{1}{4}Wl}{6EK}$
4. Two equal symmetrically spaced loads.	$M_{AB} = 2EK(2\theta_A + \theta_B) - \frac{Pa}{l}(l - a)$ $M_{BA} = 2EK(2\theta_A + \theta_B) + \frac{Pa}{l}(l - a)$ $\theta_A = \frac{2M_{AB} - M_{BA} + \frac{3Pa}{l}(l - a)}{6EK}$ $\theta_B = \frac{2M_{BA} - M_{AB} - \frac{3Pa}{l}(l - a)}{6EK}$
5. Single load at any point.	$M_{AB} = 2EK(2\theta_A + \theta_B) - \frac{Pab^2}{l^2}$ $M_{BA} = 2EK(2\theta_B + \theta_A) + \frac{Pba^2}{l^2}$ $\theta_A = \frac{2M_{AB} - M_{BA} + \frac{Pab}{l^2}(l + b)}{6EK}$ $\theta_B = \frac{2M_{BA} - M_{AB} - \frac{Pab}{l^2}(l + a)}{6EK}$

TABLE 7

CONTINUOUS GIRDER. ONE SPAN

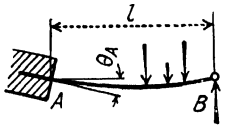
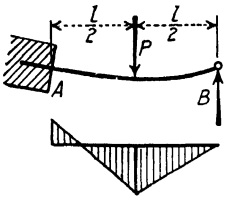
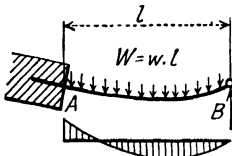
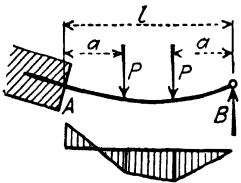
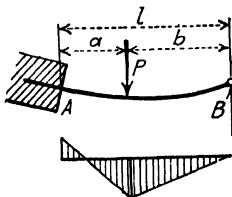
Supports all on same level
Girder hinged at B and restrained but not fixed at A

Type of loading	Moments and slopes
1. Any system of loads.	$M_{AB} = 3EK\theta_A - H_{AB}$ $\theta_A = \dfrac{M_{AB} + H_{AB}}{3EK}$ Values of H for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AB} = 3EK\theta_A - \dfrac{3}{16}Pl$ $\theta_A = \dfrac{M_{AB} + \dfrac{3}{16}Pl}{3EK}$
3. Uniform load.	$M_{AB} = 3EK\theta_A - \dfrac{1}{8}Wl$ $\theta_A = \dfrac{M_{AB} + \dfrac{1}{8}Wl}{3EK}$
4. Two equal symmetrically spaced loads.	$M_{AB} = 3EK\theta_A - \dfrac{3}{2}\dfrac{Pa}{l}(l - a)$ $\theta_A = \dfrac{M_{AB} + \dfrac{3}{2}\dfrac{Pa}{l}(l - a)}{3EK}$
5. Single load at any point.	$M_{AB} = 3EK\theta_A - \dfrac{Pab}{2l^2}(l + b)$ $\theta_A = \dfrac{M_{AB} + \dfrac{Pab}{2l^2}(l + b)}{3EK}$

Table 8

Continuous Girder. One Span

Supports on different levels
Girder fixed at the ends

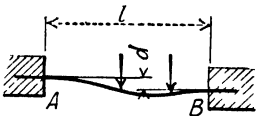
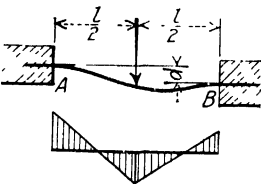
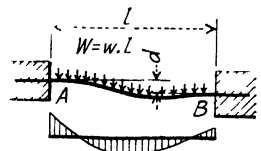
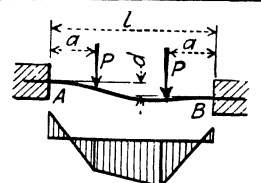
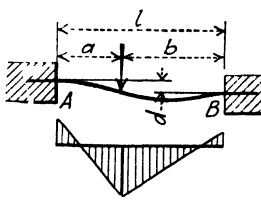
$$R = \frac{d}{l}, \quad K = \frac{I}{l}$$

Type of loading	Moments
1. Any system of loads.	$M_{AB} = -6EKR - C_{AB}$ $M_{BA} = -6EKR + C_{BA}$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AB} = -6EKR - \frac{1}{8}Pl$ $M_{BA} = -6EKR + \frac{1}{8}Pl$
3. Uniform load.	$M_{AB} = -6EKR - \frac{1}{12}Wl$ $M_{BA} = -6EKR + \frac{1}{12}Wl$
4. Two equal symmetrically spaced loads.	$M_{AB} = -6EKR - \frac{Pa}{l}(l - a)$ $M_{BA} = -6EKR + \frac{Pa}{l}(l - a)$
5. Single load at any point.	$M_{AB} = -6EKR - \frac{Pab^2}{l^2}$ $M_{BA} = -6EKR + \frac{Pa^2b}{l^2}$

TABLE 9

CONTINUOUS GIRDER. ONE SPAN

Supports on different levels
Girder fixed at A and hinged at B

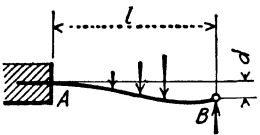
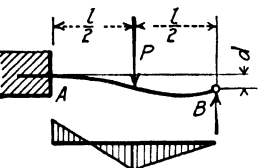
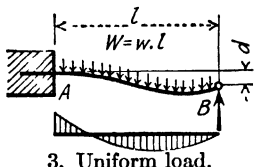
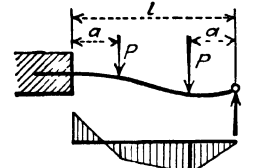
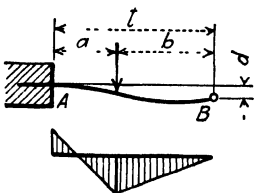
$$R = \frac{d}{l}, K = \frac{I}{l}$$

Type of loading	Moments
1. Any system of loads.	$M_{AB} = -3EKR - H_{AB}$ $M_{BA} = 0$ Values of H for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AB} = -3EKR - \frac{3}{16}Pl$
3. Uniform load.	$M_{AB} = -3EKR - \frac{1}{8}Wl$
4. Two equal symmetrically spaced loads.	$M_{AB} = -3EKR - \frac{3}{2}\frac{Pa}{l}(l - a)$
5. Single load at any point.	$M_{AB} = -3EKR - \frac{Pab}{2l^2}(l + b)$

TABLE 10

CONTINUOUS GIRDER. ONE SPAN

Supports on different levels
Girder restrained at A and B but not fixed

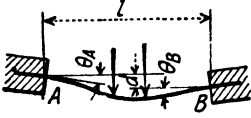
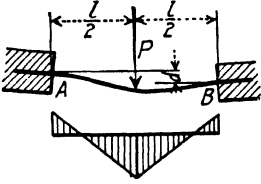
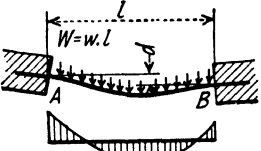
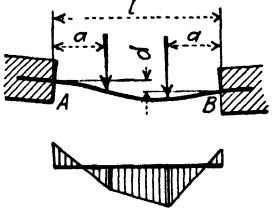
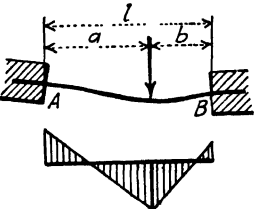
$$R = \frac{d}{l},\ K = \frac{I}{l}$$

Type of loading	Moments and slopes
1. Any system of loads.	$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - C_{AB}$ $M_{BA} = 2EK(2\theta_B + \theta_A - 3R) + C_{BA}$ $\theta_B = \frac{2M_{BA} - M_{AB} - 2C_{BA} - C_{AB} + 6EKR}{6EK}$ $\theta_A = \frac{2M_{AB} - M_{BA} + 2C_{AB} + C_{BA} + 6EKR}{6EK}$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - \frac{1}{8}Pl$ $M_{BA} = 2EK(2\theta_B + \theta_A - 3R) + \frac{1}{8}Pl$ $\theta_B = \frac{2M_{BA} - M_{AB} - \frac{3}{8}Pl + 6EKR}{6EK}$ $\theta_A = \frac{2M_{AB} - M_{BA} + \frac{3}{8}Pl + 6EKR}{6EK}$
3. Uniform load.	$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - \frac{1}{12}Wl$ $M_{BA} = 2EK(2\theta_B + \theta_A - 3R) + \frac{1}{12}Wl$ $\theta_B = \frac{2M_{BA} - M_{AB} - \frac{1}{4}Wl + 6EKR}{6EK}$ $\theta_A = \frac{2M_{AB} - M_{BA} + \frac{1}{4}Wl + 6EKR}{6EK}$
4. Two equal symmetrically spaced loads.	$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - \frac{Pa}{l}(l - a)$ $M_{BA} = 2EK(2\theta_B + \theta_A - 3R) + \frac{Pa}{l}(l - a)$ $\theta_B = \frac{2M_{BA} - M_{AB} - \frac{3Pa}{l}(l - a) + 6EKR}{6EK}$ $\theta_A = \frac{2M_{AB} - M_{BA} + \frac{3Pa}{l}(l - a) + 6EKR}{6EK}$
5. Single load at any point.	$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - \frac{Pab^2}{l^2}$ $M_{BA} = 2EK(2\theta_B + \theta_A - 3R) + \frac{Pa^2b}{l^2}$ $\theta_B = \frac{2M_{BA} - M_{AB} - \frac{Pab}{l^2}(l + a) + 6EKR}{6EK}$ $\theta_A = \frac{2M_{AB} - M_{BA} + \frac{Pab}{l^2}(l + b) + 6EKR}{6EK}$

TABLE 11

CONTINUOUS GIRDER. ONE SPAN

Supports on different levels
Girder hinged at B and restrained but not fixed at A

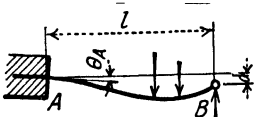
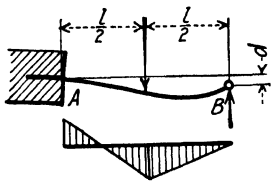
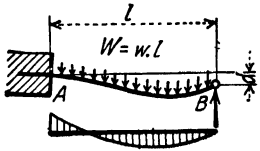
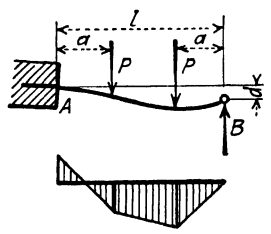
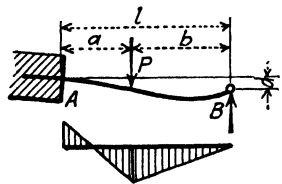
$$R = \frac{d}{l} \qquad K = \frac{I}{l}$$

Type of loading	Moments and slopes
1. Any system of loads.	$M_{AB} = 3EK(\theta_A - R) - H_{AB}$ $\theta_A = \frac{M_{AB} + H_{AB}}{3EK} + R$ Values of H for various loads are given in Tables 2 and 3
2. Single load at center.	$M_{AB} = 3EK(\theta_A - R) - \frac{3}{16}Pl$ $\theta_A = \frac{M_{AB} + \frac{3}{16}Pl}{3EK} + R$
3. Uniform load.	$M_{AB} = 3EK(\theta_A - R) - \frac{1}{8}Wl$ $\theta_A = \frac{M_{AB} + \frac{1}{8}Wl}{3EK} + R$
4. Two equal symmetrically spaced loads.	$M_{AB} = 3EK(\theta_A - R) - \frac{3}{2}\frac{Pa}{l}(l - a)$ $\theta_A = \frac{M_{AB} + \frac{3}{2}\frac{Pa}{l}(l - a)}{3EK} + R$
5. Single load at any point.	$M_{AB} = 3EK(\theta_A - R) - \frac{Pab}{2l^2}(l + b)$ $\theta_A = \frac{M_{AB} + \frac{Pab}{2l^2}(l + b)}{3EK} + R$

Table 12

Continuous Girder. Two Spans

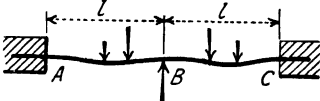
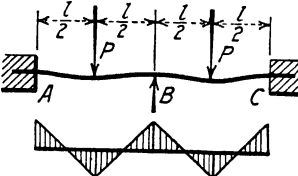
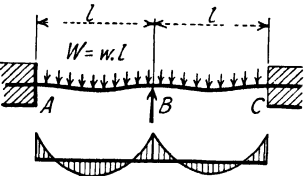
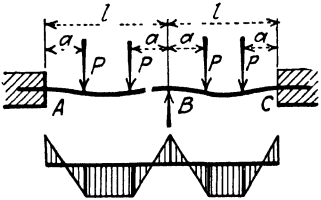
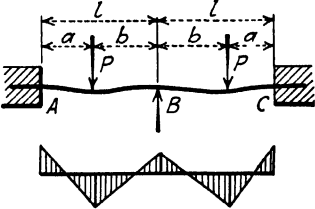
Supports all on same level
Girder fixed at the ends
Two spans identical
Load symmetrical about center support

Type of loading	Moments
1. Any system of loads symmetrical about center support.	$M_{AB} = -C_{AB}$ $M_{CB} = C_{CB}$ $M_{BC} = -C_{BA}$ Values of C for various loads are given in Tables 2 and 3.
 2. Equal loads at center of each span	$M_{AB} = M_{BC} = -M_{CB} = -\frac{1}{8}Pl$
 3. Uniform load.	$M_{AB} = M_{BC} = -M_{CB} = -\frac{1}{12}Wl$
 4. Two equal loads symmetrically spaced on each span.	$M_{AB} = M_{BC} = -M_{CB} = -\frac{Pa}{l}(l-a)$
 5. Single load on each span.	$M_{AB} = -M_{CB} = -\frac{Pab^2}{l^2}$ $M_{BC} = -\frac{Pa^2b}{l^2}$

TABLE 13

CONTINUOUS GIRDER. TWO SPANS

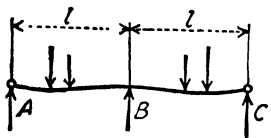
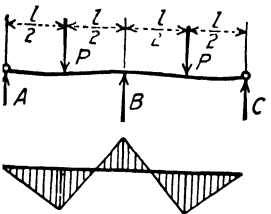
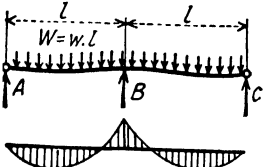
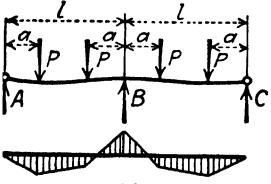
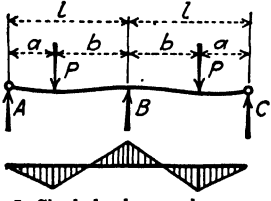
Supports all on same level
Girder hinged at the ends
Two spans identical
Load symmetrical about center support

Type of loading	Moments
. Any system of loads symmetrical about center support	$M_{BC} = -H_{BC}$ $M_{AB} = M_{CB} = 0$ Values of H for various loads are given in Tables 2 and 3.
2. Equal loads at center of each span.	$M_{BC} = -\frac{3}{16}Pl$ $M_{CB} = M_{AB} = 0$
3. Uniform load.	$M_{BC} = -\frac{1}{8}Wl$ $M_{AB} = M_{CB} = 0$
4. Two equal loads symmetrically spaced on each span.	$M_{BC} = -\frac{3}{2}\frac{Pa}{l}(l-a)$ $M_{AB} = M_{CB} = 0$
5. Single load on each span.	$M_{BC} = -\frac{Pab}{2l^2}(l+a)$ $M_{AB} = M_{CB} = 0$

TABLE 14

CONTINUOUS GIRDER. TWO SPANS

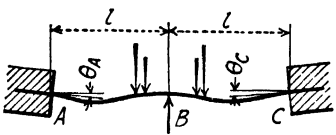
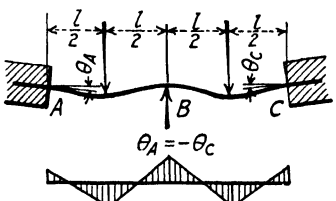
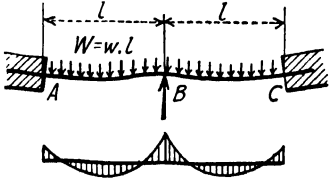
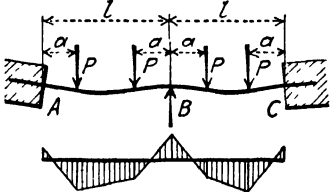
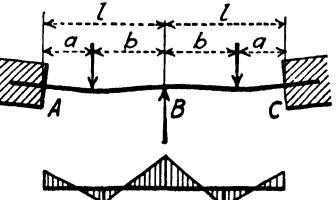
Supports all on same level
Ends of girder restrained but not fixed
Two spans identical
Load symmetrical about center support

Type of loading	Moments
1. Any system of loads symmetrical about center support.	$M_{AB} = -M_{CB} = 4EK\theta_A - C_{AB}$ $M_{BC} = 2EK\theta_C - C_{BC}$ $\theta_A = -\theta_C = \frac{M_{AB} + C_{AB}}{4EK}$ Values of C for various loads are given in Tables 2 and 3.
$\theta_A = -\theta_C$ 2. Equal loads at center of each span.	$M_{AB} = -M_{CB} = 4EK\theta_A - \frac{1}{8}Pl$ $M_{BC} = 2EK\theta_C - \frac{1}{8}Pl$ $\theta_A = \frac{M_{AB} + \frac{1}{8}Pl}{4EK}$
3. Uniform load.	$M_{AB} = -M_{CB} = 4EK\theta_A - \frac{1}{12}Wl$ $M_{BC} = 2EK\theta_C - \frac{1}{12}Wl$ $\theta_A = \frac{M_{AB} + \frac{1}{12}Wl}{4EK}$
4. Two equal loads symmetrically spaced on each span	$M_{AB} = -M_{CB} = 4EK\theta_A - \frac{Pa}{l}(l - a)$ $M_{BC} = 2EK\theta_C - \frac{Pa}{l}(l - a)$ $\theta_A = \frac{M_{AB} + \frac{Pa}{l}(l - a)}{4EK}$
5. Single load on each span	$M_{AB} = -M_{CB} = 4EK\theta_A - \frac{Pab^2}{l^2}$ $M_{BC} = 2EK\theta_C - \frac{Pa^2b}{l^2}$ $\theta_A = \frac{M_{AB} + \frac{Pab^2}{l^2}}{4EK}$

TABLE 15

CONTINUOUS GIRDER. TWO SPANS

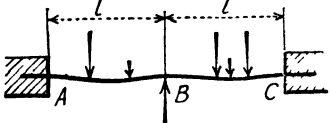
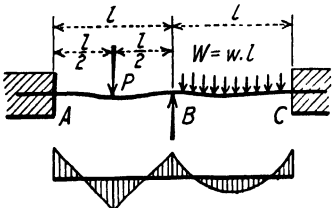
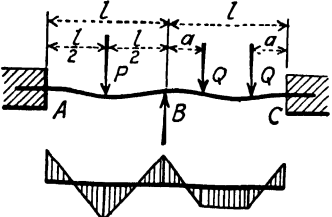
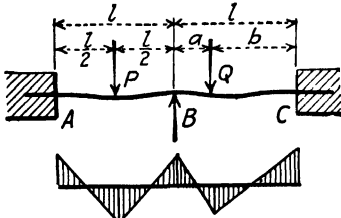
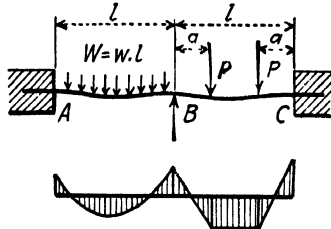
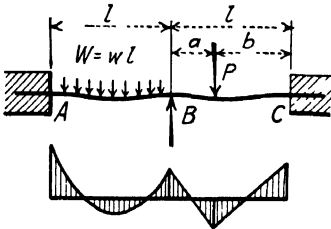
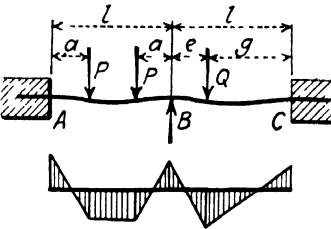
Supports all on same level
Girder fixed at the ends
Two spans identical except for load

Type of loading	Moments
1. Any system of loads.	$M_{AB} = \frac{1}{4}C_{BC} - C_{AB} - \frac{1}{4}C_{BA}$ $M_{BC} = -\frac{1}{2}C_{BA} - \frac{1}{2}C_{BC}$ $M_{CB} = C_{CB} - \frac{1}{4}C_{BA} + \frac{1}{4}C_{BC}$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at center of one span, uniform load on other span.	$M_{AB} = \frac{1}{48}Wl - \frac{5}{32}Pl$ $M_{BC} = -\frac{1}{16}Pl - \frac{1}{24}Wl$ $M_{CB} = \frac{5}{48}Wl - \frac{1}{32}Pl$
3. Single load at center of one span, two equal symmetrically spaced loads on other.	$M_{AB} = \frac{1}{4}\frac{Qa}{l}(l-a) - \frac{5}{32}Pl$ $M_{BC} = -\frac{1}{16}Pl - \frac{Qa}{2l}(l-a)$ $M_{CB} = \frac{5}{4}\frac{Qa}{l}(l-a) - \frac{1}{32}Pl$
4. Single load at center of one span, single load at any point on other.	$M_{AB} = \frac{1}{4}\frac{Qab^2}{l^2} - \frac{5}{32}Pl$ $M_{BC} = -\frac{1}{16}Pl - \frac{Qab^2}{2l^2}$ $M_{CB} = \frac{Qab(4a+b)}{4l^2} - \frac{Pl}{32}$

TABLE 15 (*Continued*)

Type of loading | **Moments**

5. Uniform load on one span, two equal symmetrically spaced loads on other.

$$M_{AB} = \frac{1}{4}\frac{Pa}{l}(l - a) - \frac{5}{48}Wl$$

$$M_{BC} = -\frac{1}{24}Wl - \frac{Pa}{2l}(l - a)$$

$$M_{CB} = \frac{5}{4}\frac{Pa}{l}(l - a) - \frac{1}{48}Wl$$

6. Uniform load on one span, single load at any point on other.

$$M_{AB} = \frac{1}{4}\frac{Pab^2}{l^2} - \frac{5}{48}Wl$$

$$M_{BC} = -\frac{1}{24}Wl - \frac{Pab^2}{2l^2}$$

$$M_{CB} = \frac{Pab}{4l^2}(4a + b) - \frac{1}{48}Wl$$

7. Two equal symmetrically spaced loads on one span, single load at any point on the other.

$$M_{AB} = \frac{1}{4}\frac{Qeg^2}{l^2} - \frac{5}{4}\frac{Pa}{l}(l - a)$$

$$M_{BC} = -\frac{Pa}{2l}(l - a) - \frac{Qeg^2}{2l^2}$$

$$M_{CB} = \frac{Qeg}{4l^2}(4e + g) - \frac{Pa}{4l}(l - a)$$

TABLE 16

CONTINUOUS GIRDER. TWO SPANS

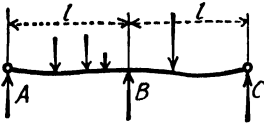
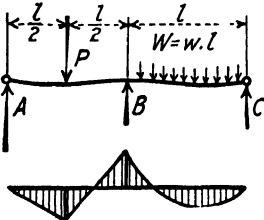
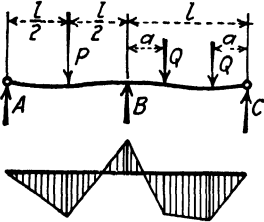
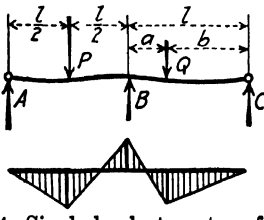
Supports all on same level
Girder hinged at ends
Two spans identical except for load

Type of loading	Moments
1. Any system of loads.	$M_{BC} = -\frac{1}{2}(H_{BA} + H_{BC})$ $M_{AB} = M_{CB} = 0$ Values of H for various loads are given in Tables 2 and 3
2. Single load at center of one span, uniform load on other.	$M_{BC} = -\frac{3}{32}Pl - \frac{1}{16}Wl$
3. Single load at center of one span, two equal symmetrically spaced loads on other.	$M_{BC} = -\frac{3}{32}Pl - \frac{3Qa}{4l}(l - a)$
4. Single load at center of one span, single load at any point on other.	$M_{BC} = -\frac{3}{32}Pl - \frac{Qab}{4l^2}(l + b)$

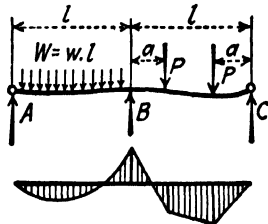
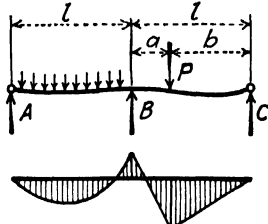
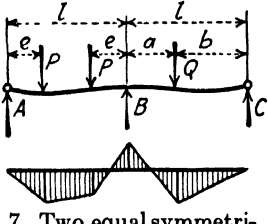
TABLE 16 (*Continued*)

Type of loading	Moments
5. Uniform load on one span, two equal symmetrically spaced loads on other.	$M_{BC} = -\frac{1}{16} Wl - \frac{3Pa}{4l}(l - a)$
6. Uniform load on one span single load at any point on other.	$M_{BC} = -\frac{1}{16} Wl - \frac{Pab}{4l^2}(l + b)$
7. Two equal symmetrically spaced loads on one span, single load at any point on other.	$M_{BC} = -\frac{3Pe}{4l}(l - e) - \frac{Qab}{4l^2}(l + b)$

TABLE 17

CONTINUOUS GIRDER. TWO SPANS

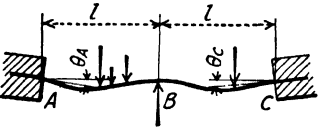
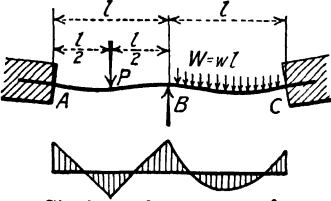
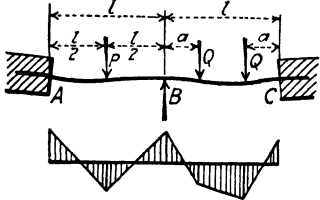
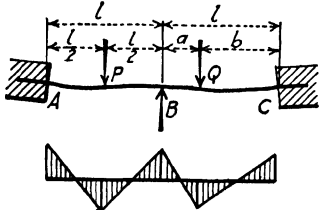
Supports all on same level
Ends of girder restrained but not fixed
Two spans identical except for load

Type of loading	Moments
$\theta_A = -\theta_C$ 1. Any system of loads.	$M_{AB} = 4EK\theta_A - \frac{1}{4}C_{BA} - C_{AB} + \frac{1}{4}C_{BC}$ $M_{BC} = -2EK\theta_A - \frac{1}{2}C_{BA} - \frac{1}{2}C_{BC}$ $M_{CB} = -4EK\theta_A - \frac{1}{4}C_{BA} + \frac{1}{4}C_{BC} + C_{CB}$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at center of one span, uniform load on other.	$M_{AB} = 4EK\theta_A - \frac{5}{32}Pl + \frac{1}{48}Wl$ $M_{BC} = -2EK\theta_A - \frac{1}{16}Pl - \frac{1}{24}Wl$ $M_{CB} = -4EK\theta_A - \frac{1}{32}Pl + \frac{5}{48}Wl$
3. Single load at center of one span, two equal symmetrically spaced loads on other.	$M_{AB} = 4EK\theta_A - \frac{5}{32}Pl + \frac{Qa}{4l}(l - a)$ $M_{BC} = -2EK\theta_A - \frac{1}{16}Pl - \frac{Qa}{2l}(l - a)$ $M_{CB} = -4EK\theta_A - \frac{1}{32}Pl + \frac{5Qa}{4l}(l - a)$
4. Single load at center of one span, single load at any point on other.	$M_{AB} = 4EK\theta_A - \frac{5}{32}Pl + \frac{Qab^2}{4l^2}$ $M_{BC} = -2EK\theta_A - \frac{1}{16}Pl - \frac{Qab^2}{2l^2}$ $M_{CB} = -4EK\theta_A - \frac{1}{32}Pl + \frac{Qab}{4l^2}(b + 4a)$

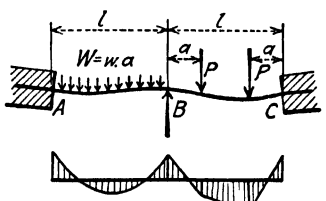
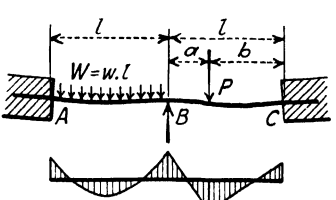
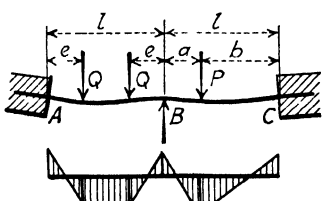
TABLE 17 (*Continued*)

Type of loading	Moments
 5. Uniform load on one span, two equal symmetrically spaced loads on other.	 $M_{AB} = 4EK\theta_A - \frac{5}{48}Wl + \frac{Pa}{4l}(l - a)$ $M_{BC} = -2EK\theta_A - \frac{1}{24}Wl - \frac{Pa}{2l}(l - a)$ $M_{CB} = -4EK\theta_A - \frac{1}{48}Wl + \frac{5Pa}{4l}(l - a)$
 6. Uniform load on one span, single load at any point on other.	 $M_{AB} = 4EK\theta_A - \frac{5}{48}Wl + \frac{Pab^2}{4l^2}$ $M_{BC} = -2EK\theta_A - \frac{1}{24}Wl - \frac{Pab^2}{2l^2}$ $M_{CB} = -4EK\theta_A - \frac{1}{48}Wl + \frac{Pab}{4l^2}(b + 4a)$
 7. Two equal symmetrically spaced loads on one span, single load at any point on other.	$M_{AB} = 4EK\theta_A - \frac{5Qe}{4l}(l - e) + \frac{Pab^2}{4l^2}$ $M_{BC} = -2EK\theta_A - \frac{Qe}{2l}(l - e) - \frac{Pab^2}{2l^2}$ $M_{CB} = -4EK\theta_A - \frac{Qe}{4l}(l - e) + \frac{Pab}{4l^2}(b + 4a)$

TABLE 18

CONTINUOUS GIRDER. TWO SPANS

Supports all on same level
Girder fixed at the ends
Lengths of two spans different

$K = \frac{I_1}{l_1}$

Moment of inertia for AB is I_0
Moment of inertia for BC is I_1

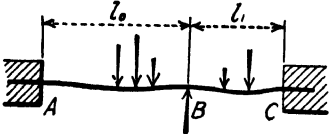
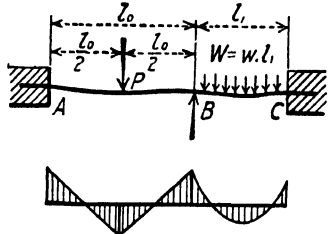
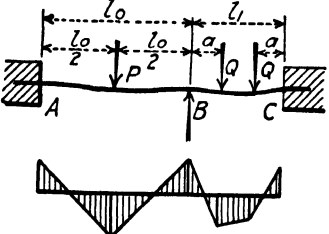
$n = \frac{I_1 l_0}{l_1 I_0}$

Type of loading	Moments
1. Any system of loads.	$M_{BC} = \frac{1}{n+1}(-nC_{BA} - C_{BC})$ $M_{AB} = -\frac{M_{BC}}{2} - H_{AB}$ $M_{CB} = \frac{M_{BC}}{2} + H_{CB}$ Values of C and H for various loads are given in Tables 2 and 3.
2. Single load at center of one span, uniform load on other.	$M_{BC} = -\frac{1}{n+1}\left(\frac{1}{8}nPl_0 + \frac{1}{12}Wl_1\right)$ $M_{AB} = -\left(\frac{M_{BC}}{2} + \frac{3}{16}Pl_0\right)$ $M_{CB} = \frac{M_{BC}}{2} + \frac{1}{8}Wl_1$
3. Single load at center of one span, two equal symmetrically spaced loads on other.	$M_{BC} = -\frac{1}{n+1}\left[\frac{1}{8}\,n\,Pl_0 + \frac{Qa}{l_1}(l_1 - a)\right]$ $M_{AB} = -\left(\frac{M_{BC}}{2} + \frac{3}{16}Pl_0\right)$ $M_{CB} = \frac{M_{BC}}{2} + \frac{3Qa}{2l_1}(l_1 - a)$

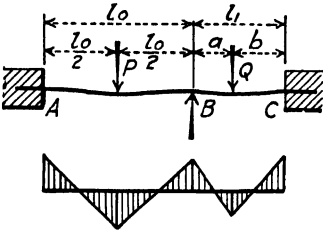
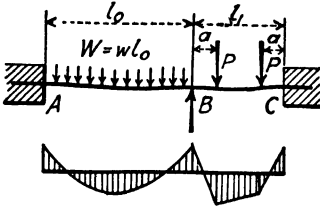
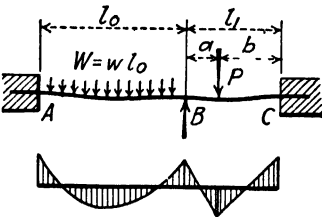
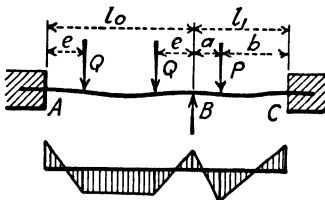
TABLE 18 (*Continued*)

Type of loading	Moments
4. Single load at center of one span, single load at any point of other	$M_{BC} = -\frac{1}{n+1}\left[\frac{1}{8}\, n\, Pl_0 + \frac{Qab^2}{l_1^2}\right]$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{3}{16}Pl_0$ $M_{CB} = \frac{M_{BC}}{2} + \frac{Qab}{2l_1^2}(l_1 + a)$
5. Uniform load on one span, two equal symmetrically spaced loads on other.	$M_{BC} = -\frac{1}{n+1}\left[\frac{n}{12}Wl_0 + \frac{Pa}{l_1}(l_1 - a)\right]$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{1}{8}Wl_0$ $M_{CB} = \frac{M_{BC}}{2} + \frac{3Pa}{2l_1}(l_1 - a)$
6. Uniform load on one span, single load at any point on other.	$M_{BC} = -\frac{1}{n+1}\left[\frac{n}{12}Wl_0 + \frac{Pab^2}{l_1^2}\right]$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{1}{8}Wl_0$ $M_{CB} = \frac{M_{BC}}{2} + \frac{Pab}{2l_1^2}(l_1 + a)$
7. Two equal symmetrically spaced loads on one span, single load at any point on other.	$M_{BC} = -\frac{1}{n+1}\left[\frac{nQe}{l_0}(l_0 - e) + \frac{Pab^2}{l_1^2}\right]$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{3Qe}{2l_0}(l_0 - e)$ $M_{CB} = \frac{M_{BC}}{2} + \frac{Pab}{2l_1^2}(l_1 + a)$

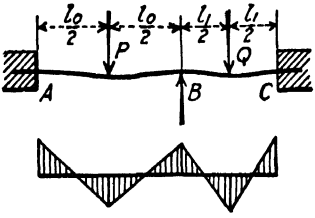
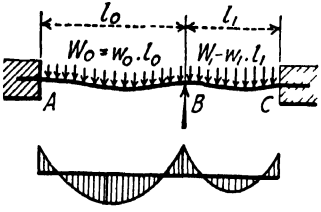
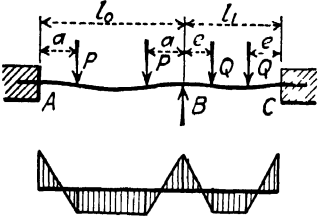
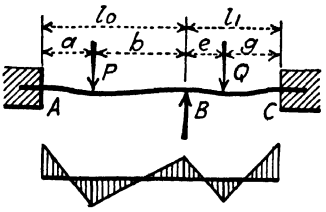
TABLE 18 (*Continued*)

Type of loading	Moments
  8. Single load at center of each span.	$M_{BC} = \frac{1}{n+1}\left(-n\frac{1}{8}Pl_0 - \frac{1}{8}Ql_1\right)$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{3}{16}Pl_0$ $M_{CB} = \frac{M_{BC}}{2} + \frac{3}{16}Ql_1$
  9. Uniform load on both spans	 $M_{BC} = \frac{1}{n+1}\left(-n\frac{1}{12}W_0l_0 - \frac{1}{12}W_1l_1\right)$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{1}{8}W_0l_0$ $M_{CB} = +\frac{M_{BC}}{2} + \frac{1}{8}W_1l_1$
  10. Two equal symmetrically spaced loads on each span.	$M_{BC} = \frac{1}{n+1}\left[-n\frac{Pa}{l_0}(l_0 - a) - \frac{Qe}{l_1}(l_1 - e)\right]$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{3Pa}{2l_0}(l_0 - a)$ $M_{CB} = \frac{M_{BC}}{2} + \frac{3Qe}{2l_1}(l_1 - e)$
  11. Single load at any point on each span.	$M_{BC} = \frac{1}{n+1}\left(-n\frac{Pba^2}{l_0^2} - \frac{Qeg^2}{l_1^2}\right)$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{Pab}{2l_0^2}(l_0 + b)$ $M_{CB} = \frac{M_{BC}}{2} + \frac{Qeg}{2l_1^2}(l_1 + e)$

TABLE 19

CONTINUOUS GIRDER. TWO SPANS

Supports all on same level
Girder hinged at the ends
Lengths of two spans different
Moment of inertia for AB is I_0
Moment of inertia for BC is I_1

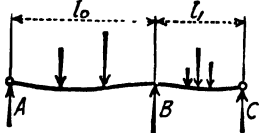
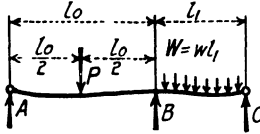
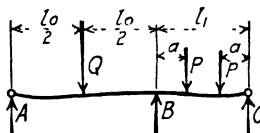
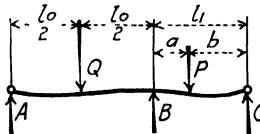
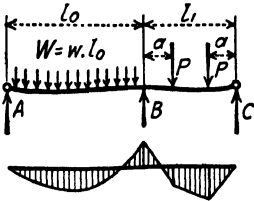
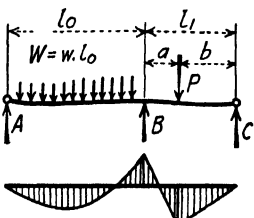
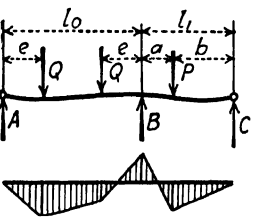
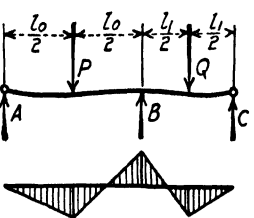
$$K = \frac{I_1}{l_1} \qquad n = \frac{I_1 l_0}{l_1 I_0}$$

Type of loading	Moments
1. Any system of loads.	$M_{BC} = -\frac{1}{n+1}(nH_{BA} + H_{BC})$ Values of H for various loads are given in Tables 2 and 3.
 2. Single load at center of one span, uniform load on other.	$M_{BC} = -\frac{1}{n+1}\left(\frac{3}{16}nPl_0 + \frac{1}{8}Wl_1\right)$
 3. Single load at center of one span, two equal symmetrically spaced loads on other.	$M_{BC} = -\frac{1}{n+1}\left[\frac{3}{16}nQl_0 + \frac{3}{2}\frac{Pa}{l_1}(l_1 - a)\right]$
 4. Single load at center of one span, single load at any point on other.	$M_{BC} = -\frac{1}{n+1}\left[\frac{3}{16}nQl_0 + \frac{Pab}{2l_1^2}(l_1 + b)\right]$

TABLE 19 (*Continued*)

Type of loading	Moments

5. Uniform load on one span, two equal symmetrically spaced loads on other.

$$M_{BC} = -\frac{1}{n+1}\left[\frac{1}{8}nWl_0 + \frac{3}{2}\frac{Pa}{l_1}(l_1 - a)\right]$$

6. Uniform load on one span, single load at any point on other.

$$M_{BC} = -\frac{1}{n+1}\left[\frac{1}{8}nWl_0 + \frac{Pab}{2l_1^2}(l_1 + b)\right]$$

7. Two equal symmetrically spaced loads on one span, single load at any point on other.

$$M_{BC} = -\frac{1}{n+1}\left[\frac{3}{2}n\frac{Qe}{l_0}(l_0 - e) + \frac{Pab}{2l_1^2}(l_1 + b)\right]$$

8. Single load at center of each span.

$$M_{BC} = -\frac{1}{n+1}\left(n\frac{3}{16}Pl_0 + \frac{3}{16}Ql_1\right)$$

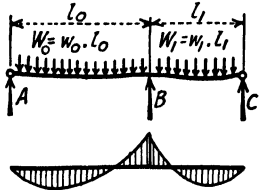
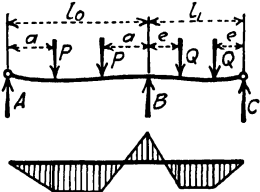
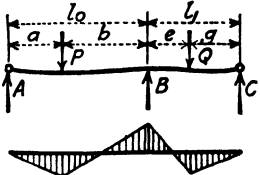
TABLE 19 (*Continued*)

Type of loading	Moments
9. Uniform load on both spans.	$M_{BC} = -\frac{1}{n+1}\left(n\frac{1}{8}W_0l_0 + \frac{1}{8}W_1l_1\right)$
10. Two equal symmetrically spaced loads on each span.	$M_{BC} = -\frac{1}{n+1}\left[n\frac{3Pa}{2l_0}(l_0 - a) + \frac{3Qe}{2l_1}(l_1 - e)\right]$
11. Single load at any point on each span.	$M_{BC} = -\frac{1}{n+1}\left[n\frac{Pab}{2l_0^2}(l_0 + a) + \frac{Qeg}{2l_1^2}(l_1 + g)\right]$

TABLE 20

CONTINUOUS GIRDER. TWO SPANS

Supports all on same level
Ends of girder restrained but not fixed
Lengths of two spans different
Moment of inertia of AB is I_0
Moment of inertia of BC is I_1

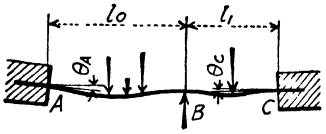
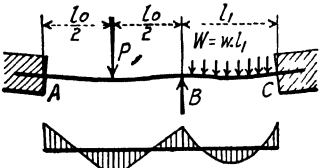
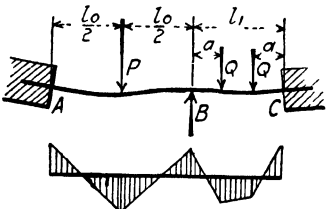
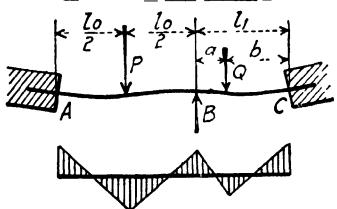
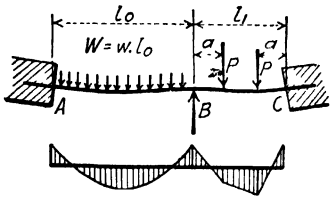
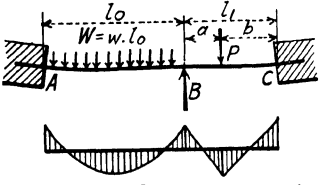
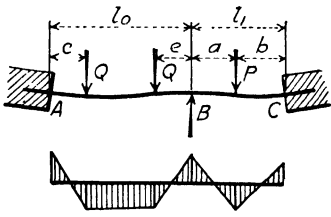
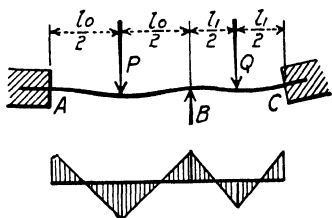
$$K = \frac{I_1}{l_1} \qquad n = \frac{I_1 l_0}{l_1 I_0}$$

Type of loading	Moments
  1 Any system of loads	$M_{BC} = -\frac{1}{n+1}[nC_{BA} + C_{BC} + 2EK(\theta_A - \theta_C)]$ $M_{AB} = -\frac{M_{BC}}{2} - H_{AB} + \frac{3EK\theta_A}{n}$ $M_{CB} = \frac{M_{BC}}{2} + H_{CB} + 3EK\theta_C$ Values of H and C for various loads are given in Tables 2 and 3.
  2. Single load at center of one span, uniform load on other.	$M_{BC} = -\frac{1}{n+1}\left[n\frac{1}{8}Pl_0 + \frac{1}{12}Wl_1 + 2EK(\theta_A - \theta_C)\right]$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{3}{16}Pl_0 + \frac{3EK\theta_A}{n}$ $M_{CB} = \frac{M_{BC}}{2} + \frac{1}{8}Wl_1 + 3EK\theta_C$
  3. Single load at center of one span, two equal symmetrically spaced loads on other.	$M_{BC} = -\frac{1}{n+1}\left[n\frac{1}{8}Pl_0 + \frac{Qa}{l_1}(l_1 - a) + 2EK(\theta_A - \theta_C)\right]$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{3}{16}Pl_0 + \frac{3EK\theta_A}{n}$ $M_{CB} = \frac{M_{BC}}{2} + \frac{3Qa}{2l_1}(l_1 - a) + 3EK\theta_C$
  4. Single load at center of one span, single load at any point on other.	$M_{BC} = -\frac{1}{n+1}\left[n\frac{1}{8}Pl_0 + \frac{Qab^2}{l_1^2} + 2EK(\theta_A - \theta_C)\right]$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{3}{16}Pl_0 + \frac{3EK\theta_A}{n}$ $M_{CB} = \frac{M_{BC}}{2} + \frac{Qab}{2l_1^2}(l_1 + a) + 3EK\theta_C$

TABLE 20 (*Continued*)

Type of loading	Moments

5. Uniform load on one span, two equal symmetrically spaced loads on other.

$$M_{BC} = -\frac{1}{n+1}\left[n\frac{1}{12}Wl_0 + \frac{Pa}{l_1}(l_1 - a) + 2EK(\theta_A - \theta_C)\right]$$

$$M_{AB} = -\frac{M_{BC}}{2} - \frac{1}{8}Wl_0 + \frac{3EK\theta_A}{n}$$

$$M_{CB} = \frac{M_{BC}}{2} + \frac{3Pa}{2l_1}(l_1 - a) + 3EK\theta_C$$

6. Uniform load on one span, single concentrated load at any point on other.

$$M_{BC} = -\frac{1}{n+1}\left[n\frac{1}{12}Wl_0 + \frac{Pab^2}{l_1^2} + 2EK(\theta_A - \theta_C)\right]$$

$$M_{AB} = -\frac{M_{BC}}{2} - \frac{1}{8}Wl_0 + \frac{3EK\theta_A}{n}$$

$$M_{CB} = \frac{M_{BC}}{2} + \frac{Pab}{2l_1^2}(l_1 + a) + 3EK\theta_C$$

7. Two equal symmetrically spaced loads on one span, single concentrated load at any point on other.

$$M_{BC} = -\frac{1}{n+1}\left[n\frac{Qe}{l_0}(l_0 - e) + \frac{Pab^2}{l_1^2} + 2EK(\theta_A - \theta_C)\right]$$

$$M_{AB} = -\frac{M_{BC}}{2} - \frac{3Qe}{2l_0}(l_0 - e) + \frac{3EK\theta_A}{n}$$

$$M_{CB} = \frac{M_{BC}}{2} + \frac{Pab}{2l_1^2}(l_1 + a) + 3EK\theta_C$$

8. Single load at center of each span.

$$M_{BC} = -\frac{1}{n+1}\left[n\frac{1}{8}Pl_0 + \frac{1}{8}Ql_1 + 2EK(\theta_A - \theta_C)\right]$$

$$M_{AB} = -\frac{M_{BC}}{2} - \frac{3}{16}Pl_0 + \frac{3EK\theta_A}{n}$$

$$M_{CB} = \frac{M_{BC}}{2} + \frac{3}{16}Ql_1 + 3EK\theta_C$$

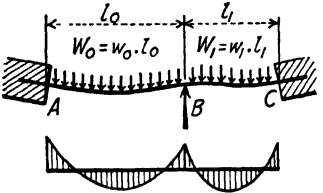
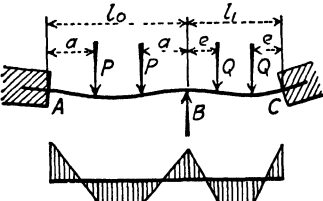
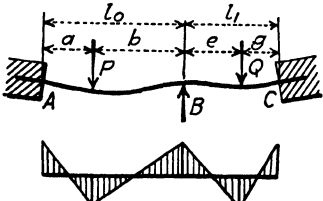
TABLE 20 (*Continued*)

Type of loading	Moments
9. Uniform load on both spans.	$M_{BC} = -\frac{1}{n+1}\left[n\frac{1}{12}W_0 l_0 + \frac{1}{12}W_1 l_1 + 2EK(\theta_A - \theta_C)\right]$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{1}{8}W_0 l_0 + \frac{3EK\theta_A}{n}$ $M_{CB} = \frac{M_{BC}}{2} + \frac{1}{8}W_1 l_1 + 3EK\theta_C$
10. Two equal symmetrically spaced loads on each span.	$M_{BC} = -\frac{1}{n+1}\left[n\frac{Pa}{l_0}(l_0 - a) + \frac{Qe}{l_1}(l_1 - e) + 2EK(\theta_A - \theta_C)\right]$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{3Pa}{2l_0}(l_0 - a) + \frac{3EK\theta_A}{n}$ $M_{CB} = \frac{M_{BC}}{2} + \frac{3Qe}{2l_1}(l_1 - e) + 3EK\theta_C$
11. Single load at any point on each span.	$M_{BC} = -\frac{1}{n+1}\left[n\frac{Pba^2}{l_0^2} + \frac{Qeg^2}{l_1^2} + 2EK(\theta_A - \theta_C)\right]$ $M_{AB} = -\frac{M_{BC}}{2} - \frac{Pab}{2l_0^2}(l_0 + b) + \frac{3EK\theta_A}{n}$ $M_{CB} = \frac{M_{BC}}{2} + \frac{Qeg}{2l_1^2}(l_1 + e) + 3EK\theta_C$

Table 21

Continuous Girder, Two Spans

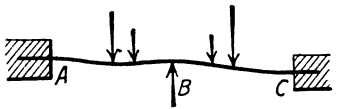
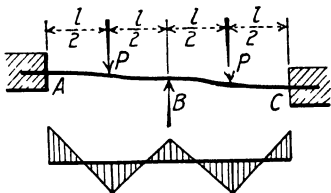
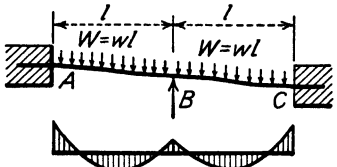
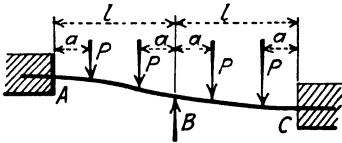
Supports all on different levels
Girder fixed at the ends
Two spans identical
Load symmetrical about center support

Type of loading	Moments
1. Any system of loads symmetrical about center support	$M_{BC} = \left[\frac{3EK}{l}(2d_B - d_A - d_C) - C_{BA}\right]$ $M_{AB} = -\frac{3EK(d_B - d_A)}{l} - \frac{M_{BC}}{2} - H_{AB}$ $M_{CB} = -\frac{3EK(d_C - d_B)}{l} + \frac{M_{BC}}{2} + H_{CB}$ Values of H and C for various loads are given in Tables 2 and 3.
2. Equal loads at center of each span.	$M_{BC} = \frac{3EK}{l}(2d_B - d_A - d_C) - \frac{1}{8}Pl$ $M_{AB} = -\frac{3EK(d_B - d_A)}{l} - \frac{M_{BC}}{2} - \frac{3}{16}Pl$ $M_{CB} = -\frac{3EK(d_C - d_B)}{l} + \frac{M_{BC}}{2} + \frac{3}{16}Pl$
3. Uniform load.	$M_{BC} = \frac{3EK}{l}(2d_B - d_A - d_C) - \frac{1}{12}Wl$ $M_{AB} = -\frac{3EK(d_B - d_A)}{l} - \frac{M_{BC}}{2} - \frac{1}{8}Wl$ $M_{CB} = -\frac{3EK(d_C - d_B)}{l} + \frac{M_{BC}}{2} + \frac{1}{8}Wl$
4. Two equal loads symmetrically spaced on each span.	$M_{BC} = \frac{3EK}{l}(2d_B - d_A - d_C) - \frac{Pa}{l}(l - a)$ $M_{AB} = -\frac{3EK(d_B - d_A)}{l} - \frac{M_{BC}}{2} - \frac{3Pa}{2l}(l - a)$ $M_{CB} = -\frac{3EK(d_C - d_B)}{l} + \frac{M_{BC}}{2} + \frac{3Pa}{2l}(l - a)$

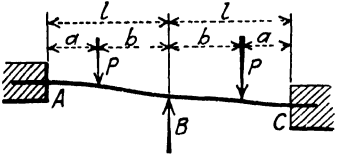
TABLE 21 (*Continued*)

Types of loading	Moments
5. Single load on each span	$M_{BC} = \frac{3EK}{l}(2d_B - d_A - d_C) - \frac{Pba^2}{l^2}$ $M_{AB} = -\frac{3EK(d_B - d_A)}{l} - \frac{M_{BC}}{2} - \frac{Pab}{2l^2}(l + b)$ $M_{CB} = -\frac{3EK(d_C - d_B)}{l} + \frac{M_{BC}}{2} + \frac{Pab}{2l^2}(l + b)$

TABLE 22

CONTINUOUS GIRDER. TWO SPANS

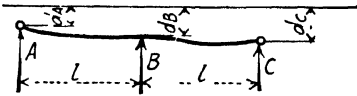
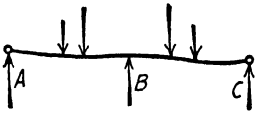
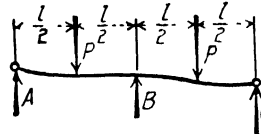
Supports all on different levels
Girder hinged at the ends
Two spans identical
Load symmetrical about center support

Type of loading	Moments
1. Any system of loads symmetrical about center support.	$M_{BC} = \frac{3EK}{2l}(2d_B - d_A - d_C) - H_{BA}$ Values of H for various loads are given in Tables 2 and 3
2. Equal loads at center of each span.	$M_{BC} = \frac{3EK}{2l}(2d_B - d_A - d_C) - \frac{3}{16}Pl$

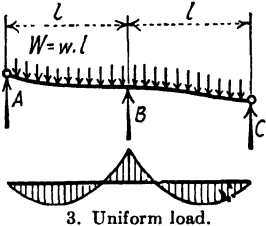
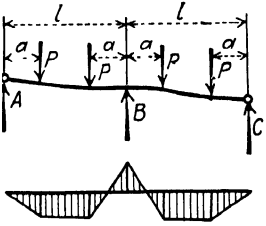
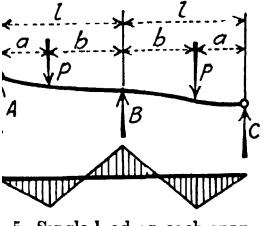
Table 22 (*Continued*)

Type of loading	Moments
 3. Uniform load.	$M_{BC} = \frac{3EK}{2l}(2d_B - d_A - d_C) - \frac{1}{8}Wl$
 4 Two equal loads symmetrically spaced on each span.	$M_{BC} = \frac{3EK}{2l}(2d_B - d_A - d_C) - \frac{3Pa}{2l}(l - a)$
 5. Single load on each span.	$M_{BC} = \frac{3EK}{2l}(2d_B - d_A - d_C) - \frac{Pab}{2l^2}(l + a)$

TABLE 23

CONTINUOUS GIRDER. TWO SPANS

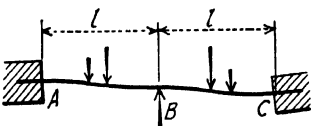
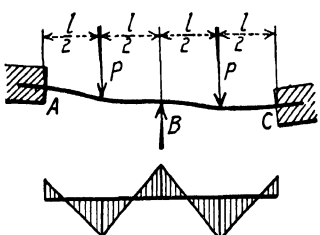
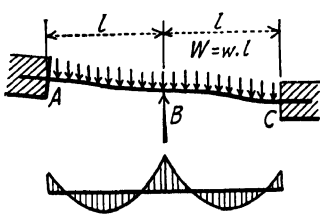
Supports on different levels
Ends of girder restrained but not fixed
Two spans identical
Load symmetrical about center support

Type of loading	Moments
1. Any system of loads symmetrical about center support.	$M_{BC} = \frac{3EK}{l}(2d_B - d_A - d_C) - C_{BA} - EK(\theta_A - \theta_C)$ $M_{AB} = 3EK\left[\theta_A - \frac{(d_B - d_A)}{l}\right] - \frac{M_{BC}}{2} - H_{AB}$ $M_{CB} = 3EK\left[\theta_C - \frac{(d_C - d_B)}{l}\right] + \frac{M_{BC}}{2} + H_{CB}$ Values of H and C for various loads are given in Tables 2 and 3.
2. Equal loads at center of each span.	$M_{BC} = \frac{3EK}{l}(2d_B - d_A - d_C) - \frac{1}{8}Pl - EK(\theta_A - \theta_C)$ $M_{AB} = 3EK\left[\theta_A - \frac{(d_B - d_A)}{l}\right] - \frac{M_{BC}}{2} - \frac{3}{16}Pl$ $M_{CB} = 3EK\left[\theta_C - \frac{(d_C - d_B)}{l}\right] + \frac{M_{BC}}{2} + \frac{3}{16}Pl$
3. Uniform load.	$M_{BC} = \frac{3EK}{l}(2d_B - d_A - d_C) - \frac{1}{12}Wl - EK(\theta_A - \theta_C)$ $M_{AB} = 3EK\left[\theta_A - \frac{(d_B - d_A)}{l}\right] - \frac{M_{BC}}{2} - \frac{1}{8}Wl$ $M_{CB} = 3EK\left[\theta_C - \frac{(d_C - d_B)}{l}\right] + \frac{M_{BC}}{2} + \frac{1}{8}Wl$

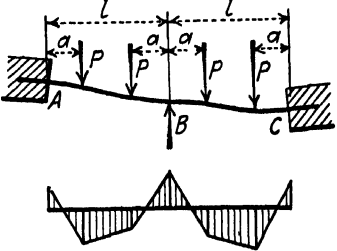
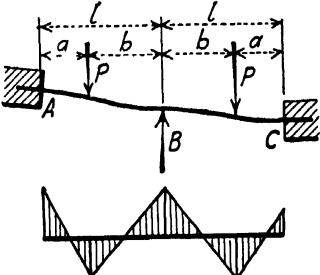
TABLE 23 (*Continued*)

Type of loading	Moments
4. Two equal loads symmetrically spaced on each span.	$M_{BC} = \frac{3EK}{l}(2d_B - d_A - d_C) - \frac{Pa}{l}(l - a) - EK(\theta_A - \theta_C)$ $M_{AB} = 3EK\left[\theta_A - \frac{(d_B - d_A)}{l}\right] - \frac{M_{BC}}{2} - \frac{3Pa}{2l}(l - a)$ $M_{CB} = 3EK\left[\theta_C - \frac{(d_C - d_B)}{l}\right] + \frac{M_{BC}}{2} + \frac{3Pa}{2l}(l - a)$
5. Single load on each span.	$M_{BC} = \frac{3EK}{l}(2d_B - d_A - d_C) - \frac{Pba^2}{l^2} - EK(\theta_A - \theta_C)$ $M_{AB} = 3EK\left[\theta_A - \frac{(d_B - d_A)}{l}\right] - \frac{M_{BC}}{2} - \frac{Pab}{2l^2}(l + b)$ $M_{CB} = 3EK\left[\theta_C - \frac{(d_C - d_B)}{l}\right] + \frac{M_{BC}}{2} + \frac{Pab}{2l^2}(l + b)$

TABLE 24

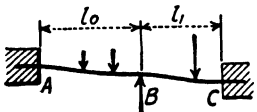
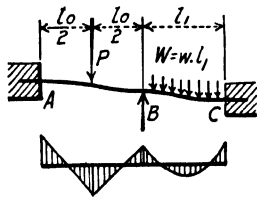
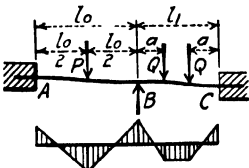
CONTINUOUS GIRDER. TWO SPANS

Supports on different levels
Girder fixed at the ends
Lengths of two spans different

Moment of inertia for AB is I_0

Moment of inertia for BC is I_1

$$K = \frac{I_1}{l_1}$$

$$n = \frac{I_1 l_0}{l_1 I_0}$$

Type of loading — Moments

1. Any system of loads.

$$M_{BC} = \frac{1}{n+1}\left[\frac{6EK}{l_0 l_1}\left\{l_1(d_B - d_A) - l_0(d_C - d_B)\right\} - nC_{BA} - C_{BC}\right]$$

$$M_{AB} = \frac{3EK}{nl_0}\left[d_A - d_B\right] - \frac{M_{BC}}{2} - H_{AB}$$

$$M_{CB} = \frac{3EK}{l_1}\left[d_B - d_C\right] + \frac{M_{BC}}{2} + H_{CB}$$

Values of H and C for various loads are given in Tables 2 and 3.

2. Single load at center of one span, uniform load on other.

$$M_{BC} = \frac{1}{n+1}\left[\frac{6EK}{l_0 l_1}\left\{l_1(d_B - d_A) - l_0(d_C - d_B)\right\} - n\frac{1}{8}Pl_0 - \frac{1}{12}Wl_1\right]$$

$$M_{AB} = \frac{3EK}{nl_0}\left[d_A - d_B\right] - \frac{M_{BC}}{2} - \frac{3}{16}Pl_0$$

$$M_{CB} = \frac{3EK}{l_1}\left[d_B - d_C\right] + \frac{M_{BC}}{2} + \frac{1}{8}Wl_1$$

3. Single load at center of one span, two equal symmetrically spaced loads on the other.

$$M_{BC} = \frac{1}{n+1}\left[\frac{6EK}{l_0 l_1}\left\{l_1(d_B - d_A) - l_0(d_C - d_B)\right\} - \frac{n}{8}Pl_0 - \frac{Qa}{l_1}(l_1 - a)\right]$$

$$M_{AB} = \frac{3EK}{nl_0}\left[d_A - d_B\right] - \frac{M_{BC}}{2} - \frac{3}{16}Pl_0$$

$$M_{CB} = \frac{3EK}{l_1}\left[d_B - d_C\right] + \frac{M_{BC}}{2} + \frac{3Qa}{2l_1}(l_1 - a)$$

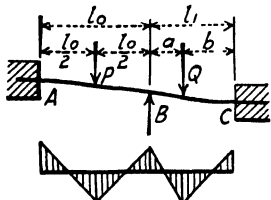
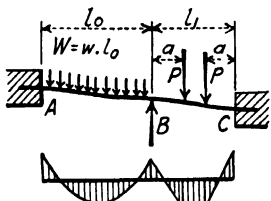
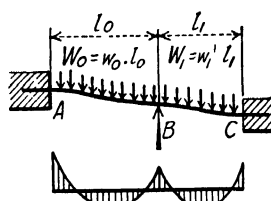
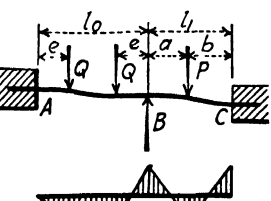
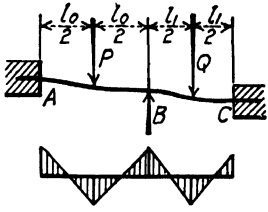
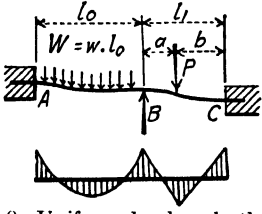
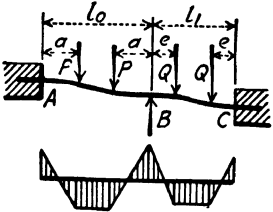
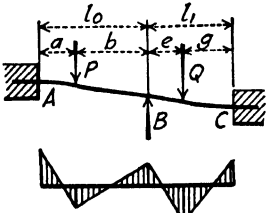
TABLE 24 (*Continued*)

Type of loading	Moments
4. Single load at center of one span, single load at any point of other.	$M_{BC} = \frac{1}{n+1}\left[\frac{6EK}{l_0 l_1}\left\{l_1(d_B - d_A) - l_0(d_C - d_B)\right\} - \frac{n}{8}Pl_0 - \frac{Qab^2}{l_1^2}\right]$ $M_{AB} = \frac{3EK}{nl_0}\left[d_A - d_B\right] - \frac{M_{BC}}{2} - \frac{3}{16}Pl_0$ $M_{CB} = \frac{3EK}{l_1}\left[d_B - d_C\right] + \frac{M_{BC}}{2} + \frac{Qab}{2l_1^2}(l_1 + a)$
5. Uniform load on one span, two equal symmetrically spaced loads on the other.	$M_{BC} = \frac{1}{n+1}\left[\frac{6EK}{l_0 l_1}\left\{l_1(d_B - d_A) - l_0(d_C - d_B)\right\} - \frac{n}{12}Wl_0 - \frac{Pa}{l_1}(l_1 - a)\right]$ $M_{AB} = \frac{3EK}{nl_0}\left[d_A - d_B\right] - \frac{M_{BC}}{2} - \frac{1}{8}Wl_0$ $M_{CB} = \frac{3EK}{l_1}\left[d_B - d_C\right] + \frac{M_{BC}}{2} + \frac{3Pa}{2l_1}(l_1 - a)$
6. Uniform load on one span, single load at any point on other.	$M_{BC} = \frac{1}{n+1}\left[\frac{6EK}{l_0 l_1}\left\{l_1(d_B - d_A) - l_0(d_C - d_B)\right\} - \frac{n}{12}Wl_0 - \frac{Pab^2}{l_1^2}\right]$ $M_{AB} = \frac{3EK}{nl_0}\left[d_A - d_B\right] - \frac{M_{BC}}{2} - \frac{1}{8}Wl_0$ $M_{CB} = \frac{3EK}{l_1}\left[d_B - d_C\right] + \frac{M_{BC}}{2} + \frac{Pab}{2l_1^2}(l_1 + a)$
7. Two equal symmetrically spaced loads on one span, single load at any point on other.	$M_{BC} = \frac{1}{n+1}\left[\frac{6EK}{l_0 l_1}\left\{l_1(d_B - d_A) - l_0(d_C - d_B)\right\} - \frac{Qen}{l_0}\left(l_0 - e\right) - \frac{Pab^2}{l_1^2}\right]$ $M_{AB} = \frac{3EK}{nl_0}\left[d_A - d_B\right] - \frac{M_{BC}}{2} - \frac{3Qe}{2l_0}(l_0 - e)$ $M_{CB} = \frac{3EK}{l_1}\left[d_B - d_C\right] + \frac{M_{BC}}{2} + \frac{Pab}{2l_1^2}(l_1 + a)$

TABLE 24 (*Continued*)

Type of loading — Moments

8. Single load at center of each span.

$$M_{BC} = \frac{1}{n+1}\left[\frac{6EK}{l_0 l_1}\left\{l_1(d_B - d_A) - l_0(d_C - d_B)\right\} - \frac{n}{8}Pl_0 - \frac{1}{8}Ql_1\right]$$

$$M_{AB} = \frac{3EK}{nl_0}\left[d_A - d_B\right] - \frac{M_{BC}}{2} - \frac{3}{16}Pl_0$$

$$M_{CC} = \frac{3EK}{l_1}\left[d_B - d_C\right] + \frac{M_{BC}}{2} + \frac{3}{16}Ql_1$$

9. Uniform load on both spans.

$$M_{BC} = \frac{1}{n+1}\left[\frac{6EK}{l_0 l_1}\left\{l_1(d_B - d_A) - l_0(d_C - d_B)\right\} - \frac{n}{12}W_0 l_0 - \frac{1}{12}W_1 l_1\right]$$

$$M_{AB} = \frac{3EK}{nl_0}\left[d_A - d_B\right] - \frac{M_{BC}}{2} - \frac{1}{8}W_0 l_0$$

$$M_{CB} = \frac{3EK}{l_1}\left[d_B - d_C\right] + \frac{M_{BC}}{2} + \frac{1}{8}W_1 l_1$$

10. Two equal symmetrically spaced loads on each span.

$$M_{BC} = \frac{1}{n+1}\left[\frac{6EK}{l_0 l_1}\left\{l_1(d_B - d_A) - l_0(d_C - d_B)\right\} - \frac{nPa}{l_0}\left(l_0 - a\right) - \frac{Qe}{l_1}(l_1 - e)\right]$$

$$M_{AB} = \frac{3EK}{nl_0}\left[d_A - d_B\right] - \frac{M_{BC}}{2} - \frac{3Pa}{2l_0}(l_0 - a)$$

$$M_{CB} = \frac{3EK}{l_1}\left[d_B - d_C\right] + \frac{M_{BC}}{2} + \frac{3Qe}{2l_1}(l_1 - e)$$

11. Single load at any point on each span.

$$M_{BC} = \frac{1}{n+1}\left[\frac{6EK}{l_0 l_1}\left\{l_1(d_B - d_A) - l_0(d_C - d_B)\right\} - \frac{nPba^2}{l_0} - \frac{Qeg^2}{l_1^2}\right]$$

$$M_{AB} = \frac{3EK}{nl_0}\left[d_A - d_B\right] - \frac{M_{BC}}{2} - \frac{Pab}{2l_0^2}(l_0 + b)$$

$$M_{CB} = \frac{3EK}{l_1}\left[d_B - d_C\right] + \frac{M_{BC}}{2} + \frac{Qeg}{2l_1^2}(l_1 + e)$$

TABLE 25

CONTINUOUS GIRDER. TWO SPANS

Supports on different levels
Girder hinged at the ends
Lengths of two spans different
Moment of inertia for AB is I_0
Moment of inertia for BC is I_1

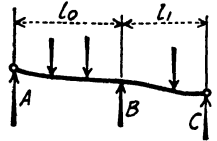
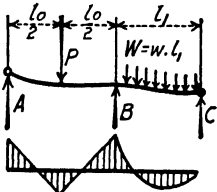
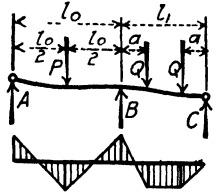
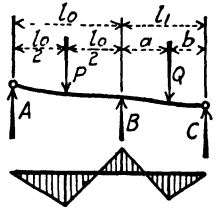
$$K = \frac{I_1}{l_1}$$
$$n = \frac{Kl_0}{I_0}$$

Type of loading	Moments
1. Any system of loads.	$M_{BC} = \frac{6EK}{2l_0l_1(n+1)}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{1}{n+1}[nH_{BA} + H_{BC}]$ Values of C and H for various loads are given in Tables 2 and 3.
2. Single load at center of one span, uniform load on other	$M_{BC} = \frac{6EK}{2l_0l_1(n+1)}\left[l_1(d_B - d_A)] - l_0(d_C - d_B)\right] - \frac{1}{n+1}\left[\frac{3nPl_0}{16} + \frac{1}{8}Wl_1\right]$
3. Single load at center of one span, two equal symmetrically spaced loads on other	$M_{BC} = \frac{6EK}{2l_0l_1(n+1)}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{1}{n+1}\left[\frac{3nPl_0}{16} + \frac{3Qa}{2l_1}(l_1 - a)\right]$
4. Single load at center of one span, single load at any point of other.	$M_{BC} = \frac{6EK}{2l_0l_1(n+1)}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{1}{n+1}\left[\frac{3nPl_0}{16} + \frac{Qab}{2l_1^2}(l_1 + b)\right]$

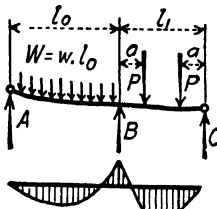
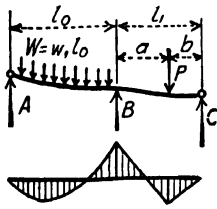
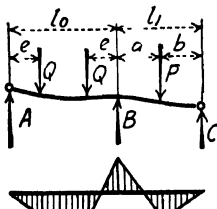
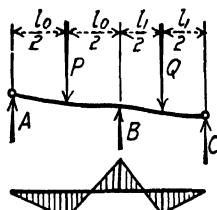
TABLE 25 (*Continued*)

Type of loading	Moments
5. Uniform load on one span, two equal symmetrically spaced loads on other.	$M_{BC} = \frac{6EK}{2l_0l_1(n+1)}\Big[l_1(d_B - d_A) - l_0(d_C - d_B) - \frac{1}{n+1}\left[\frac{nWl_0}{8} + \frac{3Pa}{2l_1}(l_1 - a)\right]$
6. Uniform load on one span, single load at any point on other.	$M_{BC} = \frac{6EK}{2l_0l_1(n+1)}\Big[l_1(d_B - d_A) - l_0(d_C - d_B) - \frac{1}{n+1}\left[\frac{nWl_0}{8} + \frac{Pab}{2l_1^2}(l_1 + b)\right]$
7. Two equal symmetrically spaced loads on one span, single load at any point on other.	$M_{BC} = \frac{6EK}{2l_0l_1(n+1)}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{1}{n+1}\left[\frac{3Qen}{2l_0}(l_0 - e) + \frac{Pab}{2l_1^2}(l_1 + b)\right]$
8. Single load at center of each span.	$M_{BC} = \frac{6EK}{2l_0l_1(n+1)}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{1}{n+1}\left[\frac{3Pl_0n}{16} + \frac{3Ql_1}{16}\right]$

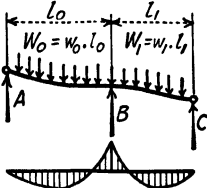
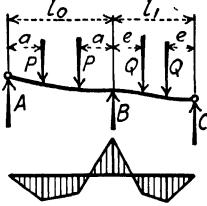
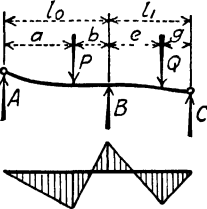
TABLE 25 (*Continued*)

Type of loading	Moments
9. Uniform load on both spans.	$M_{BC} = \frac{6EK}{2l_0l_1(n+1)}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{1}{n+1}\left[\frac{nW_0l_0}{8} + \frac{W_1l_1}{8}\right]$
10. Two equal symmetrically spaced loads in each span.	$M_{BC} = \frac{6EK}{2l_0l_1(n+1)}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{1}{n+1}\left[\frac{3Pan}{2l_0}(l_0 - a) + \frac{3Qe}{2l_1}(l_1 - e)\right]$
11. Single load at any point on each span.	$M_{BC} = \frac{6EK}{2l_0l_1(n+1)}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{1}{n+1}\left[\frac{nPab}{2l_0^2}(l_0 + a) + \frac{Qeg}{2l_1^2}(l_1 + g)\right]$

TABLE 26

CONTINUOUS GIRDER. TWO SPANS

Supports on different levels
Ends of girder restrained but not fixed
Lengths of two spans different
Moment of inertia for AB is I_0
Moment of inertia for BC is I_1

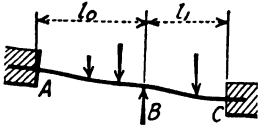
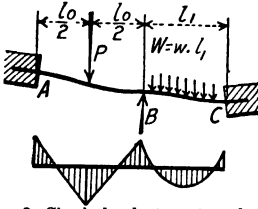
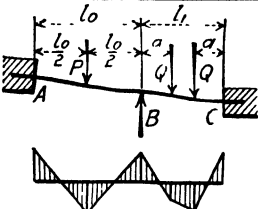
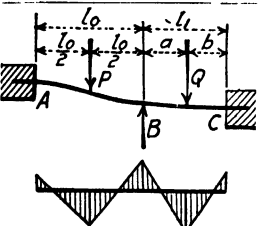
$$K = \frac{I_1}{l_1}$$
$$n = \frac{Kl_0}{I_0}$$

Type of loading	Moments
1. Any system of loads.	$M_{BC} = \frac{1}{n+1}\left\{\frac{6EK}{l_0 l_1}[l_1(d_B - d_A) - l_0(d_C - d_B)] - nC_{BA} - C_{BC} - 2EK(\theta_A - \theta_C)\right\}$ $M_{AB} = \frac{3EK}{n}\left[\theta_A - \frac{d_B - d_A}{l_0}\right] - \frac{M_{BC}}{2} - H_{AB}$ $M_{CB} = 3EK\left[\theta_C - \frac{d_C - d_B}{l_1}\right] + \frac{M_{BC}}{2} + H_{CB}$ *Values of C and H for various loads are given in Tables 2 and 3.
2. Single load at center of one span, uniform load on other.	$M_{BC} = \frac{1}{n+1}\left\{\frac{6EK}{l_0 l_1}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{n}{8}Pl_0 - \frac{1}{12}Wl_1 - 2EK(\theta_A - \theta_C)\right\}$ $M_{AB} = \frac{3EK}{n}\left[\theta_A - \frac{d_B - d_A}{l_0}\right] - \frac{M_{BC}}{2} - \frac{3}{16}Pl_0$ $M_{CB} = 3EK\left[\theta_C - \frac{d_C - d_B}{l_1}\right] + \frac{M_{BC}}{2} + \frac{1}{8}Wl_1$
3. Single load at center of one span, two equal symmetrically spaced loads on other.	$M_{BC} = \frac{1}{n+1}\left\{\frac{6EK}{l_0 l_1}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{n}{8}Pl_0 - \frac{Qa}{l_1}(l_1 - a) - 2EK(\theta_A - \theta_C)\right\}$ $M_{AB} = \frac{3EK}{n}\left[\theta_A - \frac{d_B - d_A}{l_0}\right] - \frac{M_{BC}}{2} - \frac{3}{16}Pl_0$ $M_{CB} = 3EK\left[\theta_C - \frac{d_C - d_B}{l_1}\right] + \frac{M_{BC}}{2} + \frac{3Qa}{2l_1}(l_1 - a)$
4. Single load at center of one span, single load at any point on other.	$M_{BC} = \frac{1}{n+1}\left\{\frac{6EK}{l_0 l_1}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{n}{8}Pl_0 - \frac{Qab^2}{l_1^2} - 2EK(\theta_A - \theta_C)\right\}$ $M_{AB} = \frac{3EK}{n}\left[\theta_A - \frac{d_B - d_A}{l_0}\right] - \frac{M_{BC}}{2} - \frac{3}{16}Pl_0$ $M_{CB} = 3EK\left[\theta_C - \frac{d_C - d_B}{l_1}\right] + \frac{M_{BC}}{2} + \frac{Qab}{2l_1^2}(l_1 + a)$

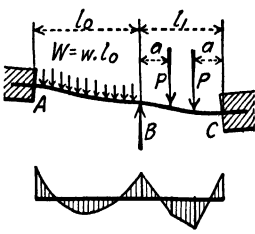
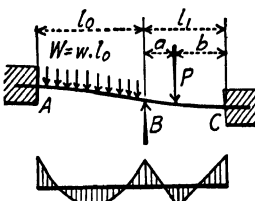
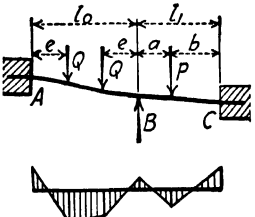
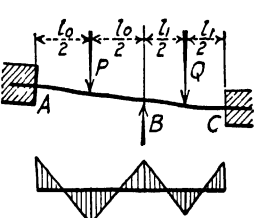
TABLE 26 (*Continued*)

Type of loading	Moments
 5. Uniform load on one span, two equal symmetrically spaced loads on other.	$M_{BC} = \frac{1}{n+1}\left\{\frac{6EK}{l_0l_1}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{n}{12}Wl_0 - \frac{Pa}{l_1}(l_1 - a)\right\} - 2EK(\theta_A - \theta_C)\Big\}$ $M_{AB} = \frac{3EK}{n}\left[\theta_A - \frac{d_B - d_A}{l_0}\right] - \frac{M_{BC}}{2} - \frac{1}{8}Wl_0$ $M_{CB} = 3EK\left[\theta_C - \frac{d_C - d_B}{l_1}\right] + \frac{M_{BC}}{2} + \frac{3Pa}{2l_1}(l_1 - a)$
 6. Uniform load on one span, single load at any point on other.	$M_{BC} = \frac{1}{n+1}\left\{\frac{6EK}{l_0l_1}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{n}{12}Wl_0 - \frac{Pab^2}{l_1^2} - 2EK(\theta_A - \theta_C)\right\}$ $M_{AB} = \frac{3EK}{n}\left[\theta_A - \frac{d_B - d_A}{l_0}\right] - \frac{M_{BC}}{2} - \frac{1}{8}Wl_0$ $M_{CB} = 3EK\left[\theta_C - \frac{d_C - d_B}{l_1}\right] + \frac{M_{BC}}{2} + \frac{Pab}{2l_1^2}(l_1 + a)$
 7. Two equal symmetrically spaced loads on one span, single load at any point on other.	 $M_{BC} = \frac{1}{n+1}\left\{\frac{6EK}{l_0l_1}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{nQe}{l_0}(l_0 - e) - \frac{Pab^2}{l_1^2} - 2EK(\theta_A - \theta_C)\right\}$ $M_{AB} = \frac{3EK}{n}\left[\theta_A - \frac{d_B - d_A}{l_0}\right] - \frac{M_{BC}}{2} - \frac{3Qe}{2l_0}(l_0 - e)$ $M_{CB} = 3EK\left[\theta_C - \frac{d_C - d_B}{l_1}\right] + \frac{M_{BC}}{2} + \frac{Pab}{2l_1^2}(l_1 + a)$
 8. Single load at center of each span.	$M_{BC} = \frac{1}{n+1}\left\{\frac{6EK}{l_0l_1}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{n}{8}Pl_0 - \frac{1}{8}Ql_1 - 2EK(\theta_A - \theta_C)\right\}$ $M_{AB} = \frac{3EK}{n}\left[\theta_A - \frac{d_B - d_A}{l_0}\right] - \frac{M_{BC}}{2} - \frac{3}{16}Pl_0$ $M_{CB} = 3EK\left[\theta_C - \frac{d_C - d_B}{l_1}\right] + \frac{M_{BC}}{2} + \frac{3}{16}Ql_1$

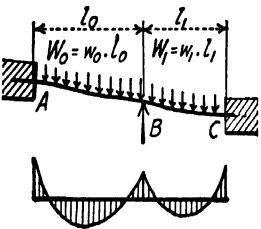
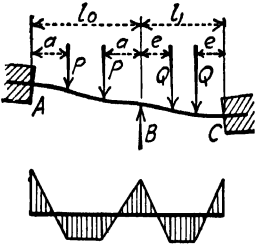
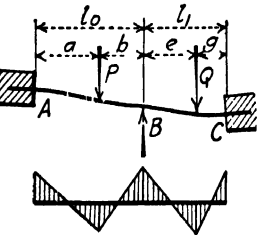
TABLE 26 (*Continued*)

Type of loading	Moments
9. Uniform load on each span.	$M_{BC} = \frac{1}{n+1}\left\{\frac{6LK}{l_0 l_1}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{n}{12}W_0 l_0 - \frac{1}{12}W_1 l_1 - 2EK(\theta_A - \theta_C)\right\}$ $M_{AB} = \frac{3EK}{n}\left[\theta_A - \frac{d_B - d_A}{l_0}\right] - \frac{M_{BC}}{2} - \frac{1}{8}W_0 l_0$ $M_{CB} = 3EK\left[\theta_C - \frac{d_C - d_B}{l_1}\right] + \frac{M_{BC}}{2} + \frac{1}{8}W_1 l_1$
10. Two equal symmetrically spaced loads on each span	$M_{BC} = \frac{1}{n+1}\left\{\frac{6EK}{l_0 l_1}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{Pan}{l_0}(l_0 - a) - \frac{Qe}{l_1}(l_1 - e) - 2EK(\theta_A - \theta_C)\right\}$ $M_{AB} = \frac{3EK}{n}\left[\theta_A - \frac{d_B - d_A}{l_0}\right] - \frac{M_{BC}}{2} - \frac{3Pa}{2l_0}(l_0 - a)$ $M_{CB} = 3EK\left[\theta_C - \frac{d_C - d_B}{l_1}\right] + \frac{M_{BC}}{2} + \frac{3Qe}{2l_1}(l_1 - e)$
11. Single load at any point on each span	$M_{BC} = \frac{1}{n+1}\left\{\frac{6EK}{l_0 l_1}\left[l_1(d_B - d_A) - l_0(d_C - d_B)\right] - \frac{nPba^2}{l_0^2} - \frac{Qeg^2}{l_1^2} - 2EK(\theta_A - \theta_C)\right\}$ $M_{AB} = \frac{3EK}{n}\left[\theta_A - \frac{d_B - d_A}{l_0}\right] - \frac{M_{BC}}{2} - \frac{Pab}{2l_0^2}(l_0 + b)$ $M_{CB} = 3EK\left[\theta_C - \frac{d_C - d_B}{l_1}\right] + \frac{M_{BC}}{2} + \frac{Qeg}{2l_1^2}(l_1 + e)$

TABLE 27

CONTINUOUS GIRDER. THREE SPANS

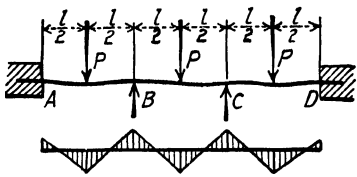
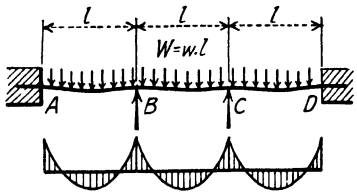
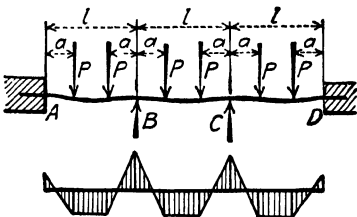
Supports all on same level
Girder fixed at the ends
All spans identical
Loads on all spans identical

Type of loading	Moments
1. Equal loads at center of each span.	$M_{DC} = -M_{AB} = -M_{BC} = -M_{CD} = C_{AB} = \frac{1}{8}Pl$
2. Uniform load on all spans.	$M_{DC} = -M_{AB} = -M_{BC} = -M_{CD} = \frac{1}{12}Wl$
3. Two equal symmetrically spaced loads on each span.	$M_{DC} = -M_{AB} = -M_{BC} = -M_{CD} = \frac{Pa}{l}(l - a)$

TABLE 28

CONTINUOUS GIRDER. THREE SPANS

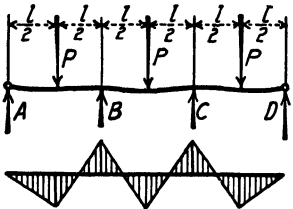
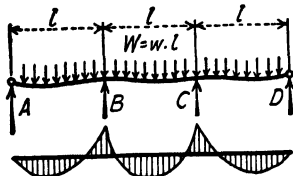
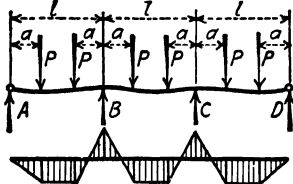
Supports all on same level
Girder hinged at the ends
All spans identical
Loads on all spans identical

Type of loading	Moments
1. Equal loaas at center of each span.	$M_{BC} = M_{CD} = -\frac{4}{5}H_{AB} = -\frac{3}{20}Pl$
 2. Uniform load on all spans.	$M_{BC} = M_{CD} = -\frac{1}{10}Wl$
 3. Two equal symmetrically spaced loads on each span.	$M_{BC} = M_{CD} = -\frac{6}{5}\frac{Pa}{l}(l-a)$

Table 29

Continuous Girder. Three Spans

Supports all on same level
Ends of girder restrained but not fixed
All spans identical
Loads on all spans identical

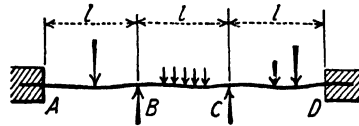
$$\theta_A = -\theta_D$$

Type of loading	Moments
1. Equal loads at center of each span.	$M_{BC} = M_{CD} = -C_{AB} - \frac{2}{3}EK\theta_A$ $M_{DC} = -M_{AB} = C_{AB} - \frac{10}{3}EK\theta_A$ $M_{BC} = M_{CD} = -\frac{1}{8}Pl - \frac{2}{3}EK\theta_A$ $M_{DC} = -M_{AB} = \frac{1}{8}Pl - \frac{10}{3}EK\theta_A$
2. Uniform load on all spans.	$M_{BC} = M_{CD} = -\frac{1}{12}Wl - \frac{2}{3}EK\theta_A$ $M_{DC} = -M_{AB} = \frac{1}{12}Wl - \frac{10}{3}EK\theta_A$
3. Two equal symmetrically spaced loads on each span.	$M_{BC} = M_{CD} = -\frac{Pa}{l}(l - a) - \frac{2}{3}EK\theta_A$ $M_{DC} = -M_{AB} = \frac{Pa}{l}(l - a) - \frac{10}{3}EK\theta_A$

TABLE 30

CONTINUOUS GIRDER. THREE SPANS

Supports all on same level
Girder fixed at the ends
All spans identical except for loads

GENERAL CASE

Any system of loading.

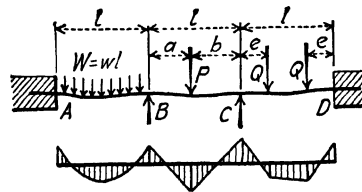
$$M_{BC} = -\frac{2}{45}[7(2H_{BA} - H_{AB} + 2H_{BC}) + 2(H_{DC} - 2H_{CD} - 2H_{CB})]$$

$$M_{CD} = \frac{2}{45}[7(H_{DC} - 2H_{CD} - 2H_{CB}) + 2(2H_{BA} + 2H_{BC} - H_{AB})]$$

$$M_{AB} = -\frac{1}{2}M_{BC} - H_{AB}$$

$$M_{DC} = \frac{1}{2}M_{CD} + H_{DC}$$

Values of H for various loads are given in Tables 2 and 3.

SPECIAL CASE

Uniform load on first span, single load at any point on second, and two equal symmetrically spaced loads on third span.

Use equations above. Take values of H from Tables 2 and 3 as follows:

For AB

$$H_{AB} = H_{BA} = \frac{1}{8}Wl$$

For BC

$$H_{BC} = \frac{Pab}{2l^2}(l + b),\ H_{CB} = \frac{Pab}{2l^2}(l + a)$$

For CD

$$H_{CD} = H_{DC} = \frac{3Qe}{2l}(l - e)$$

Substituting these values of H in the equations above, gives

$$M_{BC} = -\frac{2}{45}\left[\frac{7}{8}Wl - \frac{3Qe}{l}(l - e) + \frac{3Pab}{l^2}(l + 3b)\right]$$

$$M_{CD} = \frac{2}{45}\left[\frac{1}{4}Wl - \frac{21Qe}{2l}(l - e) - \frac{3Pab}{l^2}(l + 3a)\right]$$

$$M_{AB} = -\frac{1}{2}M_{BC} - \frac{1}{8}Wl$$

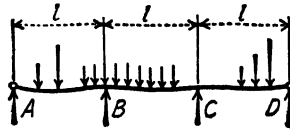
$$M_{DC} = \frac{1}{2}M_{CD} + \frac{3Qe}{2l}(l - e)$$

By substituting the proper values of H in the general equations of this table equations can be obtained in the manner illustrated, for any system of special loads. Or, in the case of numerical problems, numerical values of H may be substituted in the equations for the general case.

TABLE 31

CONTINUOUS GIRDER. THREE SPANS

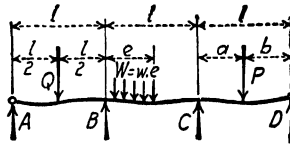
Supports all on same level
Girder hinged at the ends
All spans identical except for load

GENERAL CASE

Any system of loading.

$$M_{BC} = -\frac{2}{15}[4(H_{BA} + H_{BC}) - (H_{CB} + H_{CD})]$$

$$M_{CD} = -\frac{2}{15}[4(H_{CB} + H_{CD}) - (H_{BA} + H_{BC})]$$

Values of H for various loads are given in Tables 2 and 3.

SPECIAL CASE

Single load at center of first span, uniform load on part of second span, and single load at any point on third span.

Use equations above. Take values of H from Tables 2 and 3 as follows:

For AB

$$H_{BA} = \frac{3}{16}Ql$$

For BC

$$H_{BC} = \frac{We}{8l^2}(2l - e)^2, \quad H_{CB} = \frac{We}{8l^2}(2l^2 - e^2)$$

For CD

$$H_{CD} = \frac{Pab}{2l^2}(l + b)$$

Substituting these values of H in the equations above, gives

$$M_{BC} = -\frac{2}{15}\left[\frac{3}{4}Ql + \frac{We}{8l^2}\left\{(7l - 5e)(2l - e) + le\right\} - \frac{Pab}{2l^2}(l + b)\right]$$

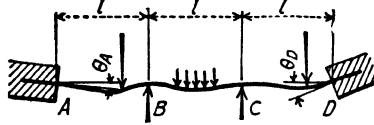
$$M_{CD} = -\frac{2}{15}\left[\frac{We}{8l^2}(4l^2 - 5e^2 + 4le) + \frac{2Pab}{l^2}(l + b) - \frac{3}{16}Ql\right]$$

By substituting the proper values of H in the general equations of this table, equations can be obtained in the manner illustrated, for any system of special loads. Or, in the case of numerical problems, numerical values of H may be substituted in the equations for the general case.

TABLE 32

CONTINUOUS GIRDER. THREE SPANS

Supports all on same level
Ends of girder restrained but not fixed
All spans identical except for load

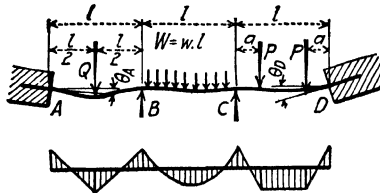
GENERAL CASE

Any system of loading.

$$M_{BC} = -\frac{2}{45}[7(2H_{BA} - H_{AB} + 2H_{BC}) + 2(H_{DC} - 2H_{CD} - 2H_{CB}) + 3EK(7\theta_A + 2\theta_D)]$$

$$M_{CD} = \frac{2}{45}[7(H_{DC} - 2H_{CD} - 2H_{CB}) + 2(-H_{AB} + 2H_{BA} + 2H_{BC}) + 3EK(7\theta_D + 2\theta_A),$$

$$M_{AB} = -\frac{1}{2}M_{BC} - H_{AB} + 3EK\theta_A$$

$$M_{DC} = \frac{1}{2}M_{CD} + H_{DC} + 3EK\theta_D$$

Values of H for various loads are given in Tables 2 and 3.

SPECIAL CASE

Single load at center of first span, uniform load on second span, two equal symmetrically spaced loads on third span.

Use equations above. Take values of H from Table 3 as follows:

For AB

$$H_{AB} = H_{BA} = \frac{3}{16}Ql$$

For BC

$$H_{BC} = H_{CB} = \frac{1}{8}Wl$$

For CD

$$H_{CD} = H_{DC} = \frac{3}{2}\frac{Pa}{l}(l - a)$$

Substituting these values of H in the equations above, gives

$$M_{BC} = -\frac{2}{45}\left[\frac{21}{16}Ql + \frac{5}{4}Wl - \frac{3Pa}{l}(l - a) + 3EK(7\theta_A + 2\theta_D)\right]$$

$$M_{CD} = \frac{2}{45}\left[-\frac{21}{2}\frac{Pa}{l}(l - a) - \frac{5}{4}Wl + \frac{3}{8}Ql + 3EK(7\theta_D + 2\theta_A)\right]$$

$$M_{AB} = -\frac{1}{2}M_{BC} - \frac{3}{16}Ql + 3EK\theta_A$$

$$M_{DC} = \frac{1}{2}M_{CD} + \frac{3}{2}\frac{Pa}{l}(l - a) + 3EK\theta_D$$

By substituting the proper values of H in the general equations of this table, equations can be obtained in the manner illustrated, for any system of special loads. Or, in the case of numerical problems, numerical values of H may be substituted in the equations for the general case.

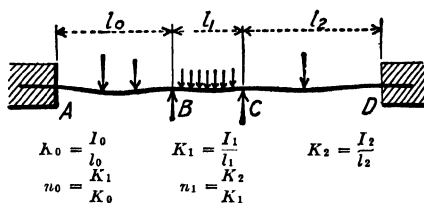
TABLE 33

CONTINUOUS GIRDER. THREE SPANS

All supports on same level
Girder fixed at the ends
Lengths of all spans different

GENERAL CASE

$$K_0 = \frac{I_0}{l_0} \qquad K_1 = \frac{I_1}{l_1} \qquad K_2 = \frac{I_2}{l_2}$$

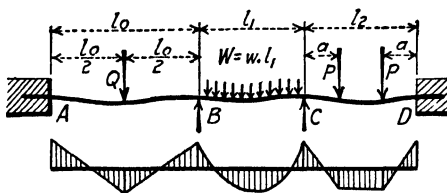
$$n_0 = \frac{K_1}{K_0} \qquad n_1 = \frac{K_2}{K_1}$$

Any system of loads.

$$M_{BC} = -\frac{2}{3}\,\frac{(4n_1 + 3)(2n_0H_{BA} - n_0H_{AB} + 2H_{BC}) + 2(H_{DC} - 2H_{CD} - 2n_1H_{CB})}{4(n_0 + 1)(n_1 + 1) - n_0}$$

$$M_{CD} = \frac{2}{3}\,\frac{(3n_0 + 4)(H_{DC} - 2H_{CD} - 2n_1H_{CB}) + 2n_1(2n_0H_{BA} - n_0H_{AB} + 2H_{BC})}{4(n_0 + 1)(n_1 + 1) - n_0}$$

$$M_{AB} = -\frac{M_{BC}}{2} - H_{AB}$$

$$M_{DC} = \frac{M_{CD}}{2} + H_{DC}$$

Values of H for various loads are given in Tables 2 and 3.

SPECIAL CASE

l_0, $\frac{l_0}{2}$, $\frac{l_0}{2}$, Q, l_1, $W = w.l_1$, l_2, a, P, a, P, A, B, C, D

Single load at center of first span, uniform load on second span, two equal symmetrically spaced loads on third span

Use equations above Take values of H from Table 3 as follows:

For AB

$$H_{AB} = H_{BA} = \frac{3}{16}Ql_0$$

For BC

$$H_{BC} = H_{CB} = \frac{1}{8}Wl_1$$

For CD

$$H_{CD} = H_{DC} = \frac{3}{2}\frac{Pa}{l_2}(l_2 - a)$$

Substituting these values of H in the equations above, gives

$$M_{BC} = -\frac{2}{3}\,\frac{(4n_1 + 3)(\frac{3}{16}n_0Ql_0 + \frac{1}{4}Wl_1) - \frac{3Pa}{l_2}(l_2 - a) - \frac{1}{2}n_1Wl_1}{4(n_0 + 1)(n_1 + 1) - n_0}$$

$$M_{CD} = \frac{2}{3}\,\frac{(3n_0 + 4)\left\{-\frac{3}{2}\frac{Pa}{l_2}(l_2 - a) - \frac{1}{4}n_1Wl_1\right\} + \frac{3}{8}n_1n_0Ql_0 + \frac{1}{2}n_1Wl_1}{4(n_0 + 1)(n_1 + 1) - n_0}$$

$$M_{AB} = -\frac{M_{BC}}{2} - \frac{3}{16}Ql_0$$

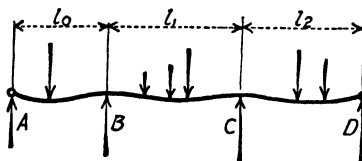
$$M_{DC} = \frac{M_{CD}}{2} + \frac{3}{2}\frac{Pa}{l_2}(l_2 - a)$$

By substituting the proper values of H in the general equations of this table, equations can be obtained in the manner illustrated, for any system of special loads. Or, in the case of numerical problems, numerical values of H may be substituted in the equations for the general case.

TABLE 34

CONTINUOUS GIRDER. THREE SPANS

Supports all on same level
Girder hinged at the ends
Lengths of all spans different

GENERAL CASE

$$K_0 = \frac{I_0}{l_0} \qquad K_1 = \frac{I_1}{l_1} \qquad K_2 = \frac{I_2}{l_2}$$

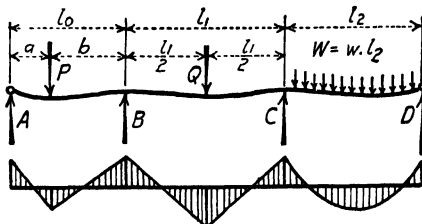
$$n_0 = \frac{K_1}{K_0} \qquad n_1 = \frac{K_2}{K_1}$$

Any system of loading

$$M_{BC} = -\frac{4(n_1 + 1)(n_0 H_{BA} + H_{BC}) - 2(n_1 H_{CB} + H_{CD})}{4(n_0 + 1)(n_1 + 1) - n_1}$$

$$M_{CD} = -\frac{4(n_0 + 1)(n_1 H_{CB} + H_{CD}) - 2n_1(n_0 H_{BA} + H_{BC})}{4(n_0 + 1)(n_1 + 1) - n_1}$$

Values of H for various loads are given in Tables 2 and 3.

SPECIAL CASE

Single load at any point on first span, single load at center of second span, uniform load on third span
Use equations above. Take values of H from Tables 2 and 3 as follows:

For AB

$$H_{BA} = \frac{Pab}{2l_0^2}(l_0 + a)$$

For BC

$$H_{BC} = H_{CB} = \frac{3}{16} Q l_1$$

For CD

$$H_{CD} = \frac{1}{8} W l_2$$

Substituting these values of H in the equations above, gives

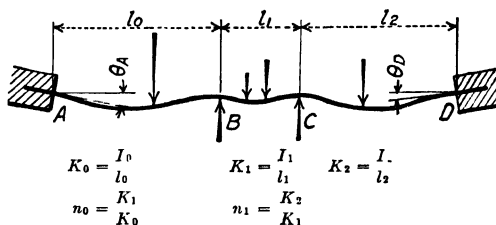
$$M_{BC} = -\frac{\frac{3}{8}Ql_1(n_1 + 2) + 4n_0(n_1 + 1)\frac{Pab}{2l_0^2}(l_0 + a) - \frac{1}{4} Wl_2}{4(n_0 + 1)(n_1 + 1) - n_1}$$

$$M_{CD} = -\frac{\frac{3}{8} n_1 Q l_1(2n_0 + 1) + \frac{1}{2}(n_0 + 1)(Wl_2) - n_0 n_1 \frac{Pab}{l_0^2}(l_0 + a)}{4(n_0 + 1)(n_1 + 1) - n_1}$$

By substituting the proper values of H in the general equations of this table, equations can be obtained in the manner illustrated, for any system of special loads. Or, in the case of numerical problems, numerical values of H may be substituted in the equations for the general case.

TABLE 35

CONTINUOUS GIRDER. THREE SPANS

Supports all on same level
Ends of girder restrained but not fixed
Lengths of all spans different

GENERAL CASE

$$K_0 = \frac{I_0}{l_0} \qquad K_1 = \frac{I_1}{l_1} \qquad K_2 = \frac{I_2}{l_2}$$

$$n_0 = \frac{K_1}{K_0} \qquad n_1 = \frac{K_2}{K_1}$$

Any system of loading.

$$M_{BC} = -\frac{2}{3}\,\frac{(4n_1+3)(2n_0H_{BA} - n_0H_{AB} + 2H_{BC}) + 2(H_{DC} - 2H_{CD} - 2n_1H_{CB}) + 3EK_1[(4n_1+3)\theta_A + 2n_1\theta_D]}{[4(n_0+1)(n_1+1) - n_0]}$$

$$M_{CD} = \frac{2}{3}\,\frac{(3n_0+4)(H_{DC} - 2H_{CD} - 2n_1H_{CB}) + 2n_1(2n_0H_{BA} - n_0H_{AB} + 2H_{BC}) + 3EK_2[(3n_0+4)\theta_D + 2\theta_A]}{4(n_0+1)(n_1+1) - n_0}$$

$$M_{AB} = -\frac{M_{BC}}{2} - H_{AB} + \frac{3EK_1\theta_A}{n_0}$$

$$M_{DC} = \frac{M_{CD}}{2} + H_{DC} + 3EK_2\theta_D$$

Values of H for various loads are given in Tables 2 and 3

SPECIAL CASE

Single load at center of first span, uniform load on second span, two equal symmetrically spaced loads on third span.

Use equations above. Take values of H from Tables 2 and 3 as follows:

For AB

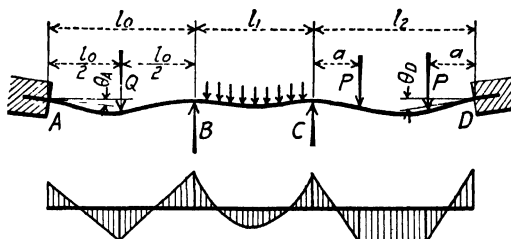
$$H_{AB} = H_{BA} = \frac{3}{16}Ql_0$$

For BC

$$H_{BC} = H_{CB} = \frac{1}{8}Wl_1$$

For CD

$$H_{CD} = H_{DC} = \frac{3}{2}\frac{Pa}{l_2}(l_2 - a)$$

Substituting these values of H in the equations above, gives

$$M_{BC} = -\frac{2}{3}\,\frac{(4n_1+3)\left(\frac{3}{16}n_0Ql_0 + \frac{1}{4}Wl_1\right) - \frac{3Pa}{l_2}\left(l_2 - a\right) - \frac{1}{2}n_1Wl_1 + 3EK_1[(4n_1+3)\theta_A + 2n_1\theta_D]}{4(n_0+1)(n_1+1) - n_0}$$

$$M_{CD} = \frac{2}{3}\,\frac{(3n_0+4)\left\{-\frac{3}{2}\frac{Pa}{l_2}\left(l_2 - a\right) - \frac{1}{4}n_1Wl_1\right\} + \frac{3}{8}n_1n_0Ql_0 + \frac{1}{2}n_1Wl_1 + 3EK_2[(3n_0+4)\theta_D + 2\theta_A]}{4(n_0+1)(n_1+1) - n_0}$$

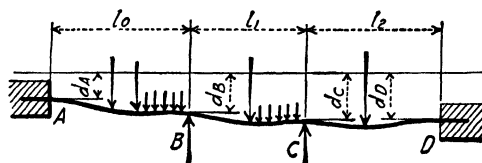
$$M_{AB} = -\frac{M_{BC}}{2} - \frac{3}{16}Ql_0 + \frac{3EK_1\theta_A}{n_0}$$

$$M_{DC} = \frac{M_{CD}}{2} + \frac{3}{2}\frac{Pa}{l_2}(l_2 - a) + 3EK_2\theta_D$$

By substituting the proper values of H in the general equations of this table, equations can be obtained in the manner illustrated, for any system of special loads. Or, in the case of numerical problems, numerical values of H may be substituted in the equations for the general case.

TABLE 36

CONTINUOUS GIRDER. THREE SPANS

Supports on different levels
Lengths of all spans different
Girder fixed at ends

$$K_0 = \frac{I_0}{l_0} \qquad K_1 = \frac{I_1}{l_1} \qquad K_2 = \frac{I_2}{l_2}$$

$$n_0 = \frac{K_1}{K_0} \qquad n_1 = \frac{K_2}{K_1}$$

Any system of loads.

$$M_{BC} = \frac{6EK_1\left[(4n_1 + 3)l_1l_2(d_B - d_A) - 2(2n_1 + 1)l_0l_2(d_C - d_B) + 2n_1l_0l_1(d_D - d_C)\right]}{l_0l_1l_2\left[4(n_0 + 1)(n_1 + 1) - n_0\right]}$$

$$- \frac{2\left\{(4n_1 + 3)(2n_0H_{BA} - n_0H_{AB} + 2H_{BC}) + 2(H_{DC} - 2H_{CD} - 2n_1H_{CB})\right\}}{3\left[4(n_0 + 1)(n_1 + 1) - n_0\right]}$$

$$M_{CD} = \frac{6EK_2\left[2(n_0 + 2)l_0l_2(d_C - d_B) - (3n_0 + 4)l_0l_1(d_D - d_C) - 2l_1l_2(d_B - d_A)\right]}{l_0l_1l_2\left[4(n_0 + 1)(n_1 + 1) - n_0\right]}$$

$$+ \frac{2\left\{(3n_0 + 4)(H_{DC} - 2H_{CD} - 2n_1H_{CB}) + 2n_1(2n_0H_{BA} - n_0H_{AB} + 2H_{BC})\right\}}{3\left[4(n_0 + 1)(n_1 + 1) - n_0\right]}$$

$$M_{AB} = -\frac{M_{BC}}{2} - H_{AB} - \frac{3EK_1}{n_0}\frac{(d_B - d_A)}{l_0}$$

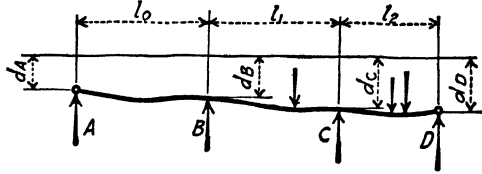
$$M_{DC} = \frac{M_{CD}}{2} + H_{DC} - 3EK_2\frac{(d_D - d_C)}{l_2}$$

Values of H for various loads are given in Tables 2 and 3.

For any particular problem, substitute the numerical values of the constants in the above equations. The method of making this substitution is illustrated by the numerical problem presented on p. 455.

TABLE 37

CONTINUOUS GIRDER. THREE SPANS

Supports on different levels
Lengths of all spans different
Girder hinged at ends

$$K_0 = \frac{I_0}{l_0} \qquad K_1 = \frac{I_1}{l_1} \qquad K_2 = \frac{I_2}{l_2}$$

$$n_0 = \frac{K_1}{K_0} \qquad n_1 = \frac{K_2}{K_1}$$

Any system of loads.

$$M_{BC} = \frac{6EK_1\left[2(n_1+1)l_1l_2(d_B-d_A) - (3n_1+2)l_0l_2(d_C-d_B) + n_1l_0l_1(d_D-d_C)\right]}{l_0l_1l_2\left[4(n_0+1)(n_1+1)-n_1\right]} - \left[\frac{4(n_1+1)(n_0H_{BA}+H_{BC}) - 2(n_1H_{CB}+H_{CD})}{4(n_0+1)(n_1+1)-n_1}\right]$$

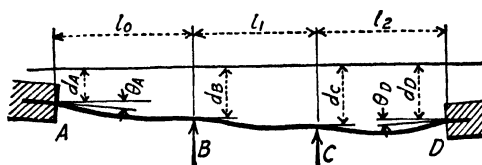
$$M_{CD} = \frac{6EK_2\left[(2n_0+3)l_0l_2(d_C-d_B) - 2(n_0+1)l_0l_1(d_D-d_C) - l_1l_2(d_B-d_A)\right]}{l_0l_1l_2\left[4(n_0+1)(n_1+1)-n_1\right]} - \left[\frac{4(n_0+1)(n_1H_{CB}+H_{CD}) - 2n_1(n_0H_{BA}+H_{BC})}{4(n_0+1)(n_1+1)-n_1}\right]$$

Values of H for various loads are given in Tables 2 and 3.

For any particular loading, substitute the numerical values of the constants in the above equations. The method of determining these constants and of making the substitutions is illustrated by the numerical problem on p. 455.

TABLE 38

CONTINUOUS GIRDERS. THREE SPANS

Supports on different levels
Lengths of all spans different
Ends of girder restrained but not fixed

$$K_0 = \frac{I_0}{l_0} \qquad K_1 = \frac{I_1}{l_1} \qquad K_2 = \frac{I_2}{l_2}$$

$$n_0 = \frac{K_1}{K_0} \qquad n_1 = \frac{K_2}{K_1}$$

Any system of loads.

$$M_{BC} = \frac{6EK_1\left[(4n_1+3)l_1l_2(d_B-d_A) - 2(2n_1+1)l_0l_2(d_C-d_B) + 2n_1l_0l_1(d_D-d_C)\right]}{l_0l_1l_2\left[4(n_0+1)(n_1+1)-n_0\right]}$$

$$-\frac{2\left\{(4n_1+3)(2n_0H_{BA}-n_0H_{AB}+2H_{BC})+2(H_{DC}-2H_{CD}-2n_1H_{CB})+3EK_1\left[(4n_1+3)\theta_A+2n_1\theta_D\right]\right\}}{3\left[4(n_0+1)(n_1+1)-n_0\right]}$$

$$M_{CD} = \frac{6EK_2\left[2(n_0+2)l_0l_2(d_C-d_B) - (3n_0+4)l_0l_1(d_D-d_C) - 2l_1l_2(d_B-d_A)\right]}{l_0l_1l_2\left[4(n_0+1)(n_1+1)-n_0\right]}$$

$$+\frac{2\left\{(3n_0+4)(H_{DC}-2H_{CD}-2n_1H_{CB})+2n_1(2n_0H_{BA}-n_0H_{AB}+2H_{BC})+3EK_2\left[(3n_0+4)\theta_D+2\theta_A\right]\right\}}{3\left[4(n_0+1)(n_1+1)-n_0\right]}$$

$$M_{AB} = -\frac{M_{BC}}{2} - H_{AB} + \frac{3EK_1}{n_0}\left[\theta_A - \frac{(d_B-d_A)}{l_0}\right]$$

$$M_{DC} = -\frac{M_{CD}}{2} + H_{DC} + 3EK_2\left[\theta_D - \frac{(d_D-d_C)}{l_2}\right]$$

Values of H for various loads are given in Tables 2 and 3.

For any particular loading, substitute the numerical values of the constants in the above equations. The method of determining these constants and of making the substitutions is illustrated by the numerical problem on p. 455. The moments can be determined only if θ_A and θ_D are known.

TABLE 39

CONTINUOUS GIRDER. ANY NUMBER OF SPANS. EQUATION OF THREE MOMENTS

Supports all on same level
Lengths of all spans different
Any system of loads

$$n = \frac{Kl_0}{I_0} \quad K = \frac{I_1}{l_1}$$

Position of spans	Equation of three moments
1. Two intermediate spans.	$nM_{AB} + 2M_{BC}(n + 1) + M_{CD} = -2\,[nH_{BA} + H_{BC}]$ Values of H for various loads are given in Tables 2 and 3.
2. Two end spans at left. End of girder hinged.	$2M_{BC}(n + 1) + M_{CD} = -2\,[nH_{BA} + H_{BC}]$
3. Two end spans at right. End of girder hinged.	$nM_{AB} + 2M_{BC}(n + 1) = -2\,[nH_{BA} + H_{BC}]$
4. Two end spans at left. End of girder restrained.	$M_{BC}(4 + 3n) + 2M_{CD} = 2nH_{AB} - 4(nH_{BA} + H_{BC}) - 6EK\theta_A$ If the girder is fixed at A, $\theta_A = 0$ and $M_{BC}(4 + 3n) + 2M_{CD} = 2nH_{AB} - 4(nH_{BA} + H_{BC})$
5. Two end spans at right. End of girder restrained.	$2nM_{AB} + M_{BC}(4n + 3) = 2H_{CB} - 4[nH_{BA} + H_{BC}] + 6EK\theta_C$ If the girder is fixed at C, $\theta_C = 0$, and $2nM_{AB} + M_{BC}(4n + 3) = 2H_{CB} - 4\,[nH_{BA} + H_{BC}]$

The method of using these equations is illustrated by the numerical problem given on p. 455.

Table 40

Continuous Girder. Any Number of Spans. Equation of Three Moments

Supports on different levels
Lengths of all spans different
Any system of loads

$$n = K\frac{l_0}{I_0} \quad K = \frac{I_1}{l_1}$$

Position of spans	Equation of three moments
 1. Two intermediate spans.	$nM_{AB} + 2M_{BC}(n+1) + M_{CD} = \frac{6EK}{l_0 l_1}[l_1(d_B - d_A) - l_0(d_C - d_B)] - 2[H_{BC} + nH_{BA}]$ Values of H for various loads are given in Tables 2 and 3.
 2. Two end spans at left. End of girder hinged.	$2M_{BC}(n+1) + M_{CD} = \frac{6EK}{l_0 l_1}[l_1(d_B - d_A) - l_0(d_C - d_B)] - 2[nH_{BA} + H_{BC}]$
 3. Two end spans at right. End of girder hinged.	$nM_{AB} + 2M_{BC}(n+1) = \frac{6EK}{l_0 l_1}[l_1(d_B - d_A) - l_0(d_C - d_B)] - 2[nH_{BA} + H_{BC}]$
 4. Two end spans at left. End of girder restrained.	If M_{AB} is known $2M_{BC}(n+1) + M_{CD} = \frac{6EK}{l_0 l_1}[l_1(d_B - d_A) - l_0(d_C - d_B)] - 2[H_{BC} + nH_{BA}] - nM_{AB}$ If θ_A is known $M_{BC}(4+3n) + 2M_{CD} = \frac{6EK}{l_0 l_1}[3l_1(d_B - d_A) - 2l_0(d_C - d_B)] + 2nH_{AB} - 4[H_{BC} + nH_{BA}] - 6EK\theta_A$ If the end A is fixed, $\theta_A = 0$.
 5. Two end spans at right. End of girder restrained.	If M_{CB} is known $nM_{AB} + 2M_{BC}(n+1) = \frac{6EK}{l_0 l_1}[l_1(d_B - d_A) - l_0(d_C - d_B)] - 2[H_{BC} + nH_{BA}] + M_{CB}$ If θ_C is known $2nM_{AB} + M_{BC}(4n+3) = \frac{6EK}{l_0 l_1}[2l_1(d_B - d_A) - 3l_0(d_C - d_B)] - 4[H_{BC} + nH_{BA}] + 2H_{CB} + 6EK\theta_C$ If the end C is fixed, $\theta_C = 0$.

The method of using these equations is illustrated by the numerical problem given on p. 456.

TABLE 41

TWO-LEGGED RECTANGULAR BENT

Two legs identical
Horizontal load on one leg
Legs hinged at the bases
M_D is moment of external load about D
Moment of inertia of AB is I
Moment of inertia of AD is I_1

$$K = \frac{I}{l}$$

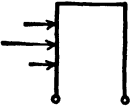
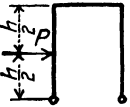
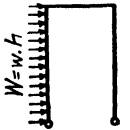
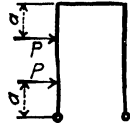
$$n = \frac{Kh}{I_1}$$

Type of loading	Moments
1. Any system of loads.	$M_{BC} = -\frac{1}{2}\left[M_D + \frac{2nH_{AD}}{2n+3}\right]$ $M_{AB} = \frac{1}{2}\left[M_D - \frac{2nH_{AD}}{2n+3}\right]$ Values of H_{AD} for various loads are given in Tables 2 and 3.
2. Single load at mid-height.	$M_{BC} = -\frac{Ph}{4}\left[1 + \frac{3n}{4(2n+3)}\right]$ $M_{AB} = \frac{Ph}{4}\left[1 - \frac{3n}{4(2n+3)}\right]$
3. Uniform load.	$M_{BC} = -\frac{Wh}{4}\left[1 + \frac{n}{4n+6}\right]$ $M_{AB} = \frac{Wh}{4}\left[1 - \frac{n}{4n+6}\right]$
4. Two equal symmetrically spaced loads.	$M_{BC} = -\frac{P}{2}\left[h + \frac{3\,an(h-a)}{h(2n+3)}\right]$ $M_{AB} = \frac{P}{2}\left[h - \frac{3\,an(h-a)}{h\,(2n+3)}\right]$

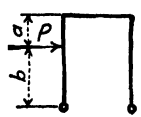
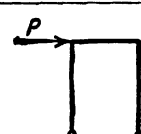
TABLE 41 (*Continued*)

Type of loading	Moments
5. Single load at any point.	$M_{BC} = -\frac{1}{2}Pb\left[1 + \frac{an(h+b)}{h^2(2n+3)}\right]$ $M_{AB} = \frac{1}{2}Pb\left[1 - \frac{an(h+b)}{h^2(2n+3)}\right]$
6. Hydraulic load. (W = Total Load)	$M_{BC} = -\frac{Wa}{6}\left[1 + \frac{n(10h^2 - 3a^2)}{10h^2(2n+3)}\right]$ $M_{AB} = \frac{Wa}{6}\left[1 - \frac{n(10h^2 - 3a^2)}{10h^2(2n+3)}\right]$
7. External couple at top.	$M_{AB} = M\left[1 - \frac{3}{2}\frac{1}{(2n+3)}\right]$ $M_{AD} = -M_{BC} = \frac{3}{2}\left(\frac{M}{2n+3}\right)$
8. Single load at top.	$M_{BC} = -M_{AB} = -\frac{Ph}{2}$

TABLE 42

TWO-LEGGED RECTANGULAR BENT

Two legs identical
Horizontal loads on both legs
Load symmetrical about vertical center line

Legs hinged at the

Moment of inertia of AB is I
Moment of inertia of AD is I_1

$$K = \frac{I}{l}$$

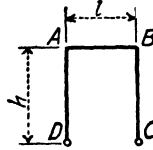
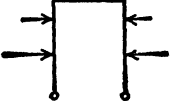
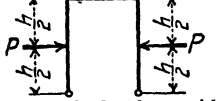
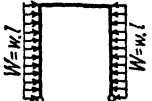
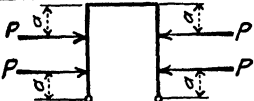
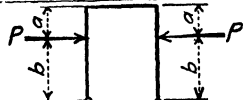
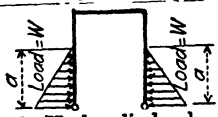
$$n = \frac{Kh}{I_1}$$

Type of loading	Moments
1. Any system of loads.	$M_{AB} = M_{BC} = -\left[\frac{2n}{2n+3}\right] H_{AD}$ Values of H_{AD} for various loads are given in Tables 2 and 3.
2. Single load at mid height.	$M_{AB} = M_{BC} = -\left[\frac{2n}{2n+3}\right]\frac{3}{16} Ph$
3. Uniform load.	$M_{AB} = M_{BC} = -\left[\frac{2n}{2n+3}\right]\frac{1}{8} Wh$
4. Two equal symmetrically spaced loads.	$M_{AB} = M_{BC} = -\left[\frac{2n}{2n+3}\right]\frac{3}{2}\frac{Pa}{h}(h-a)$
5. Single load at any point.	$M_{AB} = M_{BC} = -\left[\frac{2n}{2n+3}\right]\frac{Pab}{2h^2}(h+b)$
6. Hydraulic load.	$M_{AB} = M_{BC} = -\left[\frac{2n}{2n+3}\right]\frac{Wa}{60h^2}(10h^2 - 3a^2)$
7. External couples at top.	$M_{AB} = -M_{BA} = \frac{2nM}{2n+3}$ $M_{AD} = -M_{BC} = \frac{3M}{2n+3}$

Table 43

Two-legged Rectangular Bent

Two legs identical
Vertical load on top
Legs hinged at the bases
Moment of inertia of AB is I
Moment of inertia of AD is I_1

$$K = \frac{I}{l}$$

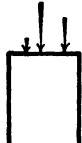
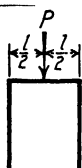
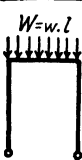
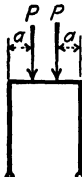
$$n = \frac{Kh}{I_1}$$

Type of loading	Moments
1. Any system of loads.	$M_{AD} = -M_{BC} = \frac{3}{2}\left[\frac{C_{AB} + C_{BA}}{2n + 3}\right]$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AD} = -M_{BC} = \frac{3}{8}\left[\frac{Pl}{2n + 3}\right]$
3. Uniform load.	$M_{AD} = -M_{BC} = \frac{1}{4}\left[\frac{Wl}{2n + 3}\right]$
4. Two equal symmetrically spaced loads.	$M_{AD} = -M_{BC} = \frac{3Pa(l-a)}{l(2n + 3)}$

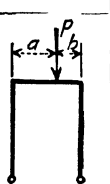
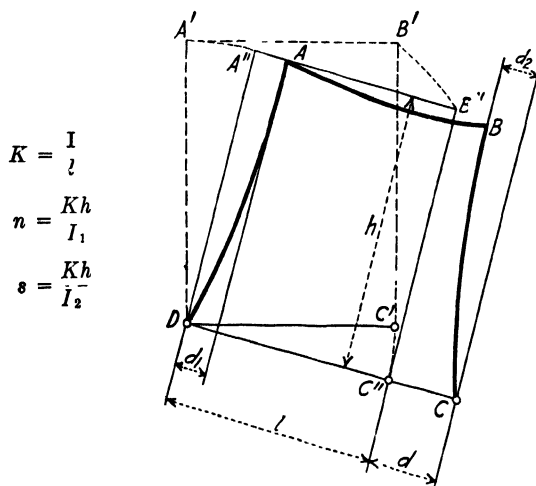
TABLE 43 (*Continued*)

Type of loading	Moments
5. Single load at any point.	$M_{AD} = -M_{BC} = \frac{3}{2}\frac{Pab}{l}\left[\frac{1}{2n+3}\right]$

TABLE 44

TWO-LEGGED RECTANGULAR BENT

Legs hinged at the bases
Settlement of foundations
Moment of inertia of AB is I
Moment of inertia of AD is I_1
Moment of inertia of BC is I_2

$$K = \frac{I}{l}$$

$$n = \frac{Kh}{I_1}$$

$$s = \frac{Kh}{I_2}$$

$$M_{AB} = M_{BC} = \frac{d}{h}\frac{3EK}{[n + s + 3]}$$

If the bent is symmetrical about a vertical center line $n = s$, and

$$M_{AB} = M_{BC} = \frac{d}{h}\frac{3EK}{2n + 3}$$

If $n = s = 1$, then

$$M_{AB} = M_{BC} = \frac{3}{5}\frac{dEK}{h}$$

TABLE 45

TWO-LEGGED RECTANGULAR BENT

Legs hinged at the bases
Cross-sections of two legs different
Horizontal load on one leg

M_D = moment of external load about D

$$K = \frac{I_{AB}}{l}$$

$$n = \frac{Kh}{I_{AD}}$$

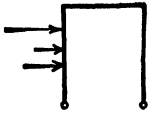
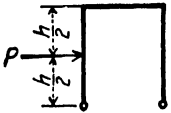
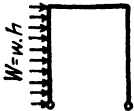
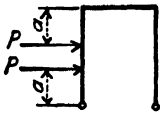
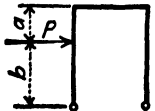
$$s = \frac{Kh}{I_{BC}}$$

Type of loading	Moments
 1. Any system of loads	 $M_{AB} = \frac{1}{2}\frac{M_D(2s+3) - 2nH_{AD}}{n+s+3}$ $M_{BC} = -\frac{1}{2}\frac{M_D(2n+3) + 2nH_{AD}}{n+s+3}$ Values of H_{AD} for various loads are given in Tables 2 and 3.
 2. Single load at mid-height.	 $M_{AB} = \frac{Ph}{16}\frac{(8s + 12 - 3n)}{n+s+3}$ $M_{BC} = -\frac{Ph}{16}\frac{(11n + 12)}{n+s+3}$
3. Uniform load.	$M_{AB} = \frac{Wh}{8}\frac{(4s + 6 - n)}{n+s+3}$ $M_{BC} = -\frac{Wh}{8}\frac{5n + 6}{n+s+3}$
 4. Two equal loads symmetrically spaced.	$M_{AB} = \frac{P}{2h}\frac{h^2(2s+3) - 3an(h-a)}{(n+s+3)}$ $M_{BC} = -\frac{P}{2h}\frac{h^2(2n+3) + 3an(h-a)}{n+s+3}$
 5. Single load at any point.	$M_{AB} = \frac{Pb}{2h^2}\frac{h^2(2s+3) - an(h+b)}{s+n+3}$ $M_{BC} = -\frac{Pb}{2h^2}\frac{h^2(2n+3) + an(h+b)}{s+n+3}$

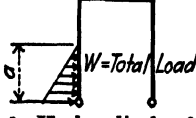
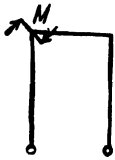
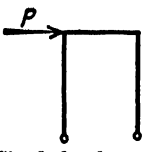
TABLE 45 (*Continued*)

Type of loading	Moments
W=Total Load, a 6. Hydraulic load.	$M_{AB} = \frac{Wa}{60h^2} \frac{10h^2(2s+3) - n(10h^2 - 3a^2)}{n+s+3}$ $M_{BC} = -\frac{Wa}{60h^2} \frac{10h^2(2n+3) + n(10h^2 - 3a^2)}{n+s+3}$
M 7. External couple at top.	$M_{AB} = M\left[1 - \frac{3}{2} \frac{1}{n+s+3}\right]$ $M_{AD} = -M_{BC} = \frac{3}{2}\left[\frac{M}{n+s+3}\right]$
P 8. Single load at top.	$M_{AB} = \frac{Ph}{2}\left[\frac{3+2s}{n+s+3}\right]$ $M_{BC} = -\frac{Ph}{2}\left[\frac{3+2n}{n+s+3}\right]$

TABLE 46

TWO-LEGGED RECTANGULAR BENT

Legs hinged at the bases
Cross sections of two legs different
Vertical load on top

$$K = \frac{I_{AB}}{l}$$

$$n = \frac{Kh}{I_{AD}}$$

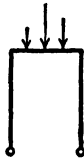
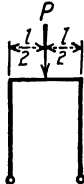
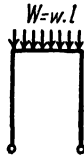
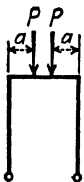
$$s = \frac{Kh}{I_{BC}}$$

Type of loading	Moments
1. Any system of loads.	$M_{AD} = -M_{BC} = \frac{3}{2}\left[\frac{C_{AB} + C_{BA}}{n + s + 3}\right]$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AD} = -M_{BC} = \frac{3Pl}{8} \frac{1}{n + s + 3}$
3. Uniform load.	$M_{AD} = -M_{BC} = \frac{Wl}{4} \frac{1}{n + s + 3}$
4. Two equal loads symmetrically spaced.	$M_{AD} = -M_{BC} = \frac{3\ Pa(l - a)}{l\ (n + s + 3)}$

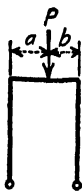
TABLE 46 (*Continued*)

Type of loading	Moments
P, a, b 5. Single load at any point.	$M_{AD} = -M_{BC} = \frac{3Pab}{2l(n + s + 3)}$

Table 47

Two-legged Rectangular Bent

Legs fixed at the bases
Cross-sections of two legs identical
Horizontal load on one leg
M_D = moment of external load about D

$$K = \frac{I_{AB}}{l}$$

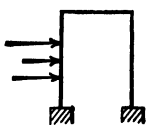
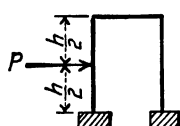
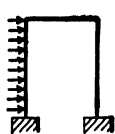
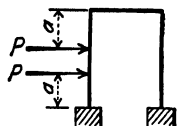
$$n = \frac{Kh}{I_{AD}} = \frac{Kh}{I_{BC}}$$

Type of loading	Moments
1. Any system of loads.	$M_{AD} = -\frac{n}{2}\left[\frac{3(M_D - C_{DA})}{6n+1} - C_{AD}\left(\frac{1}{n+2} + \frac{3}{6n+1}\right)\right]$ $M_{BC} = -\frac{n}{2}\left[\frac{3(M_D - C_{DA})}{6n+1} + C_{AD}\left(\frac{1}{n+2} - \frac{3}{6n+1}\right)\right]$ $M_{CB} = -\frac{1}{2}\left[\frac{(3n+1)(M_D - C_{DA})}{6n+1} - C_{AD}\left(\frac{1}{n+2} - \frac{3n}{6n+1}\right)\right]$ $M_{DA} = -\frac{1}{2}\left[\frac{(3n+1)(M_D - C_{DA})}{6n+1} + C_{AD}\left(\frac{1}{n+2} + \frac{3n}{6n+1}\right)\right] - C_{DA}$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at mid-height.	$M_{AD} = -\frac{nPh}{16}\left[\frac{6}{6n+1} - \frac{1}{n+2}\right]$ $M_{BC} = -\frac{nPh}{16}\left[\frac{6}{6n+1} + \frac{1}{n+2}\right]$ $M_{CB} = -\frac{3Ph}{16}\left[\frac{4n+1}{6n+1} - \frac{1}{3(n+2)}\right]$ $M_{DA} = -\frac{3Ph}{16}\left[\frac{4n+1}{6n+1} + \frac{2n+5}{3(n+2)}\right]$
 3. Uniform load.	$M_{AD} = -\frac{Whn}{2}\left[\frac{1}{6n+1} - \frac{1}{12(n+2)}\right]$ $M_{BC} = -\frac{Whn}{2}\left[\frac{1}{6n+1} + \frac{1}{12(n+2)}\right]$ $M_{CB} = -\frac{Wh}{24}\left[\frac{18n+5}{6n+1} - \frac{1}{n+2}\right]$ $M_{DA} = -\frac{Wh}{24}\left[\frac{18n+5}{6n+1} + \frac{2n+5}{n+2}\right]$
  4. Two equal loads symmetrically spaced.	$M_{AD} = \frac{Pan(h-a)}{2h}\left[\frac{6}{6n+1} + \frac{1}{n+2}\right] - \frac{3Pnh}{2(6n+1)}$ $M_{BC} = \frac{Pan(h-a)}{2h}\left[\frac{6}{6n+1} - \frac{1}{n+2}\right] - \frac{3Pnh}{2(6n+1)}$ $M_{CB} = +\frac{Pa(h-a)}{2h}\left[\frac{1}{6n+1} + \frac{1}{n+2}\right] - \frac{Ph}{2}\frac{(3n+1)}{(6n+1)}$ $M_{DA} = \frac{Pa(h-a)}{2h}\left[\frac{1}{6n+1} - \frac{2n+5}{n+2}\right] - \frac{Ph}{2}\frac{(3n+1)}{(6n+1)}$

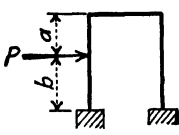
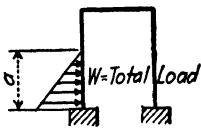
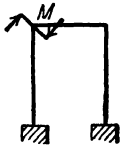
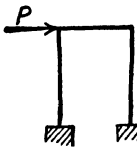
TABLE 47 (*Continued*)

Type of loading	Moments
5. Single load at any point.	$M_{AD} = \frac{Pabn}{2h^2}\left[\frac{3h}{6n+1} + \frac{b}{n+2}\right] - \frac{3Pbn}{2(6n+1)}$ $M_{BC} = \frac{Pabn}{2h^2}\left[\frac{3h}{6n+1} - \frac{b}{n+2}\right] - \frac{3Pbn}{2(6n+1)}$ $M_{CB} = \frac{Pab}{2h^2}\left[\frac{3n(a-b)+a}{6n+1} + \frac{b}{n+2}\right] - \frac{Pb(3n+1)}{2(6n+1)}$ $M_{DA} = -\frac{Pab}{2h^2}\left[\frac{3n(3a+b)+a}{6n+1} + \frac{b}{n+2}\right] - \frac{Pb(3n+1)}{2(6n+1)}$
6. Hydraulic load.	$M_{AD} = -\frac{Wan}{60h^2}\left[\frac{15ah}{6n+1} - \frac{a(5h-3a)}{n+2}\right]$ $M_{BC} = -\frac{Wan}{60h^2}\left[\frac{15ah}{6n+1} + \frac{a(5h-3a)}{n+2}\right]$ $M_{CB} = \frac{Wa^2}{60h^2}\left[\frac{18an - 45nh - 10h + 3a}{6n+1} + \frac{5h-3a}{n+2}\right]$ $M_{DA} = \frac{Wa}{60h^2}\left[\frac{-18a^2n + 75nah - 120nh^2 - 3a^2 + 10ah - 20h^2}{6n+1} - \frac{(5h-3a)a}{n+2}\right]$
7. External couple at top.	$M_{AB} = M - M_{AD} = \frac{M}{2}\left[1 + \frac{n}{n+2} - \frac{1}{6n+1}\right]$ $M_{BC} = \frac{M}{2}\left[\frac{n}{n+2} - \frac{6n}{6n+1}\right]$ $M_{CB} = -\frac{M}{2}\left[\frac{1}{n+2} + \frac{1}{6n+1}\right]$ $M_{DA} = \frac{M}{2}\left[\frac{1}{n+2} - \frac{1}{6n+1}\right]$
8. Single load at top.	$M_{AD} = M_{BC} = -\frac{Ph}{2}\frac{3n}{6n+1}$ $M_{CB} = M_{DA} = -\frac{Ph}{2}\frac{3n+1}{6n+1}$

TABLE 48

TWO-LEGGED RECTANGULAR BENT

Legs fixed at the bases
Cross-sections of two legs identical
Horizontal loads on both legs
Load symmetrical about vertical center line

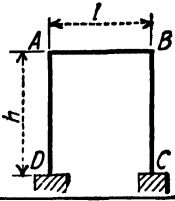
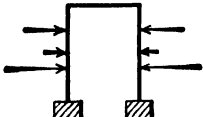
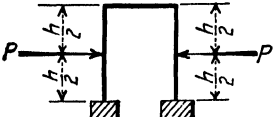
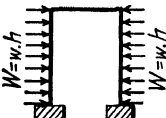
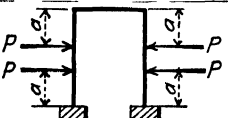
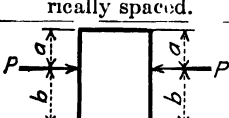
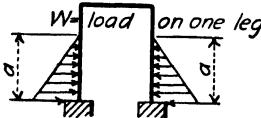
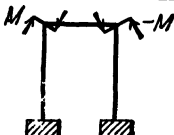
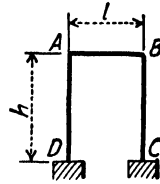
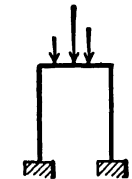
$$K = \frac{I_{AB}}{l} \qquad n = \frac{Kh}{I_{AD}} = \frac{Kh}{I_{BC}}$$

Type of loading	Moments
1. Any system of loads.	$M_{AB} = M_{BC} = -\frac{n}{n+2} C_{AD}$ $M_{CB} = -M_{DA} = \frac{1}{n+2} C_{AD} + C_{DA}$
2. Single load at mid-height.	$M_{AB} = M_{BC} = -\frac{Ph}{8} \frac{n}{n+2}$ $M_{CB} = -M_{DA} = \frac{Ph}{8}\left[\frac{n+3}{n+2}\right]$
3. Uniform load.	$M_{AB} = M_{BC} = -\frac{Wh}{12} \frac{n}{n+2}$ $M_{CB} = -M_{DA} = \frac{Wh}{12}\left[\frac{n+3}{n+2}\right]$
4. Two equal loads symmetrically spaced.	$M_{AB} = M_{BC} = -\frac{n}{n+2} \frac{Pa}{h}(h-a)$ $M_{CB} = -M_{DA} = \left[\frac{n+3}{n+2}\right] \frac{Pa}{h}(h-a)$
5. Single load at any point.	$M_{AB} = M_{BC} = -\frac{n}{n+2} \frac{Pab^2}{h^2}$ $M_{CB} = -M_{DA} = \frac{Pab}{h^2}\left[\frac{b}{n+2} + a\right]$
6. Hydraulic load.	$M_{AB} = M_{BC} = -\frac{n}{n+2} \frac{Wa^2}{30h^2}(5h-3a)$ $M_{CB} = -M_{DA} = \frac{Wa}{30h^2}\left[\frac{a(5h-3a)}{n+2} + 3a^2 - 10ah + 10h^2\right]$
7. External couple at top.	$M_{AD} = -M_{BC} = \frac{2M}{n+2}$ $M_{DA} = -M_{CB} = \frac{M}{n+2}$ $M_{AB} = -M_{BA} = \frac{Mn}{n+2}$

TABLE 49

TWO-LEGGED RECTANGULAR BENT

Legs fixed at the bases
Cross-sections of two legs identical
Vertical load on top

$$K = \frac{I_{AB}}{l}$$

$$n = \frac{Kh}{I_{AD}} = \frac{Kh}{I_{BC}}$$

Type of loading	Moments
1. Any system of loads.	$M_{AB} = -\frac{1}{2}\left\{ C_{BA}\left[\frac{2}{n+2} - \frac{1}{6n+1}\right] + C_{AB}\left[\frac{2}{n+2} + \frac{1}{6n+1}\right]\right\}$ $M_{BC} = -\frac{1}{2}\left\{ C_{BA}\left[\frac{2}{n+2} + \frac{1}{6n+1}\right] + C_{AB}\left[\frac{2}{n+2} - \frac{1}{6n+1}\right]\right\}$ $M_{CB} = -\frac{1}{2}\left\{ C_{BA}\left[\frac{1}{n+2} - \frac{1}{6n+1}\right] + C_{AB}\left[\frac{1}{n+2} + \frac{1}{6n+1}\right]\right\}$ $M_{DA} = \frac{1}{2}\left\{ C_{BA}\left[\frac{1}{n+2} + \frac{1}{6n+1}\right] + C_{AB}\left[\frac{1}{n+2} - \frac{1}{6n+1}\right]\right\}$ If the load is symmetrical about the vertical center line $C_{BA} = C_{AB}$, and $M_{AB} = M_{BC} = -\frac{2C_{AB}}{n+2}$ $M_{CB} = -M_{DA} = -\frac{C_{AB}}{n+2}$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AB} = M_{BC} = -\frac{Pl}{4(n+2)}$ $M_{CB} = -M_{DA} = -\frac{Pl}{8(n+2)}$
W=w.l 3. Uniform load.	$M_{AB} = M_{BC} = -\frac{Wl}{6(n+2)}$ $M_{CB} = -M_{DA} = -\frac{Wl}{12(n+2)}$

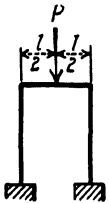
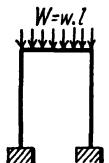



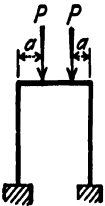
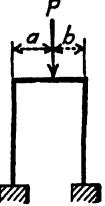
TABLE 49 (*Continued*)

Type of loading	Moments
4. Two equal loads symmetrically spaced.	$M_{AB} = M_{BC} = -\frac{2Pa(l-a)}{l(n+2)}$ $M_{CB} = -M_{DA} = -\frac{Pa(l-a)}{l(n+2)}$
5. Single load at any point.	$M_{AB} = -\frac{Pab}{2l^2}\left[\frac{2l}{n+2} + \frac{b-a}{6n+1}\right]$ $M_{BC} = -\frac{Pab}{2l^2}\left[\frac{2l}{n+2} + \frac{a-b}{6n+1}\right]$ $M_{CB} = -\frac{Pab}{2l^2}\left[\frac{l}{n+2} + \frac{b-a}{6n+1}\right]$ $M_{DA} = \frac{Pab}{2l^2}\left[\frac{l}{n+2} + \frac{a-b}{6n+1}\right]$

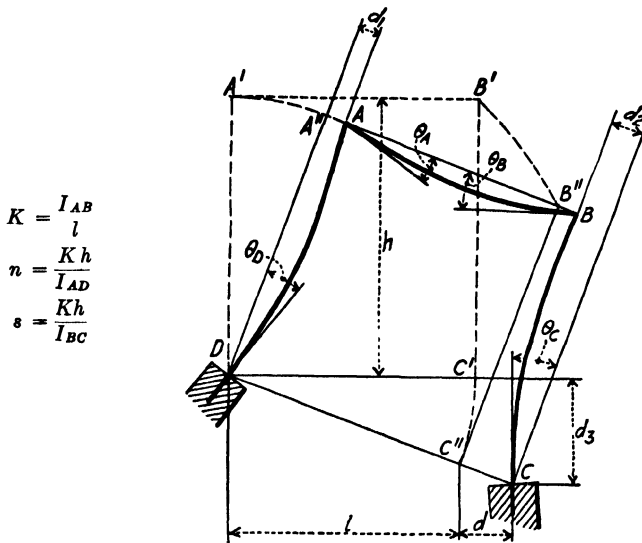
TABLE 50

TWO-LEGGED RECTANGULAR BENT

Settlement of foundation   Legs fixed at the bases

$$K = \frac{I_{AB}}{l}$$

$$n = \frac{Kh}{I_{AD}}$$

$$s = \frac{Kh}{I_{BC}}$$

Cross-sections of two legs different.

$$M_{AB} = \frac{2EK}{\Delta}\left[3(6ns + 2n - s)\frac{d}{h} + (9ns + 8n - s)\theta_C - (6ns + 2n - 7s - 3s^2)\theta_D\right]$$

$$M_{BC} = \frac{2EK}{\Delta}\left[3(6ns - n + 2s)\frac{d}{h} + (6ns - 7n + 2s - 3n^2)\theta_C - (9ns - n + 8s)\theta_D\right]$$

$$M_{CB} = \frac{2EK}{\Delta}\left[3(6ns + 5n + 2s + 1)\frac{d}{h} + (12ns + 22n + 4s + 3 + 3n^2)\theta_C - (9ns + 7n + 7s + 3)\theta_D\right]$$

$$M_{DA} = -\frac{2EK}{\Delta}\left[3(6ns + 2n + 5s + 1)\frac{d}{h} + (9ns + 7n + 7s + 3)\theta_C - (12ns + 4n + 22s + 3 + 3s^2)\theta_D\right]$$

in which $\Delta = 22ns + 2(s^2 + s + n^2 + n) + 6(n^2s + s^2n)$

Cross-sections of two legs identical, that is $n = s$

$$M_{AB} = EK\left[\frac{d}{h}\frac{3}{n+2} + \frac{\theta_C - \theta_D}{n+2} + \frac{3(\theta_C + \theta_D)}{6n+1}\right]$$

$$M_{BC} = EK\left[\frac{d}{h}\frac{3}{n+2} + \frac{\theta_C - \theta_D}{n+2} - \frac{3(\theta_C + \theta_D)}{6n+1}\right]$$

$$M_{CB} = EK\left[\frac{d}{h}\frac{3(n+1)}{n(n+2)} + \frac{2n+3}{n(n+2)}(\theta_C - \theta_D) + \frac{3(\theta_C + \theta_D)}{6n+1}\right]$$

$$M_{DA} = -EK\left[\frac{d}{h}\frac{3(n+1)}{n(n+2)} + \frac{2n+3}{n(n+2)}(\theta_C - \theta_D) - \frac{3(\theta_C + \theta_D)}{6n+1}\right]$$

If $n = s = 1$

$$M_{AB} = \frac{EK}{21}\left[21\frac{d}{h} + 16\theta_C + 2\theta_D\right]$$

$$M_{BC} = \frac{EK}{21}\left[21\frac{d}{h} - 2\theta_C - 16\theta_D\right]$$

$$M_{CB} = \frac{EK}{21}\left[42\frac{d}{h} + 44\theta_C - 26\theta_D\right]$$

$$M_{DA} = -\frac{EK}{21}\left[42\frac{d}{h} + 26\theta_C - 44\theta_D\right]$$

TABLE 51

TWO-LEGGED RECTANGULAR BENT

Legs fixed at the bases
Cross-sections of two legs different
Horizontal load on one leg

M_D = moment of external load about D

$\Delta = 2(11sn + 3sn^2 + 3s^2n + s^2 + s + n^2 + n)$

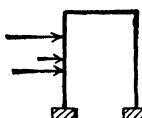
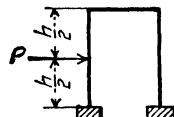
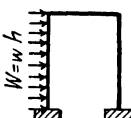
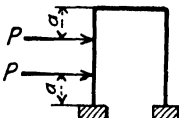
$K = \frac{I_{AB}}{l}$ $\quad n = \frac{Kh}{I_{AD}}$ $\quad s = \frac{Kh}{{}_{CB}I}$

Type of loading	Moments
 1. Any system of loads.	$M_{AD} = -\frac{n}{\Delta}[3s(s+2)(M_D - C_{DA}) - C_{AD}(6ns + 3s^2 + 5s + 2n)]$ $M_{BC} = -\frac{n}{\Delta}[3s(n+2)(M_D - C_{DA}) + C_{AD}(3ns - n - 4s)]$ $M_{CB} = -\frac{n}{\Delta}[(3ns + 2n + 5s + 2)(M_D - C_{DA}) + C_{AD}(3ns + 3n - 3s - 1)]$ $M_{DA} = -\frac{1}{\Delta}[s(3ns + 2s + 5n + 2)(M_D - C_{DA}) + nC_{AD}(3s^2 + 12s + 1)] - C_{DA}$ Values of C for various loads are given in Tables 2 and 3.
 2. Single load at mid-height.	$M_{AD} = -\frac{Phn}{8\Delta}[s(6s + 13) - 2n(3s + 1)]$ $M_{BC} = -\frac{Phn}{8\Delta}[12ns + 14s - n]$ $M_{CB} = -\frac{Phn}{8\Delta}[12s(n + 1) + 9n + 5]$ $M_{DA} = -\frac{Ph}{8\Delta}[6s^2(2n + 1) + 3s(9n + 2) + n] - \frac{Ph}{8}$
 3. Uniform load.	$M_{AD} = -\frac{Whn}{12\Delta}[s(12s + 25) - 2n(3s + 1)]$ $M_{BC} = -\frac{Whn}{12\Delta}[18ns + 26s - n]$ $M_{CB} = -\frac{Whn}{12\Delta}[2s(9n + 11) + 13n + 9]$ $M_{DA} = -\frac{Wh}{12\Delta}\left[2s^2(9n + 5) + s(37n + 10) + n\right] - \frac{Wh}{12}$
 4. Two equal loads symmetrically spaced.	$M_{AD} = -\frac{nP}{h\Delta}[3s(s + 2)(h^2 - ah + a^2) - a(h - a)(6ns + 3s^2 + 5s + 2n)]$ $M_{BC} = -\frac{nP}{h\Delta}[3s(n + 2)(h^2 - ah + a^2) + a(h - a)(3ns - n - 4s)]$ $M_{CB} = -\frac{nP}{h\Delta}[(3ns + 2n + 5s + 2)(h^2 - ah + a^2) + a(h - a)(3ns + 3n - 3s - 1)]$ $M_{DA} = -\frac{P}{h\Delta}[s(3ns + 2s + 5n + 2)(h^2 - ah + a^2)$ $na(h - a)(3s^2 + 12s + 1)] - \frac{Pa}{h}(h - a)$

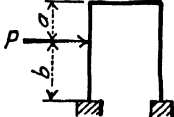
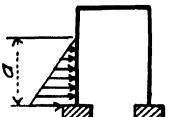
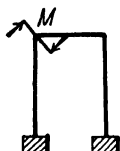
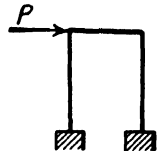
TABLE 51 (*Continued*)

Type of loading	Moments
5. Single load at any point.	$M_{AD} = -\frac{Pbn}{h^2\Delta}[3s(s+2)(h^2-a^2) - ab(6ns+3s^2+5s+2n)]$ $M_{BC} = -\frac{Pbn}{h^2\Delta}[3s(n+2)(h^2-a^2) + ab(3ns-n-4s)]$ $M_{CB} = -\frac{Pbn}{h^2\Delta}[(3ns+2n+5s+2)(h^2-a^2) + ab(3ns+3n-3s-1)]$ $M_{DA} = -\frac{Pb}{h^2\Delta}\Big[s(3ns+2s+5n+2)(h^2-a^2) + nab\Big[3s^2+12s+1\Big] - \frac{Pa^2b}{h^2}$
6. Hydraulic load.	$M_{AD} = -\frac{Wa^2n}{30h^2\Delta}[3s(s+2)(10h-3a) - (5h-3a)(6ns+3s^2+5s+2n)]$ $M_{BC} = -\frac{Wa^2n}{30h^2\Delta}[3s(n+2)(10h-3a) + (5h-3a)(3ns-n-4s)]$ $M_{CB} = -\frac{Wa^2n}{30h^2\Delta}[(3ns+2n+5s+2)(10h-3a) + (5h-3a)(3ns+3n-3s-1)]$ $M_{DA} = -\frac{Wa^2}{30h^2\Delta}[s(3ns+2s+5n+2)(10h-3a) + n(5h-3a)(3s^2+12s+1)] - \frac{Wa}{30h^2}(3a^2-10ah+10h^2)$
7. External couple at top.	$M_{AB} = -\frac{M}{\Delta}(11ns+2s^2+2s+2n) + M$ $M_{BC} = -\frac{nM}{\Delta}(10s+n)$ $M_{CB} = -\frac{nM}{\Delta}(8s-n+3)$ $M_{DA} = \frac{M}{\Delta}(7ns-2s^2-2s+n)$ $M_{AD} = \frac{M}{\Delta}(11ns+2s^2+2s+2n)$
8. Single load at top.	$M_{AD} = -\frac{Ph}{\Delta}3ns(s+2)$ $M_{BC} = -\frac{Ph}{\Delta}3ns(n+2)$ $M_{CB} = -\frac{Ph}{\Delta}n(2n+2+5s+3ns)$ $M_{DA} = -\frac{Ph}{\Delta}s(2s+2+5n+3ns)$

TABLE 52

TWO-LEGGED RECTANGULAR BENT

Legs fixed at the bases
Cross-sections of the two legs different
Vertical load on top

$$\Delta = 2(11sn + 3sn^2 + 3s^2n + s^2 + s + n^2 + n)$$

$$K = \frac{I_{AB}}{l}$$

$$n = \frac{Kh}{I_{AD}}$$

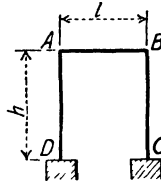
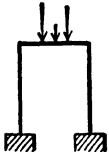
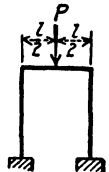
$$s = \frac{Kh}{I_{BC}}$$

Type of loading	Moments
1. Any system of loads.	$M_{AB} = -\frac{1}{\Delta}[C_{BA}(10ns + s^2) + C_{AB}(11ns + 2s^2 + 2s + 2n)]$ $M_{BC} = -\frac{1}{\Delta}[C_{BA}(11ns + 2n^2 + 2n + 2s) + C_{AB}(10ns + n^2)]$ $M_{CB} = -\frac{1}{\Delta}[C_{BA}(7ns - 2n^2 - 2n + s) + C_{AB}(8ns - n^2 + 3n)]$ $M_{DA} = \frac{1}{\Delta}[C_{BA}(8ns - s^2 + 3s) + C_{AB}(7ns - 2s^2 - 2s + n)]$ If the load is symmetrical about the vertical center line of the bent $C_{AB} = C_{BA}$, and $M_{AB} = -\frac{C_{AB}}{\Delta}(21ns + 3s^2 + 2s + 2n)$ $M_{BC} = -\frac{C_{AB}}{\Delta}(21ns + 3n^2 + 2n + 2s)$ $M_{CB} = -\frac{C_{AB}}{\Delta}(15ns - 3n^2 + n + s)$ $M_{DA} = \frac{C_{AB}}{\Delta}(15ns - 3s^2 + s + n)$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AB} = -\frac{Pl}{8\Delta}(21ns + 3s^2 + 2s + 2n)$ $M_{BC} = -\frac{Pl}{8\Delta}(21ns + 3n^2 + 2n + 2s)$ $M_{CB} = -\frac{Pl}{8\Delta}(15ns - 3n^2 + n + s)$ $M_{DA} = \frac{Pl}{8\Delta}(15ns - 3s^2 + s + n)$

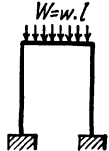
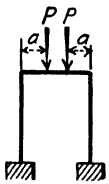
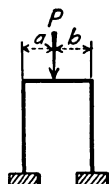
TABLE 52 (*Continued*)

Type of loading	Moments
3. Uniform load.	$M_{AB} = -\frac{Wl}{12\Delta}(21ns + 3s^2 + 2s + 2n)$ $M_{BC} = -\frac{Wl}{12\Delta}(21ns + 3n^2 + 2n + 2s)$ $M_{CB} = -\frac{Wl}{12\Delta}(15ns - 3n^2 + n + s)$ $M_{DA} = \frac{Wl}{12\Delta}(15ns - 3s^2 + s + n)$
4. Two equal loads symmetrically spaced.	$M_{AB} = -\frac{Pa(l-a)}{l\Delta}(21ns + 3s^2 + 2s + 2n)$ $M_{BC} = -\frac{Pa(l-a)}{l\Delta}(21ns + 3n^2 + 2n + 2s)$ $M_{CB} = -\frac{Pa(l-a)}{l\Delta}(15ns - 3n^2 + n + s)$ $M_{DA} = \frac{Pa(l-a)}{l\Delta}(15ns - 3s^2 + s + n)$
5. Single load at any point.	$M_{AB} = -\frac{Pab}{l^2\Delta}[ns(10l + b) + s^2(l + b) + 2b(n + s)]$ $M_{BC} = -\frac{Pab}{l^2\Delta}[ns(10l + a) + n^2(l + a) + 2a(n + s)]$ $M_{CB} = -\frac{Pab}{l^2\Delta}[ns(7l + b) + nb(3 - n) - 2na(n + 1) + sa)]$ $M_{DA} = \frac{Pab}{l^2\Delta}[ns(7l + a) + sa(3 - s) - 2sb(s + 1) + nb)]$

TABLE 53

TWO-LEGGED BENT. ONE LEG LONGER THAN THE OTHER

Legs hinged at the bases

Horizontal load on one leg

M_D = moment of external load about D

$$K = \frac{I_{AB}}{l}$$

$$n = \frac{Kh}{I_{AD}}$$

$$s = \frac{Kh}{I_{BC}q}$$

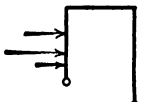
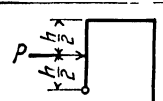
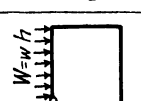
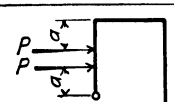
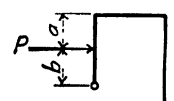
$$q = \frac{h}{BC}$$

Type of loading	Moments
1. Any system of loads.	$M_{AB} = \frac{1}{2}\frac{M_D(2s + 2 + q) - 2q^2nH_{AD}}{q^2n + s + 1 + q + q^2}$ $M_{BC} = -\frac{1}{2}\frac{M_D(2qn + 2q + 1) + 2qnH_{AD}}{q^2n + s + 1 + q + q^2}$ Values of H for various loads are given in Tables 2 and 3.
2. Single load at mid-height.	$M_{AB} = \frac{Ph}{16}\frac{4(2s + 2 + q) - 3q^2n}{q^2n + s + 1 + q + q^2}$ $M_{BC} = -\frac{Ph}{16}\frac{4(2qn + 2q + 1) + 3qn}{q^2n + s + 1 + q + q^2}$
3. Uniform load.	$M_{AB} = \frac{Wh}{8}\frac{2(2s + 2 + q) - q^2n}{q^2n + s + 1 + q + q^2}$ $M_{BC} = -\frac{Wh}{8}\frac{2(2qn + 2q + 1) + 2qn}{q^2n + s + 1 + q + q^2}$
4. Two equal loads symmetrically spaced.	$M_{AB} = \frac{P}{2h}\frac{h^2(2s + 2 + q) - 3aq^2n(h - a)}{q^2n + s + 1 + q + q^2}$ $M_{BC} = -\frac{P}{2h}\frac{h^2(2qn + 2q + 1) + 3aqn(h - a)}{q^2n + s + 1 + q + q^2}$
5. Single load at any point.	$M_{AB} = \frac{Pb}{2h^2}\frac{h^2(2s + 2 + q) - aq^2n(h + b)}{q^2n + s + 1 + q + q^2}$ $M_{BC} = -\frac{Pb}{2h^2}\frac{h^2(2qn + 2q + 1) + aqn(h + b)}{q^2n + s + 1 + q + q^2}$

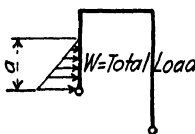
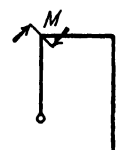
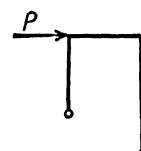
TABLE 53 (*Continued*)

Type of loading	Moments
W=Total Load; a 6. Hydraulic load.	$M_{AB} = \frac{Wa}{60h^2}\,\frac{10h^2(2s+2+q) - q^2n(10h^2-3a^2)}{q^2n+s+1+q+q^2}$ $M_{BC} = -\frac{Wa}{60h^2}\,\frac{10h^2(2qn+2q+1) + qn(10h^2-3a^2)}{q^2n+s+1+q+q^2}$
M 7. External couple at top.	$M_{AB} = \frac{M}{2}\,\frac{2s+2+q+2q^2n}{q^2n+s+1+q+q^2}$ $M_{AD} = \frac{M}{2}\,\frac{q(1+2q)}{q^2n+s+1+q+q^2}$ $M_{BC} = -\frac{M}{2}\,\frac{1+2q}{q^2n+s+1+q+q^2}$
P 8. Single load at top.	$M_{BC} = -Ph\left[\frac{2nq+2q+1}{q(2nq+2q+1)+(2s+2+q)}\right]$ $M_{AB} = Ph\left[\frac{2s+2+q}{q(2nq+2q+1)+(2s+2+q)}\right]$

TABLE 54

TWO-LEGGED BENT. ONE LEG LONGER THAN THE OTHER

Legs hinged at the bases
Vertical load on top

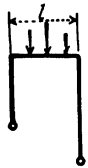
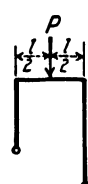
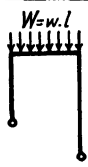
$$K = \frac{I_{AB}}{l}$$
$$n = \frac{Kh}{I_{AD}}$$
$$s = \frac{Kh}{I_{BC}q}$$
$$q = \frac{h}{BC}$$

Type of loading	Moments
1. Any system of loads.	$M_{AB} = -\frac{q}{2}\,\frac{C_{BA}(2+q) + C_{AB}(1+2q)}{q^2n + s + 1 + q + q^2}$ $M_{BC} = -\frac{1}{2}\,\frac{C_{BA}(2+q) + C_{AB}(1+2q)}{q^2n + s + 1 + q + q^2} = \frac{M_{AB}}{q}$ If the load is symmetrical about the middle of AB, then $M_{AB} = -\frac{3qC_{AB}}{2}\,\frac{1+q}{q^2n + s + 1 + q + q^2}$ $M_{BC} = -\frac{3C_{AB}}{2}\,\frac{1+q}{q^2n + s + 1 + q + q^2} = \frac{M_{AB}}{q}$
2. Single load at the center.	$M_{AB} = -\frac{3qPl}{16}\,\frac{1+q}{q^2n + s + 1 + q + q^2}$ $M_{BC} = -\frac{3Pl}{16}\,\frac{1+q}{q^2n + s + 1 + q + q^2} = \frac{M_{AB}}{q}$
3. Uniform load.	$M_{AB} = -\frac{qWl}{8}\,\frac{1+q}{q^2n + s + 1 + q + q^2}$ $M_{BC} = -\frac{Wl}{8}\,\frac{1+q}{q^2n + s + 1 + q + q^2} = \frac{M_{AB}}{q}$

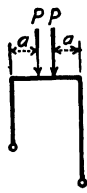
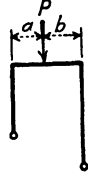
TABLE 54 (*Continued*)

Type of loading	Moments
4. Two equal loads symmetrically spaced.	$M_{AB} = -\frac{3qPa(l-a)}{2l}\frac{1+q}{q^2n+s+1+q+q^2}$ $M_{BC} = -\frac{3Pa(l-a)}{2l}\frac{1+q}{q^2n+s+1+q+q^2} = \frac{M_{AB}}{q}$
5. Single load at any point.	$M_{AB} = -\frac{qPab}{2l^2}\frac{[a(2+q)+b(1+2q)]}{q^2n+s+1+q+q^2}$ $M_{BC} = -\frac{Pab}{2l^2}\frac{[a(2+q)+b(1+2q)]}{q^2n+s+1+q+q^2} = \frac{M_{AB}}{q}$

TABLE 55

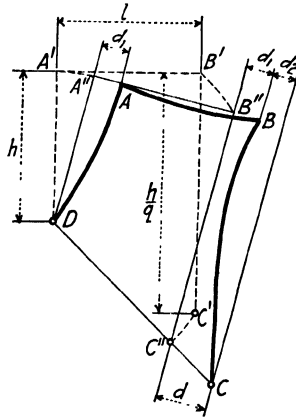
TWO-LEGGED BENT. ONE LEG LONGER THAN THE OTHER

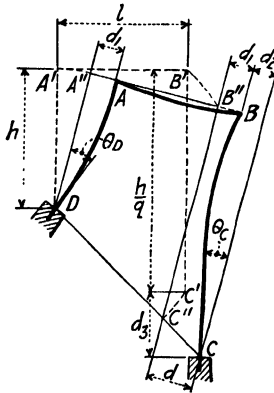
Settlement of foundations

$$K = \frac{I_{AB}}{l}$$

$$n = \frac{Kh}{I_{AD}}$$

$$s = \frac{Kh}{qI_{BC}}$$

$$q = \frac{h}{BC}$$

Legs hinged at the bases.

$$M_{AB} = \frac{qd}{h}\,\frac{3qEK}{q^2n + s + 1 + q + q^2} \qquad M_{BC} = \frac{qd}{h}\,\frac{3EK}{q^2n + s + 1 + q + q^2} = \frac{M_{AB}}{q}$$

Legs fixed at the bases.

$$M_{AB} = \frac{2EK}{\Delta}\left\{3\frac{qd}{h}[6qns + 2qn - s] + \theta_C[9qns + 2qn(3 + q) - s] - \theta_D[6q^2ns + 2q^2n - s(4 + 3q) - 3s^2]\right\}$$

$$M_{BC} = \frac{2EK}{\Delta}\left\{3\frac{qd}{h}[6ns - qn + 2s] + \theta_C[6ns - qn(3 + 4q) + 2s - 3q^2n^2] - \theta_D[9qns - q^2n + 2s(1 + 3q)]\right\}$$

$$M_{CB} = \frac{2EK}{\Delta}\left\{3\frac{qd}{h}[6ns + n(4 + q) + 2s + 1] + \theta_C[12ns + 2n(6 + 3q + 2q^2) + 4s + 3q^2n^2 + 3] - \theta_D[9qns + qn(6 + q) + s(1 + 6q) + 3q]\right\}$$

$$M_{DA} = -\frac{2EK}{\Delta}\left\{3\frac{qd}{h}[6qns + 2qn + s(1 + 4q) + q] + \theta_C[9qns + qn(6 + q) + s(1 + 6q) + 3q] - \theta_D[12q^2ns + 4q^2n + 2s(2 + 3q + 6q^2) + 3s^2 + 3q^2]\right\}$$

in which

$$\Delta = 2[ns(4 + 3q + 4q^2) + q^2n(3ns + n + 1) + s(3ns + s + 1)]$$

TABLE 56

TWO-LEGGED BENT. ONE LEG LONGER THAN THE OTHER

Legs fixed at the bases

Horizontal load on one leg

M_D = moment of external load about D

$$\Delta = 2[ns(4 + 3q + 4q^2) + s(s + 1) + q^2n(n + 1) + 3ns(q^2n + s)]$$

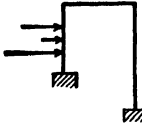
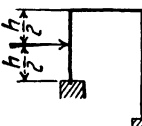
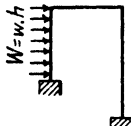
$$n = \frac{I_{AB}h}{lI_{AD}} \qquad s = \frac{I_{AB}h}{I_{BC}lq} \qquad q = \frac{h}{BC}$$

Type of loading	Moments
1. Any system of loads.	$M_{AB} = \frac{n}{\Delta}\left\{(M_D - C_{DA})s(3s + 4 + 2q) - C_{AD}(6q^2ns + 3s^2 + 4s + qs + 2q^2n)\right\}$ $M_{BC} = -\frac{n}{\Delta}\left\{(M_D - C_{DA})s(3qn + 2 + 4q) + C_{AD}(3qns - 2s - 2qs - q^2n)\right\}$ $M_{CB} = -\frac{n}{\Delta}\left\{(M_D - C_{DA})(3qns + s + 4qs + 2qn + 2q) + C_{AD}(3qns - s - 2qs + 2qn + q^2n - q)\right\}$ $M_{DA} = -\frac{1}{\Delta}\left\{(M_D - C_{DA})s(3ns + 4n + qn + 2s + 2) + C_{AD}n[3s^2 + 4s(1 + q + q^2) + q^2]\right\} - C_{DA}$
2. Single load at mid-height.	$M_{AB} = \frac{nPh}{8\Delta}\left[s(6s + 5q + 8) - 2q^2n(3s + 1)\right]$ $M_{BC} = -\frac{nPh}{8\Delta}\left[2s(6qn + 5q + 2) - q^2n\right]$ $M_{CB} = -\frac{nPh}{8\Delta}\left[q(12ns + 10s + 8n + 5 + qn) + 2s\right]$ $M_{DA} = -\frac{Ph}{8\Delta}\left[ns(12s + 16 + 7q + 4q^2) + 6s(s + 1) + q^2n\right] - \frac{Ph}{8}$
3. Uniform load.	$M_{AB} = \frac{Whn}{12\Delta}\left[s(12s + 16 + 9q) - 2q^2n(3s + 1)\right]$ $M_{BC} = -\frac{Whn}{12\Delta}\left[2s(9qn + 4 + 9q) - q^2n\right]$ $M_{CB} = -\frac{Whn}{12\Delta}\left[q(18ns + 18s + 12n + 9 + qn) + 4s\right]$ $M_{DA} = -\frac{Wh}{12\Delta}\left[ns(18s + 24 + 9q + 4q^2) + 10s(s + 1) + q^2n\right] - \frac{Wh}{12}$

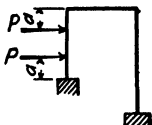
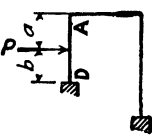
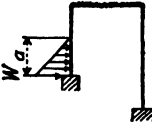
TABLE 56 (*Continued*)

Type of loading	Moments
4. Two equal symmetrically spaced loads.	$M_{AB} = \frac{Pn}{h\Delta}\left[h^2s(3s + 4 + 2q) - a(h - a)(6s^2 + 8s + 3qs + 6q^2ns + 2q^2n)\right]$ $M_{BC} = -\frac{Pn}{h\Delta}\left[h^2s(3qn + 2 + 4q) - a(h - a)(4s + 6qs + q^2n)\right]$ $M_{CB} = -\frac{Pn}{h\Delta}\left[h^2(3qns + s + 4qs + 2qn + 2q) - a(h - a)\left\{2s + q(6s + 3 - qn\right\}\right]$ $M_{DA} = -\frac{P}{h\Delta}\left[h^2s(3ns + 4n + qn + 2s + 2) + a(h - a)\left\{nqs(3 + 4q) + nq^2 - 2s(s + 1)\right\}\right] - \frac{Pa(h - a)}{h}$
5. Single load at any point.	$M_{AB} = \frac{Pbn}{h^2\Delta}\left[s(3s + 4 + 2q)(h^2 - a^2) - ab(6q^2ns + 3s^2 + 4s + qs + 2q^2n)\right]$ $M_{BC} = -\frac{Pbn}{h^2\Delta}\left[s(3qn + 2 + 4q)(h^2 - a^2) + ab(3qns - 2s - 2qs - q^2n)\right]$ $M_{CB} = -\frac{Pbn}{h^2\Delta}\left[(3qns + s + 4qs + 2qn + 2q)(h^2 - a^2) + ab(3qns - s - 2qs + 2qn + q^2n - q)\right]$ $M_{DA} = -\frac{Pb}{h^2\Delta}\left\{s(3ns + 4n + qn + 2s + 2)(h^2 - a^2) + nab\left[3s^2 + 4s(1 + q + q^2) + q^2\right]\right\} - \frac{Pa^2b}{h^2}$
6. Hydraulic load.	$M_{AB} = \frac{Wa^2n}{30h^2\Delta}\left[s(3s + 4 + 2q)(10h - 3a) - (5h - 3a)(6q^2ns + 3s^2 + 4s + qs + 2q^2n)\right]$ $M_{BC} = -\frac{Wa^2n}{30h^2\Delta}\left[s(3qn + 2 + 4q)(10h - 3a) + (5h - 3a)(3qns - 2s - 2qs - q^2n)\right]$ $M_{CB} = -\frac{Wa^2n}{30h^2\Delta}\left[(3qns + s + 4qs + 2qn + 2q)(10h - 3a) + (5h - 3a)(3qns - s - 2qs + 2qn + q^2n - q)\right]$ $M_{DA} = -\frac{Wa^2}{30h^2\Delta}\left\{s(3ns + 4n + qn + 2s + 2)(10h - 3a) + n\left[5h - 3a\right]\left[3s^2 + 4s(1 + q + q^2) + q^2\right] + \Delta\left(3a - 10h + \frac{10h^2}{a}\right)\right\}$

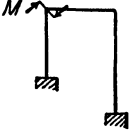
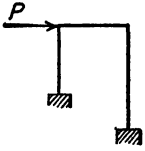
TABLE 56 (*Continued*)

Type of loading	Moments
7. External couple at top.	$M_{AB} = \frac{Mn}{\Delta}\left(6s^2 + 8s + 3qs + 6q^2ns + 2q^2n\right)$ $M_{BC} = -\frac{Mn}{\Delta}\left(4s + 6qs + q^2n\right)$ $M_{CB} = -\frac{Mn}{\Delta}\left(2s + 6qs + 3q - q^2n\right)$ $M_{DA} = \frac{M}{\Delta}\left(3qns - 2s^2 - 2s + 4q^2ns + q^2n\right)$ $M_{AD} = \frac{M}{\Delta}\left(3qns + 8q^2ns + 2s^2 + 2s + 2q^2n\right)$
8. Single load at top.	$M_{AB} = \frac{nsPh}{\Delta}\left(3s + 4 + 2q\right)$ $M_{BA} = \frac{nsPh}{\Delta}\left(3nq + 4q + 2\right)$ $M_{DA} = -\frac{sPh}{\Delta}\left(3sn + 4n + qn + 2s + 2\right)$ $M_{CB} = -\frac{nPh}{\Delta}\left(3nsq + 4sq + 2nq + s + 2q\right)$

Table 57

Two-legged Bent. One Leg Longer than the Other

Legs fixed at the bases
Vertical load on top

$$\Delta = 2[ns(4 + 3q + 4q^2) + q^2n(3ns + n + 1) + s(3ns + s + 1)]$$

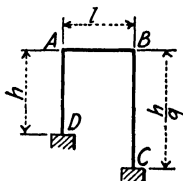
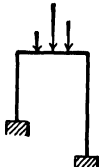
$$n = \frac{I_{AB}h}{lI_{AD}} \qquad s = \frac{I_{AB}h}{I_{BC}lq} \qquad q = \frac{h}{BC}$$

Type of loading	Moments
1. Any system of loading.	$M_{AB} = -\frac{1}{\Delta}\left\{C_{BA}[2qns(3 + 2q) + s^2] + C_{AB}[qns(3 + 8q) + 2s^2 + 2s + 2q^2n]\right\}$ $M_{BC} = -\frac{1}{\Delta}\left\{C_{BA}[ns(8 + 3q) + 2q^2n^2 + 2q^2n + 2s] + C_{AB}[2ns(2 + 3q) + q^2n^2]\right\}$ $M_{CB} = -\frac{1}{\Delta}\left\{C_{BA}[ns(4 + 3q) - 2q^2n^2 - 2q^2n + s] + C_{AB}[2ns(1 + 3q) - q^2n^2 + 3qn]\right\}$ $M_{DA} = \frac{1}{\Delta}\left\{C_{BA}[2qns(3 + q) - s^2 + 3qs] + C_{AB}[qns(3 + 4q) - 2s^2 - 2s + q^2n]\right\}$ If the load is symmetrical about the center of AB, $C_{BA} = C_{AB}$ and $M_{AB} = -\frac{1}{\Delta}C_{AB}[3qns(3 + 4q) + 3s^2 + 2s + 2q^2n]$ $M_{BC} = -\frac{1}{\Delta}C_{AB}[3ns(4 + 3q) + 3q^2n^2 + 2q^2n + 2s]$ $M_{CB} = -\frac{1}{\Delta}C_{AB}[3ns(2 + 3q) - 3q^2n^2 + qn(3 - 2q) + s]$ $M_{AD} = \frac{1}{\Delta}C_{BA}[3qns(3 + 2q) - 3s^2 + s(3q - 2) + q^2n]$

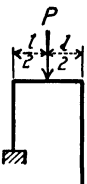
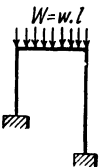
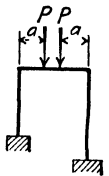
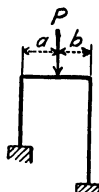
TABLE 57 (*Continued*)

Type of loading	Moments
2. Single load at center.	$M_{AB} = -\frac{Pl}{8\Delta}[3qns(3 + 4q) + 3s^2 + 2s + 2q^2n]$ $M_{BC} = -\frac{Pl}{8\Delta}[3ns(4 + 3q) + 3q^2n^2 + 2q^2n + 2s]$ $M_{CB} = -\frac{Pl}{8\Delta}[3ns(2 + 3q) - 3q^2n^2 + qn(3 - 2q) + s]$ $M_{DA} = \frac{Pl}{8\Delta}[3qns(3 + 2q) - 3s^2 + s(3q - 2) + q^2n]$
3. Uniform load.	$M_{AB} = -\frac{Wl}{12\Delta}[3qns(3 + 4q) + 3s^2 + 2s + 2q\,n]$ $M_{BC} = -\frac{Wl}{12\Delta}[3ns(4 + 3q) + 3q^2n^2 + 2q^2n + 2s]$ $M_{CB} = -\frac{Wl}{12\Delta}[3ns(2 + 3q) - 3q^2n^2 + qn(3 - 2q) + s]$ $M_{DA} = \frac{Wl}{12\Delta}[3qns(3 + 2q) - 3s^2 + s(3q - 2) + q^2n]$
4. Two equal loads symmetrically spaced.	$M_{AB} = -\frac{Pa(l - a)}{l\Delta}[3qns(3 + 4q) + 3s^2 + 2s + 2q^2n]$ $M_{BC} = -\frac{Pa(l - a)}{l\Delta}[3ns(4 + 3q) + 3q^2n^2 + 2q^2n + 2s]$ $M_{CB} = -\frac{Pa(l - a)}{l\Delta}[3ns(2 + 3q) - 3q^2n^2 + qn(3 - 2q) + s]$ $M_{DA} = \frac{Pa(l - a)}{l\Delta}[3qns(3 + 2q) - 3s^2 + s(3q - 2) + q^2n]$
5. Single load at any point.	$M_{AB} = -\frac{Pab}{l^2\Delta}[3qns(l + a) + 4q^2ns(l + b) + s^2(l + b) + 2b(s + q^2n)$ $M_{BC} = -\frac{Pab}{l^2\Delta}[4ns(l + a) + 3qns(l + b) + q^2n^2(l + a) + 2a(s + q^2n)]$ $M_{CB} = -\frac{Pab}{l^2\Delta}[2ns(l + a) + 3nsq(l + b) - q^2n^2(l + a) + qn(3b - 2aq) + sa]$ $M_{DA} = \frac{Pab}{l^2\Delta}[3qns(l + a) + 2q^2ns(l + b) - s^2(l + b) + 3qsa - 2sb + q^2nb]$

TABLE 58

TWO-LEGGED TRAPEZOIDAL BENT

Legs hinged at the bases
Bent and load symmetrical about vertical center line
Vertical load on top

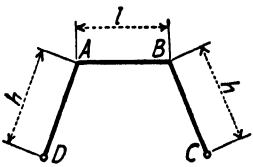
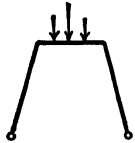
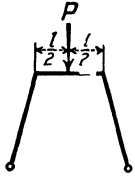
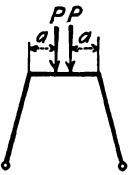
$$n = \frac{I_{AB}h}{lI_{AD}}$$

Type of loading	Moments
1. Any system of loads.	$M_{AD} = -M_{BC} = \frac{3C_{AB}}{3 + 2n}$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AD} = -M_{BC} = \frac{3Pl}{8(3 + 2n)}$
3. Uniform load.	$M_{AD} = -M_{BC} = \frac{Wl}{4(3 + 2n)}$
4. Two equal loads symmetrically spaced.	$M_{AD} = -M_{BC} = \frac{3Pa(l - a)}{l(3 + 2n)}$

Table 59

Two-legged Trapezoidal Bent

Legs hinged at the bases
Bent and load symmetrical about vertical center line
Normal loads on the two legs

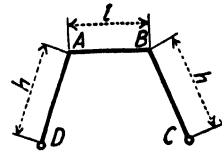
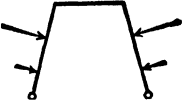
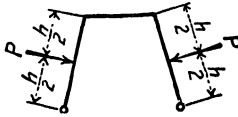
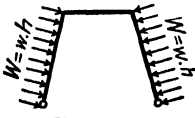
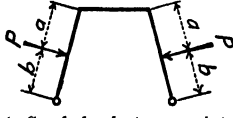
$$n = \frac{I_{AB}h}{lI_{AD}}$$

Type of loading	Moments
1. Any system of loading.	$M_{AD} = -M_{BC} = \frac{2nH_{AD}}{2n+3}$ Values of H for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AD} = -M_{BC} = \frac{3nPh}{8(2n+3)}$
3. Uniform load.	$M_{AD} = -M_{BC} = \frac{nWh}{4(2n+3)}$
4. Single load at any point.	$M_{AD} = -M_{BC} = \frac{nPab(h+b)}{h^2(2n+3)}$
5. Hydraulic load.	$M_{AD} = -M_{BC} = \frac{nWa(10h^2 - 3a^2)}{30h^2(2n+3)}$
6. External couples at top.	$M_{AD} = -M_{BC} = \frac{3M}{2n+3}$ $M_{AB} = -M_{BA} = \frac{2nM}{2n+3}$

TABLE 60

TWO-LEGGED TRAPEZOIDAL BENT

Legs fixed at the bases
Bent and load symmetrical about vertical center line
Vertical load on top

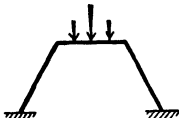
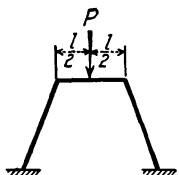
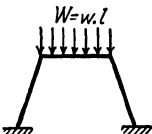
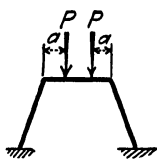
$$n = \frac{I_{AB}h}{lI_{AD}}$$

Type of loading	Moments
1. Any system of loads	$M_{AD} = -M_{BC} = \frac{2C_{AB}}{n+2}$ $M_{DA} = -M_{CB} = \frac{C_{AB}}{n+2}$ If $n = 1$ $M_{AD} = -M_{BC} = \frac{2}{3}C_{AB}$ $M_{DA} = -M_{CB} = \frac{1}{3}C_{AB}$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AD} = -M_{BC} = \frac{Pl}{4(n+2)}$ $M_{DA} = -M_{CB} = \frac{Pl}{8(n+2)}$
3. Uniform load.	$M_{AD} = -M_{BC} = \frac{Wl}{6(n+2)}$ $M_{DA} = -M_{CB} = \frac{Wl}{12(n+2)}$
4. Two equal loads symmetrically spaced.	$M_{AD} = -M_{BC} = \frac{2Pa(l-a)}{l(n+2)}$ $M_{DA} = -M_{CB} = \frac{Pa(l-a)}{l(n+2)}$

Table 61

Two-legged Trapezoidal Bent

Legs fixed at the bases
Bent and load symmetrical about vertical center line
Normal loads on the two legs

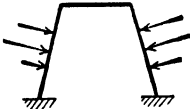
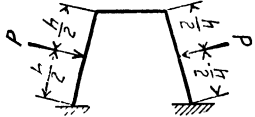
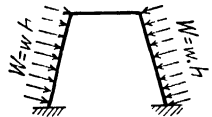
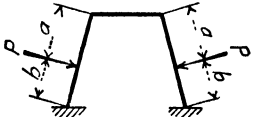
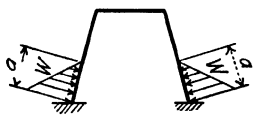
$$n = \frac{I_{AB}h}{lI_{AD}}$$

Type of loading	Moments
1. Any system of loads.	$M_{AD} = -M_{BC} = \left(\frac{n}{n+2}\right)C_{AD}$ $M_{DA} = -M_{CB} = -\left(\frac{C_{AD}}{n+2} + C_{DA}\right)$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at center.	$M_{AD} = -M_{BC} = \frac{nPh}{8(n+2)}$ $M_{DA} = -M_{CB} = -\frac{Ph}{8}\left(\frac{n+3}{n+2}\right)$
3. Uniform load.	$M_{AD} = -M_{BC} = \frac{nWh}{12(n+2)}$ $M_{DA} = -M_{CB} = -\frac{Wh}{12}\left(\frac{n+3}{n+2}\right)$
4. Single load at any point.	$M_{AD} = -M_{BC} = \frac{nPab^2}{h^2(n+2)}$ $M_{DA} = -M_{CB} = -\frac{Pab}{h^2}\left(\frac{b + na + 2a}{n+2}\right)$
5. Hydraulic load.	$M_{AD} = -M_{BC} = \frac{Wa^2n}{30h^2}\,\frac{5h-3a}{n+2}$ $M_{DA} = -M_{CB} =$ $-\frac{Wa}{30h^2}\left[\frac{3a^2(1+n) - 5ah(3+2n) + 10h^2(n+2)}{n+2}\right]$

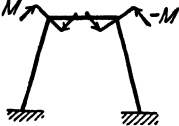
TABLE 61 (*Continued*)

Type of loading	Moments
M, -M 6. External couples at top.	$M_{AD} = -M_{BC} = \dfrac{2M}{n+2}$ $M_{AB} = -M_{BA} = \dfrac{nM}{n+2}$ $M_{DA} = -M_{CB} = \dfrac{M}{n+2}$

TABLE 62

THREE-LEGGED BENT

Legs hinged at the bases
Lengths and cross-sections of two top members identical
Lengths and cross-sections of all legs identical
Any system of loads

M_D = moment of external loads on AD about D
M_E = moment of external loads on EF about E

$$n = \frac{hI_{AB}}{lI_{AD}}$$

Type of loading	Moments
 1. Any system of vertical loads on left-hand span	 $M_{AD} = +\frac{C_{AB}(10n+9) + C_{BA}(4n+3)}{4(n+1)(4n+3)}$ $M_{BA} = +\frac{(2n+3)[C_{AB}(2n+1) + C_{BA}(4n+3)]}{4(n+1)(4n+3)}$ $M_{BC} = -\frac{C_{AB} + C_{BA}}{2(n+1)}$ $M_{BF} = -\frac{C_{AB}(4n^2-3) + C_{BA}(4n+3)(2n+1)}{4(n+1)(4n+3)}$ $M_{FB} = +\frac{C_{AB}(2n+3) - C_{BA}(4n+3)}{4(n+1)(4n+3)}$ Values of C and H for various loads are given in Tables 2 and 3.
2. Any system of vertical loads on right-hand span.	$M_{AB} = +\frac{C_{BF}(4n+3) - C_{FB}(2n+3)}{4(n+1)(4n+3)}$ $M_{BA} = +\frac{C_{BF}(4n+3)(2n+1) + C_{FB}(4n^2-3)}{4(n+1)(4n+3)}$ $M_{BC} = +\frac{C_{BF} + C_{FB}}{2(n+1)}$ $M_{FB} = +\frac{C_{BF}(4n+3) + C_{FB}(10n+9)}{4(n+1)(4n+3)}$
 3. Any system of horizontal loads on left-hand leg.	$M_{AD} = +\frac{2nH_{AD}(16n+15) - M_D(4n+3)^2}{12(n+1)(4n+3)}$ $M_{BA} = -\frac{(2n+3)[2nH_{AD} - M_D(4n+3)]}{12(n+1)(4n+3)}$ $M_{BC} = -\frac{2nH_{AD} + M_D(2n+3)}{6(n+1)}$ $M_{FB} = +\frac{2nH_{AD}(8n+9) + M_D(4n+3)^2}{12(n+1)(4n+3)}$

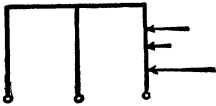
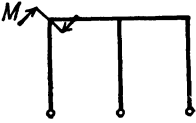
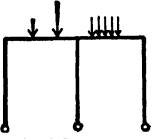
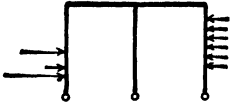
TABLE 62 (*Continued*)

Type of loading	Moments
4. Any system of horizontal loads on right-hand leg	$M_{AD} = +\frac{2nH_{FE}(8n+9) - M_E(4n+3)^2}{12(n+1)(4n+3)}$ $M_{BA} = -\frac{2nH_{FE}(10n+9) - M_E(4n+3)(2n+3)}{12(n+1)(4n+3)}$ $M_{BC} = +\frac{2nH_{FE} - M_E(2n+3)}{6(n+1)}$ $M_{FB} = +\frac{2nH_{FE}(16n+15) + M_E(4n+3)^2}{12(n+1)(4n+3)}$
5. External couple at upper left-hand corner.	$M_{AD} = +\frac{M(10n+9)}{4(n+1)(4n+3)}$ $M_{AB} = +\frac{M(16n^2+18n+3)}{4(n+1)(4n+3)}$ $M_{BA} = +\frac{M(2n+1)(2n+3)}{4(n+1)(4n+3)}$ $M_{BC} = -\frac{M}{2(n+1)}$ $M_{FB} = +\frac{M(2n+3)}{4(n+1)(4n+3)}$
6. Vertical loads on both spans.	$M_{AD} = \frac{C_{AB}(10n+9) + (C_{BA} - C_{BF})(4n+3) + C_{FB}(2n+3)}{4(n+1)(4n+3)}$ $M_{BA} = \frac{[2n+3][C_{AB}(2n+1) + C_{BA}(4n+3)] + C_{BF}(4n+3)(2n+1) + C_{FB}(4n^2-3)}{4(n+1)(4n+3)}$ $M_{BC} = \frac{-C_{BA} - C_{AB} + C_{BF} + C_{FB}}{2(n+1)}$ $M_{FB} = \frac{C_{AB}(2n+3) + (C_{BF} - C_{BA})(4n+3) + C_{FB}(10n+9)}{4(n+1)(4n+3)}$
7. Horizontal loads on both outside legs.	$M_{AD} = \frac{2nH_{AD}(16n+15) - (M_D+M_E)(4n+3)^2 + 2nH_{FE}(8n+9)}{12(n+1)(4n+3)}$ $M_{AB} = -M_{AD}$ $M_{BC} = \frac{2nH_{FE} - (M_E + M_D)(2n+3) - 2nH_{AD}}{6(n+1)}$ $M_{BF} = \frac{2nH_{AD}(10n+9) + M_D(4n+3)(2n+3)}{12(n+1)(4n+3)} + \frac{(2n+3)[2nH_{FE} + M_E(4n+3)]}{12(n+1)(4n+3)}$ $M_{BA} = -M_{BC} - M_{BF}$ $M_{FB} = \frac{2nH_{AD}(8n+9) + (M_D + M_E)(4n+3)^2 + 2nH_{FE}(16n+15)}{12(n+1)(4n+3)}$ $M_{FE} = -M_{FB}$

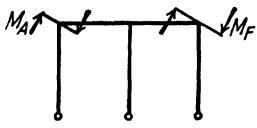
TABLE 62 (*Continued*)

Type of loading	Moments
8. External couples at both upper corners.	$M_{AD} = \dfrac{M_A(10n+9) - M_F(2n+3)}{4(n+1)(4n+3)}$ $M_{AB} = M_A - M_{AD}$ $M_{BA} = \dfrac{M_A(2n+1)(2n+3) - M_F(4n^2-3)}{4(n+1)(4n+3)}$ $M_{BC} = -\dfrac{M_A + M_F}{2(n+1)}$ $M_{BF} = -[M_{BA} + M_{BC}]$ $M_{FE} = \dfrac{M_F(10n+9) - M_A(2n+3)}{4(n+1)(4n+3)}$ $M_{FB} = M_F - M_{FE}$

Table 63

Three-legged Bent

Legs hinged at the bases
Lengths and cross-sections of two top members identical
Lengths and cross-sections of all legs identical
Vertical load on top
Load symmetrical about middle leg

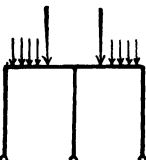
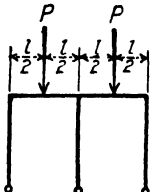
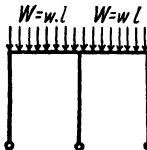
$$n = \frac{hI_{AB}}{lI_{AD}}$$

Type of loading	Moments
1. Any system of loads symmetrical about middle leg.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{3C_{AB}}{4n+3}$ $M_{BA} = -M_{BF} = \frac{2nC_{AB}}{4n+3} + C_{BA}$ $M_{BC} = 0$ Values of C for various loads are given in Tables 2 and 3
2. Two equal loads, one at center of each span.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{3Pl}{8(4n+3)}$ $M_{BA} = -M_{BF} = \frac{3Pl(2n+1)}{8(4n+3)}$ $M_{BC} = 0$
3. Uniform load.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{W}{4(4n+3)}$ $M_{BA} = -M_{BF} = \frac{Wl}{4}\frac{2n+1}{4n+3}$ $M_{BC} = 0$

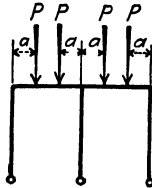
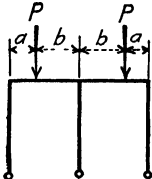
TABLE 63 (*Continued*)

Type of loading	Moments
4. Two equal loads symmetrically spaced on each span.	$M_{AD} = -M_{AB} = M_{FB} - M_{FE} = \dfrac{3Pa(l-a)}{l(4n+3)}$ $M_{BA} = -M_{BF} = \dfrac{3Pa(l-a)(2n+1)}{l(4n+3)}$ $M_{BC} = 0$
 5. Two equal loads, one on each span.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \dfrac{3Pab^2}{l^2(4n+3)}$ $M_{BA} = -M_{BF} = \dfrac{Pab}{l^2}\left[\dfrac{2nb}{4n+3} + a\right]$ $M_{BC} = 0$

TABLE 64

THREE-LEGGED BENT

Legs hinged at the bases
Lengths and cross-sections of two top members identical
Lengths and cross-sections of all legs identical
Horizontal loads on the two external legs
Load symmetrical about middle leg

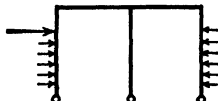
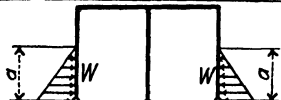
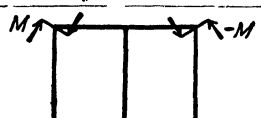
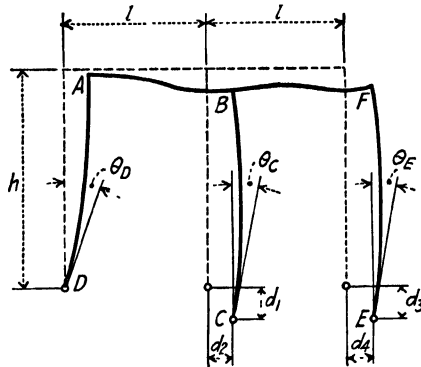
$$n = \frac{hI_{AB}}{lI_{AD}}$$

Type of loading	Moments
1. Any system of loads symmetrical about middle leg.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{4nH_{AD}}{4n+3}$ $M_{BF} = -M_{BA} = \frac{2nH_{AD}}{4n+3}$ $M_{BC} = 0$ Values of H for various loads are given in Tables 2 and 3.
2. Single load on each leg at mid-height.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{3nPh}{4(4n+3)}$ $M_{BF} = -M_{BA} = \frac{3nPh}{8(4n+3)}$ $M_{CB} = 0$
3. Uniform load.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{nWh}{2(4n+3)}$ $M_{BF} = -M_{BA} = \frac{nWh}{4(4n+3)}$ $M_{BC} = 0$
4. Single load at any point on each leg.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{2nPab(h+b)}{h^2(4n+3)}$ $M_{BF} = -M_{BA} = \frac{nPab(h+b)}{h^2(4n+3)}$ $M_{BC} = 0$
5. Hydraulic loads.	$M_{AD} = -M_{AB} = M_{FB} - M_{FE} = \frac{Wan(10h^2 - 3a^2)}{15h^2(4n+3)}$ $M_{BF} = -M_{BA} = \frac{Wan(10h^2 - 3a^2)}{30h^2(4n+3)}$ $M_{BC} = 0$
6. External couples at upper corners.	$M_{AD} = -M_{FE} = \frac{3M}{4n+3}$ $M_{AB} = -M_{FB} = M - M_{AD}$ $M_{BA} = -M_{BF} = \frac{2nM}{4n+3}$ $M_{BC} = 0$

TABLE 65

THREE-LEGGED BENT

Legs hinged at the bases
Lengths and cross-sections of two top members identical
Lengths and cross-sections of all legs identical
Settlement and sliding of foundations

$$K = \frac{I_{AB}}{l}$$

$$n = \frac{hI_{AB}}{lI_{AD}}$$

$$M_{AD} = +\frac{EK}{2hl} \cdot \frac{18h(n+1)(2d_1 - d_3) - l[d_2(8n+6) + d_4(8n+9)]}{(n+1)(4n+3)}$$

$$M_{BA} = -\frac{EK}{2hl} \cdot \frac{6h(n+1)(2n+3)(2d_1 - d_3) + l[d_2(8n+6) - d_4(10n+9)]}{(n+1)(4n+3)}$$

$$M_{BC} = +\frac{EK}{h} \cdot \frac{2d_2 - d_4}{n+1}$$

$$M_{BF} = +\frac{EK}{2hl} \cdot \frac{6h(n+1)(2n+3)(2d_1 - d_3) - l[d_2(8n+6) + d_4(2n+3)]}{(n+1)(4n+3)}$$

$$M_{FB} = +\frac{EK}{2hl} \cdot \frac{18h(n+1)(2d_1 - d_3) + l[d_2(8n+6) - d_4(16n+15)]}{(n+1)(4n+3)}$$

Table 66

Three-legged Bent

Legs fixed at the bases
Lengths and cross-sections of two top members identical
Lengths and cross-sections of all legs identical
Any system of loads
M_D = moment of external loads on AD about D
M_E = moment of external loads on EF about E

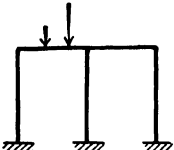
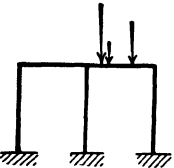
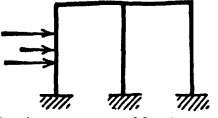
$$n = \frac{hI_{AB}}{lI_{AD}}$$

Type of loading	Moments
1. Any system of vertical loads on left-hand span	$M_{AD} = \frac{C_{AB}(11n^2+15n+2)+4nC_{BA}(n+1)}{2(n+1)(6n^2+9n+1)}$ $M_{BA} = \frac{nC_{AB}(3n^2+8n+4)+C_{BA}(n+1)(6n^2+13n+2)}{2(n+1)(6n^2+9n+1)}$ $M_{BC} = -\frac{7nC_{AB}+2C_{BA}(4n+1)}{2(6n^2+9n+1)}$ $M_{FB} = +\frac{nC_{AB}(n+3)-4nC_{BA}(n+1)}{2(n+1)(6n^2+9n+1)}$ $M_{DA} = +\frac{nC_{AB}(4n+5)+C_{BA}(n+1)(2n+1}{2(n+1)(6n^2+9n+1)}$ $M_{CB} = -\frac{C_{AB}(5n+1)+4nC_{BA}}{2(6n^2+9n+1)}$ $M_{EF} = -\frac{C_{AB}(2n^2+4n+1)-C_{BA}(n+1)(2n+1)}{2(n+1)(6n^2+9n+1)}$ Values of C for various loads are given in Tables 2 and 3
2. Any system of vertical loads on right-hand span.	$M_{AD} = -\frac{4nC_{BF}(n+1)-nC_{FB}(n+3)}{2(n+1)(6n^2+9n+1)}$ $M_{BA} = +\frac{nC_{BF}(n+1)(6n+5)+nC_{FB}(3n^2+n-3)}{2(n+1)(6n^2+9n+1)}$ $M_{BC} = +\frac{2C_{BF}(4n+1)+7nC_{FB}}{2(6n^2+9n+1)}$ $M_{FB} = +\frac{4nC_{BF}(n+1)+C_{FB}(11n^2+15n+2)}{2(n+1)(6n^2+9n+1)}$ $M_{DA} = -\frac{C_{BF}(n+1)(2n+1)-C_{FB}(2n^2+4n+1)}{2(n+1)(6n^2+9n+1)}$ $M_{CB} = +\frac{4nC_{BF}+C_{FB}(5n+1)}{2(6n^2+9n+1)}$ $M_{EF} = -\frac{C_{BF}(n+1)(2n+1)+nC_{FB}(4n+5)}{2(n+1)(6n^2+9n+1)}$
3. Any system of horizontal loads on left-hand leg.	$M_{AD} = \frac{nC_{AD}(10n^2+15n+3)-2n(M_D-C_{DA})(n+1)^2}{2(n+1)(6n^2+9n+1)}$ $M_{BA} = -\frac{n(n+2)[C_{AD}(2n+1)-(M_D-C_{DA})(n+1)]}{2(n+1)(6n^2+9n+1)}$ $M_{BC} = -\frac{nC_{AD}(2n-3)+2n(M_D-C_{DA})(n+2)}{2(6n^2+9n+1)}$ $M_{FB} = +\frac{nC_{AD}(2n^2+3n-1)+2n(M_D-C_{DA})(n+1)^2}{2(n+1)(6n^2+9n+1)}$ $M_{DA} = -\frac{C_{AD}(6n^3+27n^2+26n+2)}{6(n+1)(6n^2+9n+1)} - \frac{M_D(6n^2+9n+2)+C_{DA}(30n^2+45n+4)}{6(6n^2+9n+1)}$ $M_{CB} = -\frac{C_{AD}(6n^2-3n-1)+2(M_D-C_{DA})(3n^2+6n+1)}{6(6n^2+9n+1)}$ $M_{EF} = -\frac{C_{AD}(6n^3+9n^2-n-1)+(M_D-C_{DA})(n+1)(6n^2+9n+2)}{6(n+1)(6n^2+9n+1)}$

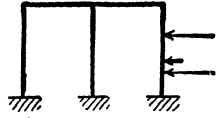
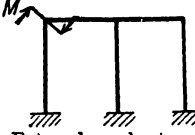
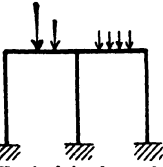
Table 66 (*Continued*)

Type of loading	Moments
4. Any system of horizontal loads on right-hand leg	$M_{AD} = \frac{nC_{FE}(2n^2+3n-1) - 2n(M_E + C_{EF})(n+1)^2}{2(n+1)(6n^2+9n+1)}$ $M_{BA} = -\frac{nC_{FE}(4n^2+4n-1) - n(M_E + C_{EF})(n+1)(n+2)}{2(n+1)(6n^2+9n+1)}$ $M_{BC} = \frac{nC_{FE}(2n-3) - 2n(M_E + C_{EF})(n+2)}{2(6n^2+9n+1)}$ $M_{FB} = \frac{nC_{FE}(10n^2+15n+3) + 2n(M_E + C_{EF})(n+1)^2}{2(n+1)(6n^2+9n+1)}$ $M_{DA} = \frac{C_{FE}(6n^3+9n^2-n-1) - (M_E + C_{EF})(n+1)(6n^2+9n+2)}{6(n+1)(6n^2+9n+1)}$ $M_{CB} = \frac{C_{FE}(6n^2-3n-1) - 2(M_E + C_{EF})(3n^2+6n+1)}{6(6n^2+9n+1)}$ $M_{EF} = \frac{C_{FE}(6n^3+27n^2+26n+2)}{6(n+1)(6n^2+9n+1)}$ $\frac{M_F(6n^2+9n+2) - C_{EF}(30n^2+45n+4)}{6(6n^2+9n+1)}$
5. External couple at upper left-hand corner.	$M_{AD} = \frac{M(11n^2+15n+2)}{2(n+1)(6n^2+9n+1)}$ $M_{BA} = \frac{Mn(n+2)(3n+2)}{2(n+1)(6n^2+9n+1)}$ $M_{BC} = -\frac{7Mn}{2(6n^2+9n+1)}$ $M_{FB} = \frac{Mn(n+3)}{2(n+1)(6n^2+9n+1)}$ $M_{DA} = +\frac{Mn(4n+5)}{2(n+1)(6n^2+9n+1)}$ $M_{CB} = -\frac{M(5n+1)}{2(6n^2+9n+1)}$ $M_{EF} = -\frac{M(2n^2+4n+1)}{2(n+1)(6n^2+9n+1)}$
6 Vertical loads on both spans.	$M_{AD} = +\frac{C_{AB}(11n^2+15n+2) + 4nC_{BA}(n+1)}{2(n+1)(6n^2+9n+1)} - \frac{4nC_{BF}(n+1) - nC_{FB}(n+3)}{2(n+1)(6n^2+9n+1)}$ $M_{BA} = +\frac{nC_{AB}(3n^2+8n+4) + C_{BA}(n+1)(6n^2+13n+2)}{2(n+1)(6n^2+9n+1)} + \frac{nC_{BF}(n+1)(6n+5) + nC_{FB}(3n^2+n-3)}{2(n+1)(6n^2+9n+1)}$ $M_{BC} = -\frac{7nC_{AB} + 2C_{BA}(4n+1)}{2(6n^2+9n+1)} + \frac{2C_{BF}(4n+1) + 7nC_{FB}}{2(6n^2+9n+1)}$ $M_{FB} = +\frac{nC_{AB}(n+3) - 4nC_{BA}(n+1)}{2(n+1)(6n^2+9n+1)} + \frac{4nC_{BF}(n+1) + C_{FB}(11n^2+15n+2)}{2(n+1)(6n^2+9n+1)}$ $M_{DA} = +\frac{nC_{AB}(4n+5) + C_{BA}(n+1)(2n+1)}{2(n+1)(6n^2+9n+1)} - \frac{C_{BF}(n+1)(2n+1) - C_{FB}(2n^2+4n+1)}{2(n+1)(6n^2+9n+1)}$ $M_{CB} = -\frac{C_{AB}(5n+1) + 4nC_{BA}}{2(6n^2+9n+1)} + \frac{4nC_{BF} + C_{FB}(5n+1)}{2(6n^2+9n+1)}$ $M_{EF} = -\frac{C_{AB}(2n^2+4n+1) - C_{BA}(n+1)(2n+1)}{2(n+1)(6n^2+9n+1)} - \frac{C_{BF}(n+1)(2n+1) + nC_{FB}(4n+5)}{2(n+1)(6n^2+9n+1)}$

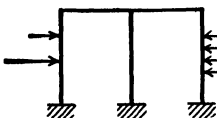
TABLE 66 (*Continued*)

Type of loading	Moments
7. Horizontal loads on both outside legs.	$M_{AD} = +\frac{nC_{AD}(10n^2 + 15n + 3) - 2n(M_D - C_{DA})(n + 1)^2}{2(n + 1)(6n^2 + 9n + 1)} + \frac{nC_{FE}(2n^2 + 3n - 1) - 2n(M_E + C_{EF})(n + 1)^2}{2(n + 1)(6n^2 + 9n + 1)}$ $M_{BA} = -\frac{n(n + 2)[C_{AD}(2n + 1) - (M_D - C_{DA})(n + 1)]}{2(n + 1)(6n^2 + 9n + 1)} - \frac{nC_{FE}(4n^2 + 4n - 1) - n(M_E + C_{EF})(n + 1)(n + 2)}{2(n + 1)(6n^2 + 9n + 1)}$ $M_{BC} = -\frac{nC_{AD}(2n - 3) + 2n(M_D - C_{DA})(n + 2)}{2(6n^2 + 9n + 1)} + \frac{nC_{FE}(2n - 3) - 2n(M_E + C_{EF})(n + 2)}{2(6n^2 + 9n + 1)}$ $M_{FB} = +\frac{nC_{AD}(2n^2 +$ [illegible] $n - 1) + 2n(M_D - C_{DA})(n + 1)^2}{2(n + 1)(6n^2 + 9n + 1)} + \frac{nC_{FE}(10n^2 + 15n + 3) + 2n(M_E + C_{EF})(n + 1)^2}{2(n + 1)(6n^2 + 9n + 1)}$ $M_{DA} = -\frac{C_{AD}(6n^3 + 27n^2 + 26n + 2)}{6(n + 1)(6n^2 + 9n + 1)} - \frac{M_D(6n^2 + 9n + 2) + C_{DA}(30n^2 + 45n + 4)}{6(6n^2 + 9n + 1)} + \frac{C_{FE}(6n^3 + 9n^2 - n - 1) - (M_E + C_{EF})(n + 1)(6n^2 + 9n + 2)}{6(n + 1)(6n^2 + 9n + 1)}$ $M_{CB} = -\frac{C_{AD}(6n^2 - 3n - 1) + 2(M_D - C_{DA})(3n^2 + 6n + 1)}{6(6n^2 + 9n + 1)} + \frac{C_{FE}(6n^2 - 3n - 1) - 2(M_E + C_{EF})(3n^2 + 6n + 1)}{6(6n^2 + 9n + 1)}$ $M_{EF} = -\frac{C_{AD}(6n^3 + 9n^2 - n - 1) + (M_D - C_{DA})(n + 1)(6n^2 + 9n + 2)}{6(n + 1)(6n^2 + 9n + 1)} + \frac{C_{FE}(6n^3 + 27n^2 + 26n + 2)}{6(n + 1)(6n^2 + 9n + 1)} - \frac{M_E(6n^2 + 9n + 2) - C_{EF}(30^2 + 45n + 4)}{6(6n^2 + 9n + 1)}$

TABLE 67

THREE-LEGGED BENT

Legs fixed at the bases
Bent symmetrical about vertical center line
Lengths and cross-sections of all legs identical
Vertical load on top
Load symmetrical about middle leg

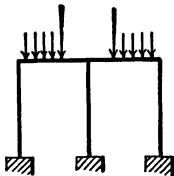
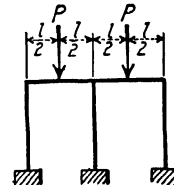
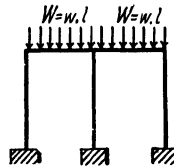
$$n = \frac{hI_{AB}}{lI_{AD}}$$

Type of loading	Moments
 1. Any system of vertical loads symmetrical about middle leg.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{C_{AB}}{n+1}$ $M_{BA} = -M_{BF} = \frac{nC_{AB}}{2(n+1)} + C_{BA}$ $M_{BC} = 0$ $M_{DA} = -M_{EF} = \frac{C_{AB}}{2(n+1)}$ Values of C for various loads are given in Tables 2 and 3.
2. Two equal loads, one at center of each span.	 $M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{Pl}{8(n+1)}$ $M_{BA} = -M_{BF} = \frac{Pl(3n+2)}{16(n+1)}$ $M_{BC} = 0$ $M_{DA} = -M_{EF} = \frac{Pl}{16(n+1)}$
 3. Uniform load.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{Wl}{12(n+1)}$ $M_{BA} = -M_{BF} = \frac{Wl(3n+2)}{24(n+1)}$ $M_{BC} = 0$ $M_{DA} = -M_{EF} = \frac{Wl}{24(n+1)}$

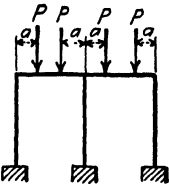
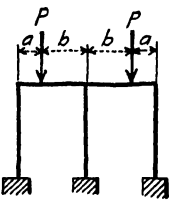
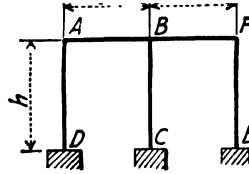
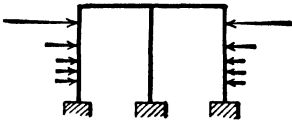
TABLE 67 (*Continued*)

Type of loading	Moments
4. Two equal loads symmetrically spaced on each span.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{Pa(l-a)}{l(n+1)}$ $M_{BA} = -M_{BF} = \frac{Pa(l-a)(3n+2)}{2l(n+1)}$ $M_{BC} = 0$ $M_{DA} = -M_{EF} = \frac{Pa(l-a)}{2l(n+1)}$
5. Two equal loads, one on each span.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{Pab^2}{l^2(n+1)}$ $M_{BA} = -M_{BF} = \frac{Pab}{l^2}\left[\frac{nb}{2(n+1)} + a\right]$ $M_{BC} = 0$ $M_{DA} = -M_{EF} = \frac{Pab^2}{2l^2(n+1)}$

TABLE 68

THREE-LEGGED BENT

Legs fixed at the bases
Bent symmetrical about vertical center line
Lengths and cross-sections of all legs identical
Horizontal loads on both external legs
Load symmetrical about middle leg

$$n = \frac{hI_{AB}}{lI_{AD}}$$

Type of loading	Moments
1. Any system of horizontal loads symmetrical about middle leg.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{nC_{AD}}{n+1}$ $M_{BF} = -M_{BA} = \frac{nC_{AD}}{2(n+1)}$ $M_{BC} = M_{CB} = 0$ $M_{EF} = -M_{DA} = \frac{C_{AD}}{2(n+1)} + C_{DA}$ Values of C for various loads are given in Tables 2 and 3.
2. Single load on each leg at mid-height.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{nPh}{8(n+1)}$ $M_{BF} = -M_{BA} = \frac{nPh}{16(n+1)}$ $M_{BC} = M_{CB} = 0$ $M_{EF} = -M_{DA} = \frac{Ph(2n+3)}{16(n+1)}$
3. Uniform load.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{nWh}{12(n+1)}$ $M_{BF} = -M_{BA} = \frac{nWh}{24(n+1)}$ $M_{BC} = M_{CB} = 0$ $M_{EF} = -M_{DA} = \frac{Wh(2n+3)}{24(n+1)}$
4. Single load at any point on each leg.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \frac{nPab^2}{h^2(n+1)}$ $M_{BF} = -M_{BA} = \frac{nPab^2}{2h^2(n+1)}$ $M_{BC} = M_{CB} = 0$ $M_{EF} = -M_{DA} = \frac{Pab}{h^2}\left[\frac{b}{2(n+1)} + a\right]$

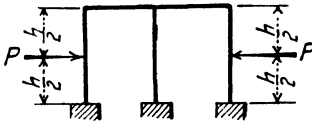
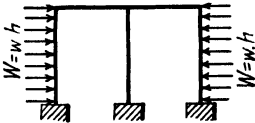
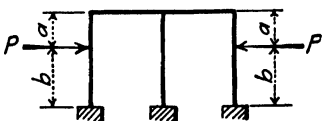



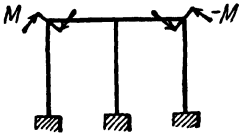
TABLE 68 (*Continued*)

Type of loading	Moments
5. Hydraulic loads.	$M_{AD} = -M_{AB} = M_{FB} = -M_{FE} = \dfrac{nWa^2(5h - 3a)}{30h^2(n + 1)}$ $M_{BF} = -M_{BA} = \dfrac{nWa^2(5h - 3a)}{60h^2(n + 1)}$ $M_{BC} = M_{CB} = 0$ $M_{EF} = -M_{DA} = \dfrac{Wa}{30h^2}\left[\dfrac{5ah - 3a^2}{2(n + 1)} + (3a^2 - 10ah + 10h^2)\right]$
6. External couples at upper corners.	$M_{AD} = -M_{FE} = \dfrac{M}{n + 1}$ $M_{AB} = -M_{FB} = \dfrac{Mn}{n + 1}$ $M_{BA} = -M_{BF} = \dfrac{Mn}{2(n + 1)}$ $M_{BC} = M_{CB} = 0$ $M_{DA} = -M_{EF} = \dfrac{M}{2(n + 1)}$

TABLE 69

RECTANGULAR FRAME

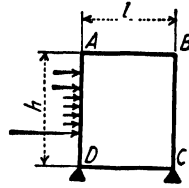
Cross-sections of all sides different
Horizontal load on one vertical side

M_D = moment of external load about D

$$K = \frac{I_{AB}}{l}$$

$$n = \frac{hK}{I_{AD}}$$

$$s = \frac{hK}{I_{BC}}$$

$$p = \frac{Kl}{I_{DC}}$$

$$M_{AB} = \frac{1}{\Delta}\left\{ M_D(3s^2n + 5nps + 2s^2p + 2sp^2 + 6ns + 6pn + 5ps + 3p^2) - C_{AD}n(6sn + 2pn + 3s^2 + 17ps + 2n + 5s + 11p + 2p^2) - C_{DA}n(3s^2 + 12ps + 6s + 10p + p^2) \right\}$$

$$M_{BC} = \frac{1}{\Delta}\left\{ -M_D(3sn^2 + 2n^2p + 5nps + 2np^2 + 6ns + 5pn + 6ps + 3p^2) - C_{AD}n(3ns + 2pn + 5ps - n - 7p - 4s + 2p^2) + C_{DA}n(3ns + 3pn - 3ps + 8p + 6s - p^2) \right\}$$

$$M_{CD} = \frac{1}{\Delta}\left\{ M_D(3sn^2 + 2n^2 + 6nps + 5pn + 5sn + 6ps + 2n + 3p) + C_{AD}n(3ns + 6ps + 3n + 8p - 3s - 1) - C_{DA}n(3ns - 4ps + 2n - 7p - pn + 5s + 2) \right\}$$

$$M_{DA} = \frac{1}{\Delta}\left\{ -M_D(5ns + 6nps + 3ns^2 + 2s^2 + 5ps + 6pn + 2s + 3p) - C_{AD}n(3s^2 + 6ps + 12s + 10p + 1) - C_{DA}n(3s^2 + 5ps + 17s + 11p + 6ns + 2pn + 2n + 2) \right\}$$

in which

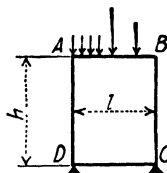
$$\Delta = 22(spn + sp + sn + np) + 2(sp^2 + s^2p + n^2p + p^2n + s^2 + s + n^2 + n) + 6(sn^2 + s^2n + p^2 + p)$$

Values of C for various loads are given in Tables 2 and 3.

Table 70

Rectangular Frame

Cross-sections of all sides different
Vertical load on top

$$K = \frac{I_{AB}}{l}$$
$$n = \frac{Kh}{I_{AD}}$$
$$s = \frac{Kh}{I_{BC}}$$
$$p = \frac{Kl}{I_{DC}}$$

Values of C for various loads are given in Tables 2 and 3.

$$M_{AB} = -\frac{1}{\Delta}\Big\{C_{BA}(10ns + s^2 + 12ps + 6pn + 3p^2) + C_{AB}(11ns + 2s^2 + 2s + 2n + 17ps + 5pn + 3p^2 + 6p)\Big\}$$

$$M_{BC} = -\frac{1}{\Delta}\Big\{C_{BA}(11ns + 2n^2 + 2n + 2s + 17pn + 5ps + 3p^2 + 6p) + C_{AB}(10ns + n^2 + 12pn + 6ps + 3p^2)\Big\}$$

$$M_{CD} = \frac{1}{\Delta}\Big\{C_{BA}(7ns - 2n^2 - 2n + s - 5pn + 4ps - 3p) + C_{AB}(8ns - n^2 + 3n - 3pn + 6ps + 3p)\Big\}$$

$$M_{DA} = \frac{1}{\Delta}\Big\{C_{BA}(8ns - s^2 + 3s - 3ps + 6pn + 3p) + C_{AB}(7ns - 2s^2 - 2s + n - 5ps + 4pn - 3p)\Big\}$$

in which

$$\Delta = 22(spn + sp + sn + np) + 2(sp^2 + s^2p + n^2p + np^2 + s^2 + s + n^2 + n) + 6(sn^2 + s^2n + p^2 + p)$$

If the load is symmetrical about the center of AB, that is, if $C_{AB} = C_{BA}$,

$$M_{AB} = -\frac{C_{AB}}{\Delta}\Big[21ns + 3s^2 + 2s + 2n + 29ps + 11pn + 6p + 6p^2\Big]$$

$$M_{BC} = -\frac{C_{AB}}{\Delta}\Big[21ns + 3n^2 + 2n + 2s + 29pn + 11ps + 6p + 6p^2\Big]$$

$$M_{CD} = \frac{C_{AB}}{\Delta}\Big[15ns - 3n^2 + n + s - 8pn + 10ps\Big]$$

$$M_{DA} = \frac{C_{AB}}{\Delta}\Big[15ns - 3s^2 + s + n - 8ps + 10pn\Big]$$

Table 71

Rectangular Frame

Cross-sections of all sides different
Single horizontal load at top

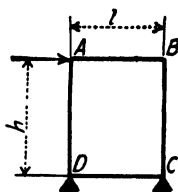
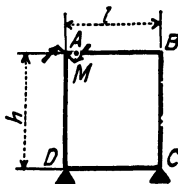
$$K = \frac{I_{AB}}{l}$$
$$n = \frac{Kh}{I_{AD}}$$
$$s = \frac{Kh}{I_{BC}}$$
$$p = \frac{Kl}{I_{DC}}$$

$$M_{AB} = \frac{Ph}{\Delta}\left(3s^2n + 5nps + 2s^2p + 2sp^2 + 6ns + 6pn + 5ps + 3p^2\right)$$
$$M_{BC} = -\frac{Ph}{\Delta}\left(3n^2s + 5nps + 2n^2p + 2np^2 + 6ns + 6ps + 5pn + 3p^2\right)$$
$$M_{CD} = \frac{Ph}{\Delta}\left(3n^2s + 6nps + 5ns + 6ps + 5pn + 2n + 3p + 2n^2\right)$$
$$M_{DA} = -\frac{Ph}{\Delta}\left(3ns^2 + 6nps + 5ns + 6pn + 5ps + 2s + 3p + 2s^2\right)$$
$$\Delta = 22(spn + sp + sn + np) + 2(s^2p + sp^2 + n^2p + p^2n + s^2 + s + n^2 + n) + 6(sn^2 + s^2n + p^2 + p)$$

Table 72

Rectangular Frame

Cross-sections of all sides different
External couple at one corner

$$K = \frac{I_{AB}}{l}$$
$$n = \frac{Kh}{I_{AD}}$$
$$s = \frac{Kh}{I_{BC}}$$
$$p = \frac{Kl}{I_{DC}}$$

$$M_{AB} = -\frac{M}{\Delta}\left(11ns + 2s^2 + 2s + 2n + 17ps + 5pn + 6p + 3p^2\right) + M$$
$$M_{BC} = -\frac{M}{\Delta}\left(n^2 + 10ns + 12pn + 6ps + 3p^2\right)$$
$$M_{CD} = +\frac{M}{\Delta}\left(3p - n^2 - 3pn + 8ns + 6ps + 3n\right)$$
$$M_{DA} = -\frac{M}{\Delta}\left(-7ns + 2s^2 + 5ps - 4pn + 2s + 3p - n\right)$$
$$M_{AD} = +\frac{M}{\Delta}\left(11ns + 2s^2 + 2s + 2n + 17ps + 5pn + 6p + 3p^2\right)$$

in which

$$\Delta = 22(pns + sp + sn + np) + 2(sp^2 + s^2p + n^2p + p^2n + s^2 + s + n^2 + n) + 6(sn^2 + s^2n + p^2 + p)$$

TABLE 73

RECTANGULAR FRAME

Frame symmetrical about vertical center line
Horizontal load on vertical side

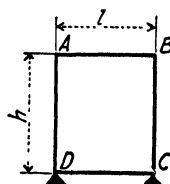
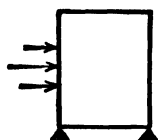
M_D = moment of external load about D

$\alpha = n^2 + 2pn + 2n + 3p$

$\beta = 6n + p + 1$

$K = \dfrac{I_{AB}}{l}$

$n = \dfrac{Kh}{I_{AD}}$

$p = \dfrac{Kl}{I_{DC}}$

Type of loading	Moments
 1. Any system of horizontal loads on left-hand side.	$M_{AB} = -\frac{1}{2}\left\{C_{AD}n\left[\frac{n+2p}{\alpha}+\frac{3}{\beta}\right] + C_{DA}n\left[\frac{p}{\alpha}+\frac{3}{\beta}\right] - M_D\frac{3n+p}{\beta}\right\}$ $M_{BC} = -\frac{1}{2}\left\{C_{AD}n\left[\frac{n+2p}{\alpha}-\frac{3}{\beta}\right] + C_{DA}n\left[\frac{p}{\alpha}-\frac{3}{\beta}\right] + M_D\frac{3n+p}{\beta}\right\}$ $M_{CD} = -\frac{1}{2}\left\{C_{AD}n\left[\frac{1}{\alpha}-\frac{3}{\beta}\right] + C_{DA}n\left[\frac{n+2}{\alpha}-\frac{3}{\beta}\right] - M_D\frac{3n+1}{\beta}\right\}$ $M_{DA} = -\frac{1}{2}\left\{C_{AD}n\left[\frac{1}{\alpha}+\frac{3}{\beta}\right] + C_{DA}n\left[\frac{n+2}{\alpha}+\frac{3}{\beta}\right] + M_D\frac{3n+1}{\beta}\right\}$ If load is symmetrical about middle of AD $C_{AD} = C_{DA}$, and $M_{AB} = -\frac{1}{2}\left\{nC_{AD}\left[\frac{n+3p}{\alpha}+\frac{6}{\beta}\right] - M_D\frac{3n+p}{\beta}\right\}$ $M_{BC} = -\frac{1}{2}\left\{nC_{AD}\left[\frac{n+3p}{\alpha}-\frac{6}{\beta}\right] + M_D\frac{3n+p}{\beta}\right\}$ $M_{CD} = -\frac{1}{2}\left\{nC_{AD}\left[\frac{n+3}{\alpha}-\frac{6}{\beta}\right] - M_D\frac{3n+1}{\beta}\right\}$ $M_{DA} = -\frac{1}{2}\left\{nC_{AD}\left[\frac{n+3}{\alpha}+\frac{6}{\beta}\right] + M_D\frac{3n+1}{\beta}\right\}$ Values of C for various loads are given in Tables 2 and 3.
 2. **Single load at mid-height.**	$M_{AB} = -\frac{Ph}{16}\left[\frac{n^2+3pn}{\alpha} - \frac{6n+4p}{\beta}\right]$ $M_{BC} = -\frac{Ph}{16}\left[\frac{n^2+3pn}{\alpha} + \frac{6n+4p}{\beta}\right]$ $M_{CD} = -\frac{Ph}{16}\left[\frac{n^2+3n}{\alpha} - \frac{18n+4}{\beta}\right]$ $M_{DA} = -\frac{Ph}{16}\left[\frac{n^2+3n}{\alpha} + \frac{18n+4}{\beta}\right]$

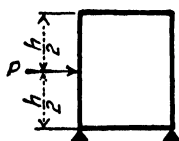



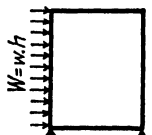
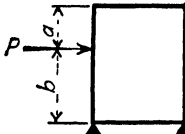
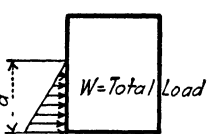
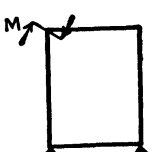
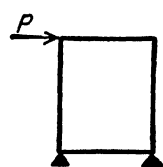
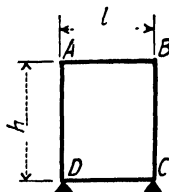
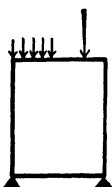
TABLE 73 (*Continued*)

Type of loading	Moments
3. Uniform load	$M_{AB} = -\frac{Wh}{24}\left[\frac{n^2 + 3pn}{\alpha} - \frac{12n + 6p}{\beta}\right]$ $M_{BC} = -\frac{Wh}{24}\left[\frac{n^2 + 3pn}{\alpha} + \frac{12n + 6p}{\beta}\right]$ $M_{CD} = -\frac{Wh}{24}\left[\frac{n^2 + 3n}{\alpha} - \frac{24n + 6}{\beta}\right]$ $M_{DA} = -\frac{Wh}{24}\left[\frac{n^2 + 3n}{\alpha} + \frac{24n + 6}{\beta}\right]$
4. Single load at any point.	$M_{AB} = -\frac{Pb}{2}\left\{\frac{an}{h^2}\left[\frac{bn + bp + hp}{\alpha} + \frac{3h}{\beta}\right] - \frac{3n + p}{\beta}\right\}$ $M_{BC} = -\frac{Pb}{2}\left\{\frac{an}{h^2}\left[\frac{bn + bp + hp}{\alpha} - \frac{3h}{\beta}\right] + \frac{3n + p}{\beta}\right\}$ $M_{CD} = -\frac{Pb}{2}\left\{\frac{an}{h^2}\left[\frac{na + a + h}{\alpha} - \frac{3h}{\beta}\right] - \frac{3n + 1}{\beta}\right\}$ $M_{DA} = -\frac{Pb}{2}\left\{\frac{an}{h^2}\left[\frac{na + a + h}{\alpha} + \frac{3h}{\beta}\right] + \frac{3n + 1}{\beta}\right\}$
5. Hydraulic load.	$M_{AB} = -\frac{Wan}{60h^2}\left[\frac{5h(na + 2ph) - 3a^2(n + p)}{\alpha} + \frac{15h(2h - a)}{\beta}\right] + \frac{Wa}{6}\frac{3n + p}{\beta}$ $M_{BC} = -\frac{Wan}{60h^2}\left[\frac{5h(na + 2ph) - 3a^2(n + p)}{\alpha} - \frac{15h(2h - a)}{\beta}\right] - \frac{Wa}{6}\frac{3n + p}{\beta}$ $M_{CD} = -\frac{Wan}{60h^2}\left[\frac{3a^2(n + 1) + 10nh(h - a) + 5h(4h - 3a)}{\alpha} - \frac{15h(2h - a)}{\beta}\right] + \frac{Wa}{6}\frac{3n + 1}{\beta}$ $M_{DA} = -\frac{Wan}{60h^2}\left[\frac{3a^2(n + 1) + 10nh(h - a) + 5h(4h - 3a)}{\alpha} + \frac{15h(2h - a)}{\beta}\right] - \frac{Wa}{6}\frac{3n + 1}{\beta}$
6 External couple at upper left-hand corner	$M_{AB} = \frac{M}{2}\left\{1 + \frac{n(n + 2p)}{\alpha} - \frac{1}{\beta}\right\}$ $M_{BC} = \frac{M}{2}\left\{\frac{n(n + 2p)}{\alpha} - \frac{6n + p}{\beta}\right\}$ $M_{CD} = \frac{M}{2}\left\{\frac{n}{\alpha} + \frac{1}{\beta}\right\}$ $M_{DA} = \frac{M}{2}\left\{\frac{n}{\alpha} - \frac{1}{\beta}\right\}$ $M_{AD} = \frac{M}{2}\left\{1 - \frac{n(n + 2p)}{\alpha} + \frac{1}{\beta}\right\}$
7. Single horizontal load at top.	$M_{AB} = -M_{BC} = \frac{Ph}{2}\frac{3n + p}{\beta}$ $M_{CD} = -M_{DA} = \frac{Ph}{2}\frac{3n + 1}{\beta}$

TABLE 74

RECTANGULAR FRAME

Frame symmetrical about vertical center line
Vertical load on top

$$\alpha = n^2 + 2pn + 2n + 3p$$
$$\beta = 6n + p + 1$$
$$K = \frac{I_{AB}}{l}$$
$$n = \frac{Kh}{I_{AD}}$$
$$p = \frac{Kl}{I_{DC}}$$

Type of loading	Moments
 1. Any system of loads.	$M_{AB} = -\frac{1}{2}\left\{ C_{BA}\left[\frac{2n+3p}{\alpha} - \frac{1}{\beta}\right] + C_{AB}\left[\frac{2n+3p}{\alpha} + \frac{1}{\beta}\right]\right\}$ $M_{BC} = -\frac{1}{2}\left\{ C_{BA}\left[\frac{2n+3p}{\alpha} + \frac{1}{\beta}\right] + C_{AB}\left[\frac{2n+3p}{\alpha} - \frac{1}{\beta}\right]\right\}$ $M_{CD} = \frac{1}{2}\left\{ C_{BA}\left[\frac{n}{\alpha} - \frac{1}{\beta}\right] + C_{AB}\left[\frac{n}{\alpha} + \frac{1}{\beta}\right]\right\}$ $M_{DA} = \frac{1}{2}\left\{ C_{BA}\left[\frac{n}{\alpha} + \frac{1}{\beta}\right] + C_{AB}\left[\frac{n}{\alpha} - \frac{1}{\beta}\right]\right\}$ Values of C for various loads are given in Tables 2 and 3
2. Any system of loads symmetrical about vertical center line.	$M_{AB} = M_{BC} = -C_{AB}\left[\frac{2n+3p}{\alpha}\right]$ $M_{CD} = M_{DA} = C_{AB}\left[\frac{n}{\alpha}\right]$
 3. Single load at center.	$M_{AB} = M_{BC} = -\frac{Pl}{8}\frac{2n+3p}{\alpha}$ $M_{CD} = M_{DA} = \frac{Pl}{8}\frac{n}{\alpha}$

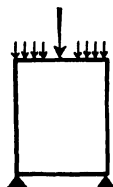
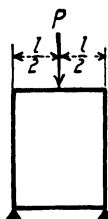



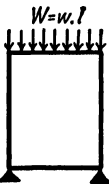
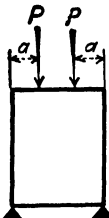
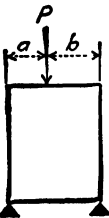
TABLE 74 (*Continued*)

Type of loading	Moments
$W = w.l$ 4. Uniform load.	$M_{AB} = M_{BC} = -\frac{Wl}{12}\frac{2n + 3p}{\alpha}$ $M_{CD} = M_{DA} = \frac{Wl}{12}\frac{n}{\alpha}$
P P a a 5. Two equal loads symmetrically spaced.	$M_{AB} = M_{BC} = -\frac{Pa}{l}(l - a)\frac{2n + 3p}{\alpha}$ $M_{CD} = M_{DA} = \frac{Pa}{l}(l - a)\frac{n}{\alpha}$
P a b 6. Single load at any point.	$M_{AB} = -\frac{Pab}{2l^2}\left[\frac{l(2n + 3p)}{\alpha} + \frac{(b - a)}{\beta}\right]$ $M_{BC} = -\frac{Pab}{2l^2}\left[\frac{l(2n + 3p)}{\alpha} - \frac{(b - a)}{\beta}\right]$ $M_{CD} = \frac{Pab}{2l^2}\left[\frac{nl}{\alpha} + \frac{b - a}{\beta}\right]$ $M_{DA} = \frac{Pab}{2l^2}\left[\frac{nl}{\alpha} - \frac{b - a}{\beta}\right]$

Table 75

Rectangular Frame

Frame and load symmetrical about vertical center line
Horizontal load on each vertical side

$\alpha = n^2 + 2pn + 2n + 3p$

$\beta = 6n + p + 1$

$K = \frac{I_{AB}}{l}$

$n = \frac{Kh}{I_{AD}}$

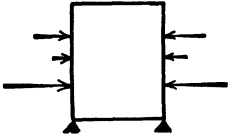
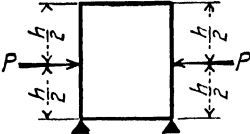
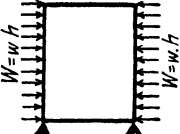
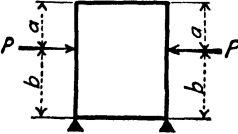
$p = \frac{Kh}{I_{DC}}$

Type of loading	Moments
1. Any system of horizontal loads symmetrical about vertical center line of frame.	$M_{AB} = M_{BC} = -\frac{n}{\alpha}[C_{AD}(n + 2p) + pC_{DA}]$ $M_{CD} = M_{DA} = -\frac{n}{\alpha}[C_{AD} + C_{DA}(n + 2)]$
2. Single load at mid-height of each vertical side.	$M_{AB} = M_{BC} = -\frac{Phn}{8}\left(\frac{n + 3p}{\alpha}\right)$ $M_{CD} = M_{DA} = -\frac{Phn}{8}\left(\frac{n + 3}{\alpha}\right)$
3. Uniform loads on both vertical sides.	$M_{AB} = M_{BC} = -\frac{Whn}{12}\left(\frac{n + 3p}{\alpha}\right)$ $M_{CD} = M_{DA} = -\frac{Whn}{12}\left(\frac{n + 3}{\alpha}\right)$
4. Single load at any point.	$M_{AB} = M_{BC} = -\frac{Pabn}{\alpha h^2}(nb + pb + ph)$ $M_{CD} = M_{DA} = -\frac{Pabn}{\alpha h^2}(a + an + h)$

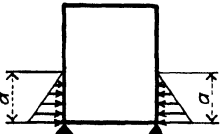
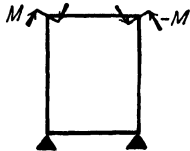
TABLE 75 (*Continued*)

Type of loading	Moments
W= Total Load on one side 5. Hydraulic load.	$M_{AB} = M_{BC} = \frac{Wan}{30h^2\alpha}[5h(an + 2ph) - 3a^2(n + p)]$ $M_{CD} = M_{DA} = \frac{Wan}{30h^2\alpha}[3a^2(n + 1) + 10nh(h - a) + 5h(4h - 3a)]$
6. External couples at both upper corners.	$M_{AB} = -M_{BA} = \frac{Mn}{\alpha}(n + 2p)$ $M_{AD} = -M_{BC} = \frac{M}{\alpha}(2n + 3p)$ $M_{CD} = M_{DA} = \frac{Mn}{\alpha}$

TABLE 76

RECTANGULAR FRAME

All sides identical in length and cross-section
Vertical load on top

$K = \frac{I_{AB}}{l} = \frac{I_{AD}}{h}$

$l = h$

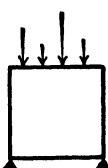
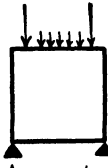
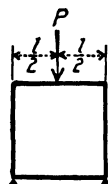
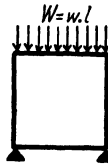
$I_{AB} = I_{BC} = I_{CD} = I_{AD}$

Type of loading	Moments
1. Any system of loads.	$M_{AB} = -\frac{1}{8}[2C_{BA} + 3C_{AB}]$ $M_{BC} = -\frac{1}{8}[3C_{BA} + 2C_{AB}]$ $M_{CD} = \frac{1}{8}C_{AB}$ $M_{DA} = \frac{1}{8}C_{BA}$ Values of C for various loads are given in Table 2 and 3.
2. Any system of loads symmetrical about vertical center line.	$M_{AB} = M_{BC} = -\frac{5}{8}C_{AB}$ $M_{CD} = M_{DA} = \frac{1}{8}C_{AB}$
P $\frac{l}{2}$ $\frac{l}{2}$ 3. Single load at center.	$M_{AB} = M_{BC} = -\frac{5Pl}{64}$ $M_{CD} = M_{DA} = \frac{Pl}{64}$
$W = w.l$ 4. Uniform load.	$M_{AB} = M_{BC} = -\frac{5Wl}{96}$ $M_{CD} = M_{DA} = \frac{Wl}{96}$

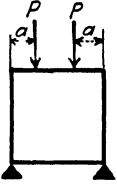
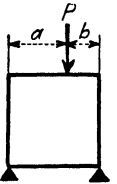
TABLE 76 (*Continued*)

Type of loading	Moments
5. Two equal loads symmetrically spaced.	$M_{AB} = M_{BC} = -\frac{5Pa(l-a)}{8l}$ $M_{CD} = M_{DA} = \frac{Pa(l-a)}{8l}$
6. Single load at any point.	$M_{AB} = -\frac{Pab(2l+b)}{8l^2}$ $M_{BC} = -\frac{Pab(2l+a)}{8l^2}$ $M_{CD} = \frac{Pab^2}{8l^2}$ $M_{DA} = \frac{Pa^2b}{8l^2}$

TABLE 77

RECTANGULAR FRAME

All sides identical in length and cross-section
Horizontal load on one vertical leg

M_D = moment of external load about D

$K = \frac{I_{AB}}{l} = \frac{I_{AD}}{h}$

$l = h$

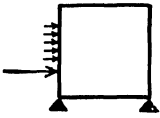
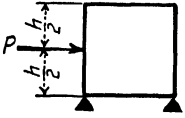
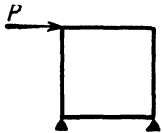
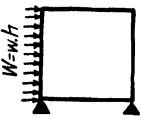
$I_{AB} = I_{BC} = I_{CD} = I_{AD}$

Type of loading	Moments
1. Any system of loads.	$M_{AB} = -\frac{1}{8}[3C_{AD} + 2C_{DA} - 2M_D]$ $M_{BC} = -\frac{1}{8}[-C_{DA} + 2M_D]$ $M_{CD} = \frac{1}{8}[C_{AD} + 2M_D]$ $M_{DA} = -\frac{1}{8}[2C_{AD} + 3C_{DA} + 2M_D]$ Values of C for various loads are given in Tables 2 and 3.
2. Single load at mid-height.	$M_{AB} = +\frac{3}{64}Ph$ $M_{BC} = -\frac{7}{64}Ph$ $M_{CD} = +\frac{9}{64}Ph$ $M_{DA} = -\frac{13}{64}Ph$
3. Single load at top.	$M_{AB} = M_{CD} = -M_{BC} = -M_{DA} = \frac{Ph}{4}$
4. Uniform load.	$M_{AB} = \frac{7Wh}{96}$ $M_{BC} = -\frac{11Wh}{96}$ $M_{CD} = \frac{13Wh}{96}$ $M_{DA} = -\frac{17Wh}{96}$

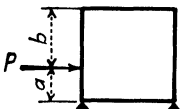
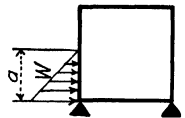
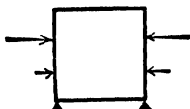
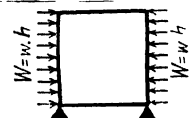
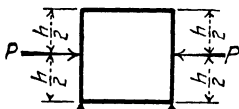
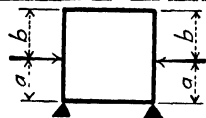
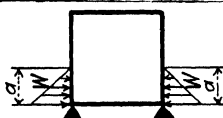
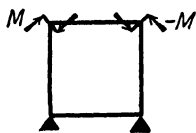
TABLE 77 (*Continued*)

Type of loading	Moments
5. Single load at any point.	$M_{AB} = \frac{Pa}{8h^2}[2h^2 - 3ab - 2b^2]$ $M_{BC} = \frac{Pa}{8}\left[\frac{b^2}{h^2} - 2\right]$ $M_{CD} = \frac{Pa}{8}\left[\frac{ab}{h^2} + 2\right]$ $M_{DA} = -\frac{Pa}{8h^2}[2h^2 + 2ab + 3b^2]$
6. Hydraulic load.	$M_{AB} = \frac{Wa^2}{240h^2}(3a + 5h)$ $M_{BC} = \frac{Wa}{240h^2}(3a^2 - 10ah - 10h^2)$ $M_{CD} = \frac{Wa}{240h^2}(5ah - 3a^2 + 20h^2)$ $M_{DA} = -\frac{Wa}{240h^2}(50h^2 - 20ah + 3a^2)$

Table 78

Rectangular Frame

All sides identical in length and cross-section
Horizontal load on each vertical side
Load symmetrical about vertical center line

$K = \frac{I_{AB}}{l} = \frac{I_{AD}}{h}$ $\quad l = h \quad$ $I_{AB} = I_{BC} = I_{CD} = I_{AD}$

Type of loading	Moments
1. Any system of loads symmetrical about vertical center line.	$M_{AB} = M_{BC} = -\frac{1}{8}[3C_{AD} + C_{DA}]$ $M_{CD} = M_{DA} = -\frac{1}{8}[C_{AD} + 3C_{DA}]$ Values of C for various loads are given in Tables 2 and 3.
2. Uniform load.	$M_{AB} = M_{BC} = M_{CD} = M_{DA} = -\frac{Wh}{24}$
3. Single load at mid-height of each vertical side.	$M_{AB} = M_{BC} = M_{CD} = M_{DA} = -\frac{Ph}{16}$
4. Single load at any point on each vertical side.	$M_{AB} = M_{BC} = -\frac{Pab}{8h^2}[3a + b]$ $M_{CD} = M_{DA} = -\frac{Pab}{8h^2}[a + 3b]$
5. Hydraulic load.	$M_{AB} = M_{BC} = -\frac{Wa(10h^2 + 5ah - 6a^2)}{240h^2}$ $M_{CD} = M_{DA} = -\frac{Wa(30h^2 - 25ah + 6a^2)}{240h^2}$
6. Couple at each upper corner.	$M_{AB} = -M_{BA} = \frac{3M}{8}$ $M_{AD} = -M_{BC} = \frac{5M}{8}$ $M_{CD} = M_{DA} = \frac{M}{8}$

APPENDIX B

DERIVATION OF FUNDAMENTAL EQUATIONS FOR ANALYSIS OF STATICALLY INDETERMINATE FRAMES

The derivation of the fundamental equations is based upon two principles, commonly known as the principles of the moment-area method. These principles are as follows:

(1) When a member is subjected to flexure, the difference in the slope of the elastic curve between any two points is equal in magnitude to the area of the $\frac{M}{EI}$ diagram for the portion of the member between the two points.

(2) When a member is subjected to flexure, the distance of any point Q on the elastic curve, measured normal to the initial position of member, from a tangent drawn to the elastic curve at any other point P, is equal in magnitude to the first or statical moment of the area of the $\frac{M}{EI}$ diagram between the two points, about the point Q.

Conventional Signs.—The signs of the quantities used in the equations in Appendix A are determined by the following conventional rules:

When the tangent to the elastic curve of a member has been turned in a clockwise direction, measured *from* its initial position, the change in slope, or the angular deformation, is positive.

When the line joining the ends of a member is rotated, the movement of one end of the member relative to the other, measured perpendicular to the initial position of member is called a deflection and is so used throughout the following discussion. The deflection is positive when such rotation is in a clockwise direction from the initial position of member.

The resisting moment, or moment of the internal stresses, on a section is positive when the internal or resisting couple acts in a clockwise direction upon *the portion of the member considered.* According to this rule the portion of the member considered must always be specified, and will be indicated by the subscripts used with the moments. For example, if C is a point on a member between the ends A and B, M_{CA} is equal to $-M_{CB}$.

The moment of an external force or couple is positive if it tends to cause a clockwise rotation.

Derivation of Equations for Moments at Ends of Members in Flexure—Member Restrained at the Ends with No Intermediate Loads.—The line AB in Fig. 1 represents the elastic curve of a member that is not acted upon by any external forces or couples except at the ends. The resisting moment at A is represented by M_{AB} and at B by M_{BA}. The change in the slope of the elastic curve at A from its initial position is represented by θ_A, and that at B by θ_B. The deflection of A from its original position A' is d. The distance of B from the tangent drawn to the curve at A is equal to $(d - l\theta_A)$.

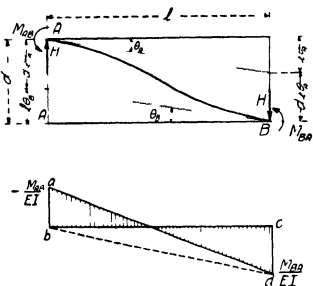



Fig. 1.

From proposition (2), $(d - l\theta_A)$ may be expressed as the statical moment of the $\frac{M}{EI}$ diagram for member AB about the end B. The quantities E and I will here be considered as constant throughout the length AB. If M represents the resisting moment on the portion of member to the left of a section, M is equal to $-M_{AB}$ at A, and to $+M_{BA}$ at B. The $\frac{M}{EI}$ diagram of Fig. 1 can best be treated as the algebraic sum of the two triangles bad and bcd. Hence the statical moment of the $\frac{M}{EI}$ diagram about B is equal to the area of triangle bad times the distance to its centroid, $\frac{2}{3}l$, plus the area of triangle bcd times the distance to its centroid, $\frac{1}{3}l$. This gives

$$d - l\theta_A = \frac{-M_{AB}l^2}{3EI} + \frac{M_{BA}l^2}{6EI} \tag{3}$$

From proposition (1), $\theta_B - \theta_A$ is equal to the area of the $\frac{M}{EI}$ diagram for member AB, or the algebraic sum of areas bad and bcd. This gives

$$\theta_B - \theta_A = \frac{-M_{AB}l}{2EI} + \frac{M_{BA}l}{2EI} \tag{4}$$

Combining Eqs. (3) and (4) to eliminate M_{BA}, letting $\frac{I}{l} = K$ and $\frac{d}{l} = R$, gives

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) \tag{5}$$

Similarly combining Eqs. (3) and (4) to eliminate M_{AB} gives

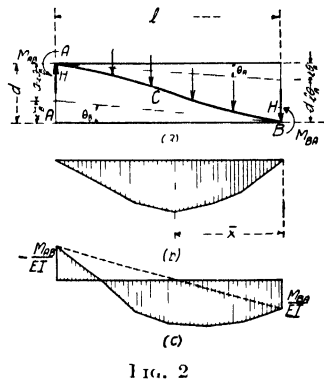
$$M_{BA} = 2EK(2\theta_B + \theta_A - 3R) \tag{6}$$

Since the signs of all quantities in Eqs. (3) and (4) are independent of the sense of the quantities themselves it follows that Eqs. (3) and

(4) are general; and they give the sense as well as magnitude of the moments, no matter what the senses of θ_A, θ_B, and R may be, provided the method of determining signs given above is followed. As before noted, M_{AB} is the resisting moment acting at the end A of the member AB. The moment which AB exerts upon the support at A is equal in magnitude but opposite in sense to M_{AB}. A and B are not necessarily supports of a member but may be any two points along the length of a member, provided there is no intermediate load on the member between them.

Equations (5) and (6) are fundamental equations.[1] They may be expressed as follows: The moment at the end of any member carrying no intermediate loads is equal to $2EK$ times the quantity: Twice the change in slope at the near end plus the change in slope at the far end minus three times the ratio of deflection to length. E is the modulus of elasticity of the material, and K is the ratio of moment of inertia to length of member.

Fig. 2

Derivation of Equations for Moments at Ends of Members in Flexure—Member Restrained at the Ends with Any System of Intermediate Loads.—The line AB, Fig. 2a, represents the elastic curve of the member of Fig. 1, but acted upon by a system of intermediate loads. The moments, slopes, and deflections at A and B are similar to those of Fig. 1. The $\frac{M}{EI}$ diagram, however, is affected by the intermediate loads. The quantity EI will again be considered constant. From well known principles of mechanics, the $\frac{M}{EI}$ diagram of Fig. 2c may be obtained by superimposing the $\frac{M}{EI}$ diagram for a simple beam under the same intermediate loads (see Fig. 2b) upon the $\frac{M}{EI}$ diagram of Fig. 1. This is merely the algebraic

[1] The slope-deflection equations for a member acted upon only by forces and couples at the ends were deduced by Manderla in 1878. See *Annual Report* of the Technische Hochschule, Munich, 1879 and *Allgemeine Bauzeitung*, 1880. The use of these equations has been developed by several writers, among whom are:

Mohr, Otto, "Abhandlungen aus dem Gebiete der Technischen Mechanik," 2nd Ed., 1914.

Kunz, F. C., Secondary Stresses, *Engineering News*, Vol. 66, p. 397, Oct 5, 1911

Wilson and Maney, Wind Stresses in the Steel Frames of Office Buildings, Univ. of Ill Eng. Exp. Sta., *Bull.* 80, 1915.

Wilson, Richart and Weiss, Analysis of Statically Indeterminate Structures by the Slope Deflection Method, Univ. of Ill. Eng. Exp. Sta , *Bull* 108, 1918.

addition of the different moments at any section, just as in an algebraic analysis the moment at the end of a girder is combined with the moment of the shear at the end and of the external loads about the given section. Denote the area of the simple beam diagram of Fig. 2b by F, and the distance of its centroid from B by $\bar{x}$. Then, using the propositions (1) and (2) as before, the statical moment of the $\frac{M}{EI}$ diagram about B is equal to $(d - \theta_A l)$.

$$(d - \theta_A l) = \frac{-M_{AB}l^2}{3EI} + \frac{M_{BA}l^2}{6EI} - \frac{Fx}{EI} \tag{7}$$

The area of the $\frac{M}{EI}$ diagram is equal to $\theta_B - \theta_A$.

$$(\theta_B - \theta_A) = \frac{-M_{AB}l}{2EI} + \frac{M_{BA}l}{2EI} - \frac{F}{EI} \tag{8}$$

Combining Eqs. (7) and (8) to eliminate M_{BA}, letting $\frac{I}{l} = K$ and $\frac{d}{l} = R$, gives

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - \frac{2F}{l^2}(3x - l) \tag{9}$$

Similarly, combining Eqs. (7) and (8) to eliminate M_{AB} gives

$$M_{BA} = 2EK(2\theta_B + \theta_A - 3R) + \frac{2F}{l^2}(2l - 3\bar{x}) \tag{10}$$

It is seen that Eqs. (9) and (10) are identical with Eqs. (5) and (6) except that they contain an additional term in the right-hand member. This additional term is independent of the slopes and deflections of the member, and depends solely upon the intermediate loads. Further significance is given to this term if the slopes and deflections are made equal to zero, as is true in a fixed beam with supports on same level. The last term then becomes the resisting moment acting on the end of the fixed beam. Hence it is seen that in general *the resisting moment at the end of a member with any system of intermediate loads can be expressed as the algebraic sum of the resisting moment at the end of a member with no intermediate loads, as given by Eqs.* (5) *and* (6), *and the resisting moment at the end of a fixed beam with an equal span and carrying the same system of intermediate loads.*

If the resisting moment at the end of a fixed beam with supports on same level be expressed by C, with subscripts similar to those used for moments, M, Eqs. (9) and (10) may be written in the following general form

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - C_{AB} \tag{11}$$

$$M_{BA} = 2EK(2\theta_B + \theta_A - 3R) + C_{BA} \tag{12}$$

These are the general *slope deflection* equations that apply to any condition of loading and restraint.

The sign of the constant C may be determined as follows: *In a fixed beam the sign of the resisting moment at the end of a member is opposite to that of the moment of external loads.* For instance, in Fig. 2 the moment of external loads about the end A is clockwise, so the resisting moment C_{AB} is counter-clockwise or negative; and since the moment of the loads is counter-clockwise about B, C_{BA} is clockwise or positive. If the loads were upward instead of downward, the signs of C_{AB} and C_{BA} would be reversed. With signs thus treated, C becomes merely a numerical, or scalar, quantity.

It has been noted that Eqs. (11) and (12) apply to any condition of restraint of the ends of a member. Figure 3 shows a member restrained at A and hinged to the support at B, so that the resisting moment at B is zero. Equations (11) and (12) may be written:

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - C_{AB}$$
$$0 = M_{BA} = 2EK(2\theta_B + \theta_A - 3R) + C_{BA}$$

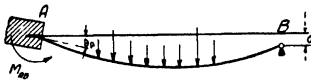



FIG. 3.

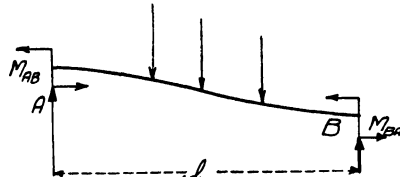



FIG. 4.

Combining these two equations to eliminate θ_B gives

$$M_{AB} = EK(3\theta_A - 3R) - \left(C_{AB} + \frac{C_{BA}}{2}\right) \tag{13}$$

If the beam is fixed at A and hinged at B, with the supports on the same level, θ_A and R in Eq. (13) are zero, and therefore the term $-\left(C_{AB} + \frac{C_{BA}}{2}\right)$ represents the resisting moment at the end A, and can be readily calculated for any given loading.

By similar reasoning, when a beam is restrained at the end B and hinged to support at A, it is found that

$$M_{BA} = EK(3\theta_B - 3R) + \left(C_{BA} + \frac{C_{AB}}{2}\right) \tag{14}$$

For more convenient reference let the quantity $\left(C_{AB} + \frac{C_{BA}}{2}\right)$ be denoted by H_{AB}, and the quantity $\left(C_{BA} + \frac{C_{AB}}{2}\right)$ by H_{BA}. Equations (13) and (14) then take the general form

$$M_{AB} = EK(3\theta_A - 3R) - H_{AB} \tag{15}$$
$$M_{BA} = EK(3\theta_B - 3R) + H_{BA} \tag{16}$$

The term H represents the resisting moment at the fixed end of a beam which is fixed at one end and hinged to the support at the other, with supports at same level. The sign of H is determined in the same way as the sign of C in Eqs. (11) and (12). That is, the sign of H is always opposite to the sign of the moment of the external loads about the fixed end of the member. If the external loads act upward instead of downward, the values of H in eqs. (15) and (16) must be reversed.

Equations (11) and (12) are the general equation for the ends of a member in flexure. Equations (15) and (16) are special forms of Eqs. (11) and (12), applicable to members having one end hinged. For convenience in reference these four equations are given in Table 1 where they are denoted as Eqs. (A), (B), (C) and (D), respectively.

TABLE 1

GENERAL EQUATIONS FOR THE MOMENTS AT THE ENDS OF A MEMBER AB IN FIG. 4

$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) \mp C_{AB}$	(A)
$M_{BA} = 2EK(2\theta_B + \theta_A - 3R) \pm C_{BA}$	(B)
If end B is hinged,	
$M_{AB} = EK(3\theta_A - 3R) \mp H_{AB}$	(C)
If end A is hinged,	
$M_{BA} = EK(3\theta_B - 3R) \pm H_{BA}$	(D)

NOTE.—The signs of the quantities used in these equations are determined by the following rules:

θ is positive (+) when the tangent to the elastic curve is turned in a clockwise direction.

R is positive (+) when the member is deflected in a clockwise direction.

The moment of the internal stresses on a section is positive (+) when the internal couple acts in a clockwise direction upon the *portion of the member considered.*

If the moment of the external forces on the member about the end at which the moment is to be determined is positive (+), the sign before the constant, C or H, is minus (−); if the moment of the external forces about the end at which the moment is to be determined is negative (−), the sign before the constant is plus (+). With the forces acting downward as shown in the sketch, for the moment at A, C_{AB} and H_{AB} are preceded by a minus (−) sign, but for the moment at B, C_{BA} and H_{BA} are preceded by a plus (+) sign.

Derivation of Equations for Moments at Ends of Members in Flexure—Member Restrained at the Ends, with Special Cases of Loading.—One method of determining the quantities C and H in Eqs. (A), (B), (C), and (D) of Table 1 has been explained. To illustrate the method, some special cases will be considered here.

Figure 5 shows a member restrained at the ends with a concentrated load at a distance a from A, and a distance b from B. In the simple beam moment diagram, the maximum ordinate is $\frac{-Pab}{l}$, the area F is $\frac{-Pab}{2}$, and the distance $\bar{x}$ of the centroid of the area F from B is $\frac{1}{3}(l + b)$

Hence putting these values in the last terms of Eqs. (9) and (10) gives

$$-\frac{2F}{l^2}(3x - l) = \frac{-Pab^2}{l^2} = -C_{AB} \tag{17}$$

and

$$\frac{2F}{l^2}(2l - 3x) = \frac{+Pa^2b}{l^2} = C_{BA} \tag{18}$$

If the member had been hinged instead of being restrained at B, the value of H_{AB} could have been found from the last term of Eq. (13), in which

$$\left(C_{AB} + \frac{C_{BA}}{2}\right) = \frac{-Pab}{2l^2}(l + b) = -H_{AB} \tag{19}$$

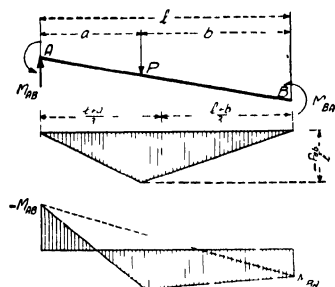
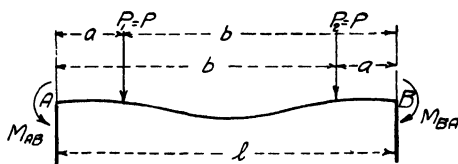



FIG. 5 FIG. 6

Similarly, if the member had been hinged at A and restrained at B, the value of H_{BA} could have been found from the last term of Eq. (14) in which

$$+\left(C_{BA} + \frac{C_{AB}}{2}\right) = \frac{Pab(l + a)}{2l^2} = H_{BA} \tag{20}$$

As another common case, consider a loading that is symmetrical about the middle of the member, as shown by Fig. 6. It is obvious that the centroid of the simple beam moment diagram will be at the middle of the member, so that $x = \frac{l}{2}$. Substituting this in the last term of Eqs. (9) and (10),

$$-\frac{2F}{l^2}(3x - l) = -\frac{F}{l} = -C_{AB} \tag{21}$$

and

$$+\frac{2F}{l^2}(2l - 3x) = +\frac{F}{l} = +C_{BA} \tag{22}$$

Similarly, for a member having the end B hinged, the last term of Eq. (13) gives

$$-\left(C_{AB} + \frac{C_{BA}}{2}\right) = -\frac{3}{2}\frac{F}{l} = -H_{AB} \tag{23}$$

For a member having the end A hinged, the last term of Eq. (14) gives

$$+\left(C_{BA} + \frac{C_{AB}}{2}\right) = +\frac{3}{2}\frac{F}{l} = +H_{BA} \tag{24}$$

A geometrical meaning is attached to the term $\frac{F}{l}$ since it represents the average ordinate of the moment diagram for a simple beam under the given loading.

From these illustrations it is seen that values of C and H may be found by the use of Eqs. (9), (10), (13) and (14).

Another method of determining C and H may be readily applied to any kind of loading. For a member carrying a single concentrated load P, as shown in Fig. 5, the value of C_{AB} is $\frac{Pab^2}{l^2}$ and the value of C_{BA} is $\frac{Pa^2b}{l^2}$ as given in Eqs. (17) and (18). If there are several concentrated loads on the member, by summation $C_{AB} = \sum\frac{Pab^2}{l^2}$, and $C_{BA} = \sum\frac{Pa^2b}{l^2}$.

If there is a distributed load on the member the same method may be used, by performing an integration in place of the summation. Let w be the unit loading on an element of length dx, which is at a distance x from the left end, and a distance $l - x$ from the right end of the member. In the expression $\frac{Pab^2}{l^2}$, replace P by wdx, a by x, and b by $l - x$, whence $C_{AB} = \int\frac{wx(l - x)^2dx}{l^2}$. Similarly, $C_{BA} = \int\frac{wx^2(l - x)dx}{l^2}$. The limits of the definite integral are fixed by the length of the member under load.

If the unit load w is not constant, its variation may be expressed in terms of x, and the general value for the total load on a length dx thus found substituted for P in the given expression for a single concentrated load, after which the integration may be performed as just indicated.

Values of C and H for different systems of loads are given for reference in Tables 2 and 3, pp. 500 and 503, respectively.

APPENDIX C

CHARTS FOR CONCRETE BEAMS AND COLUMNS WITH VARIABLE MOMENT OF INERTIA[1]

The charts presented here give coefficients for determination of *f.e.m.*,[2] stiffness, and carry-over factors. The members considered have various shapes which may be classified in the four groups illustrated in Fig. 1: *symmetrical* members with (*a*) straight haunches and with (*b*) parabolic haunches at both ends; *unsymmetrical* members with (*c*) straight haunch and with (*d*) parabolic haunch at one end. The "parabolic haunch" is in all cases a segment of a second degree parabola as indicated in Fig. 1. For each of the four types, the shape of a member is characterized by two *ratios*, a representing the length and b the depth of the haunch. The value of $b = \frac{\text{min. } I}{\text{max. } I}$ (see Fig. 1) is assumed to equal the ratio of $\left(\frac{\text{min. } d}{\text{max. } d}\right)^3$, the beam width being considered constant and reinforcement disregarded.

When the shape of a member is known, the procedure is to determine ratios of a and b, $\left(\text{or } \frac{\text{min. } d}{\text{max. } d}\right)$, enter the proper chart with these values and select coefficients of f, k, and C. Then compute $f.e.m. = f \times W \times L$ and $K = k \times E \times \frac{\text{min. } I}{L}$ (absolute value) or $K = k \times B \times \frac{(\text{min. } d)^3}{L}$ (relative value), in which B denotes beam width. The value of C from the chart is used without change in the analysis.

The same k- and C-values apply at both ends of symmetrical members; but for unsymmetrical members, the values of k_H and C_H (at haunched end) differ from k_S and C_S (at small end). Careful distinction is made on the charts between haunched or small and left or right end of a member, but an additional check may be obtained on the selection of k- and C-values by observing the following rules:

(*a*) The stiffness factor is greater at haunched than at small end: $k_H > k_S > 4$.

(*b*) The carry-over factor is smaller at haunched than at small end: $C_H < 0.5 < C_S$.

(*c*) The product of $k \times C$ is the same for both ends: $k_H C_H = k_S C_S$.

[1] Reproduced from pamphlet ST41, with the permission of the Portland Cement Association.

[2] Fixed end moment.

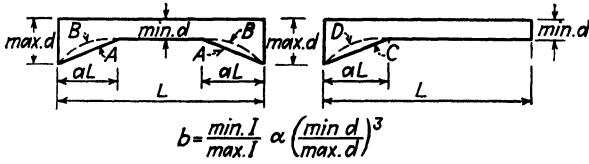
In addition, it should be kept in mind that C_H is the factor used for multiplication of the distributed moment, M_H, at haunched end; *i.e.*, $C_H \times M_H$ gives the carry-over moment induced at small end; and transposing the subscripts, the product of $C_S \times M_S$ gives carry-over moment induced at haunched end due to a distributed moment, M_S, at the small end.

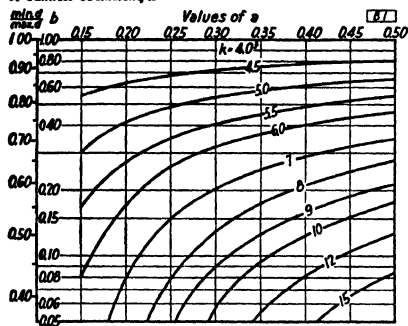
Fig. 1.

SYMMETRICAL MEMBERS WITH STRAIGHT HAUNCHES

2. Carry-over Factor, C

3. Uniform Load f.e.m. Coefficient, f

4. Concentrated Load f.e.m. Coefficient, f

SYMMETRICAL MEMBERS WITH PARABOLIC HAUNCHES

1. Stiffness Coefficient, k

Values of a

2. Carry-over Factor, C

Values of a

3. Uniform Load f.e.m. Coefficient, f

Values of a

4. Concentrated Load f.e.m. Coefficient, f

Left max d — Right max d

$b = \left(\frac{\text{min } d}{\text{max } d}\right)^3$

Fixed end moment = f × W × L

Left f.e.m. – Obtain f-values below using bottom scale

Right f.e.m. – Obtain f-values below using top scale

Values of v (for f.e.m. at right end)

Values of f

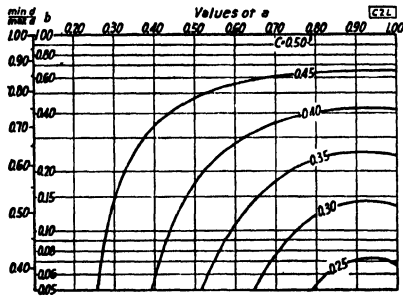
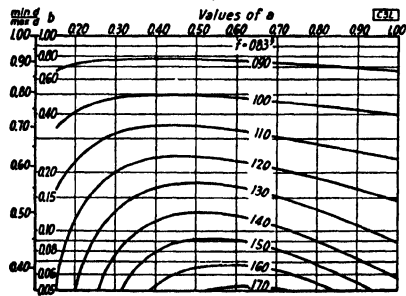
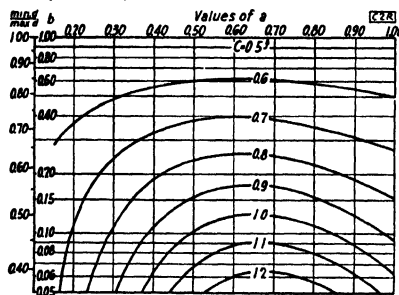
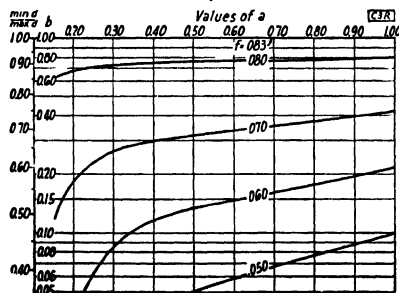
Legend: a = 0.50; a = 0.25; a = 0.15

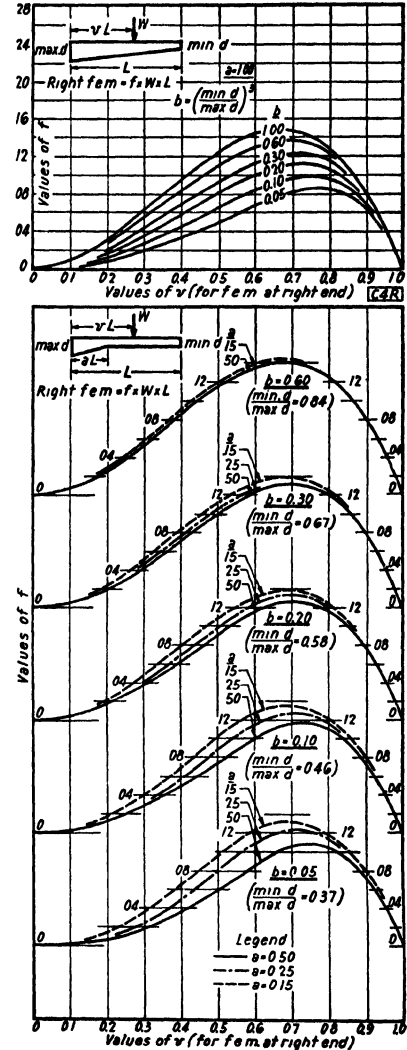
Values of v (for f.e.m. at left end)

UNSYMMETRICAL MEMBERS WITH STRAIGHT HAUNCH AT ONE END

Coefficients at Haunched End

1. Stiffness Coefficient, k

2. Carry-over Factor, C

3. Uniform Load f e m Coefficient, f

4. Concentrated Load f e.m. Coefficient, f

UNSYMMETRICAL MEMBERS WITH STRAIGHT HAUNCH AT ONE END

Coefficients at Small End

1. Stiffness Coefficient, k

4. Concentrated Load f.e.m. Coefficient, f

2. Carry-over Factor, C

3. Uniform Load f.e.m. Coefficient, f

UNSYMMETRICAL MEMBERS WITH PARABOLIC HAUNCH AT ONE END

Coefficients at Small End

UNSYMMETRICAL MEMBERS WITH PARABOLIC HAUNCH AT ONE END

Coefficients at Haunched End

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